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M. Ashordia, N. Kekelia, and T. Chilachava

CRITERIA OF STABILITY FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Consider a linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t), \tag{1}$$

where $A : [0, +\infty[\to R^{n\times n}$ is a real matrix-function with locally bounded variation components.

In this paper we give necessary and sufficient conditions guaranteling stability of the system (1) in Liapunov sense.

The following notation and definitions will be used throughout the paper:

$$R =]-\infty; +\infty[, \quad R_+ = [0; +\infty[, \quad [a, b](a, b \in R) \quad \text{is a closed segment;}$$

R^n is the space of all real column n-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$R^{n\times n}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^n$ with the norm

$$\|x\| = \max_{j=1,\dots,n} \sum_{i=1}^n |x_{ij}|.$$

If $x \in R^{n\times n}$, then X^{-1} and $\det(X)$ are respectively the matrix inverse to X and the determinant of X; $|X| = (|x_{ij}|)_{i,j=1}^n$. I_n is the identity $n \times n$-matrix. $\vee_0^{+\infty}(X) = \sup_{a,b\in R_+} \vee_a^b(x)$, where $\vee_a^b(X)$ is the sum of total variations on $[a, b]$ of the components $x_{ij} (i, j = 1, \dots, n)$ of the matrix-function $X : R_+ \to R^{n\times n}$; $V(x)(t) = v(x_{ij})(t))_{i,j=1}^n$, where $v(x_{ij})(t) = \vee_0^t(x_{ij})$ for $t \in R_+$ $(i, j = 1, \dots, n)$.

$X(t-)$ and $X(t+)$ $(X(0-) = X(0))$ are the left and the right limits of the matrix-function $X : R_+ \to R^{n\times n}$ at the point t;

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$

$BV_{\text{loc}}(R_+; R^{n\times n})$ is the set of all matrix-functions of bounded variation on every closed segment from R_+.

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If $g : [a,b] \to R$ is a nondecreasing function, $x : [a,b] \to R$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)dg_1(\tau) - \int_{]s,t[} x(\tau)dg_2(\tau) +$$
$$+ \sum_{s<\tau\le t} x(\tau)d_1g(\tau) - \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $g_j : [a,b] \to R$ $(j=1,2)$ are continuous nondecreasing functions such that the function $g_1 - g_2$ is identicaly equal to the continuons part of g, and $\int_{]s,t[} x(\tau)dg_j(\tau)$ is Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function g_j $(j=1,2)$ (if $s=t$, then $\int_s^t x(\tau)dg(\tau) = 0$);

If $G = (g_{ij})_{i,j=1}^n \in BV_{\rm loc}(R_+; R^{n\times n})$, $x = (x_i)_{i=1}^n$: $R_+ \to R^n$, $X = (x_{ij})_{i,j=1}^n$: $R_+ \to R^{n\times n}$ and $0 \le s \le t < +\infty$, then

$$\int_s^t dG(\tau)x(\tau) = \Big(\sum_{k=1}^n \int_s^t x_k(\tau)dg_{ik}(\tau)\Big)_{i=1}^n,$$
$$\int_s^t dG(\tau)X(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)dg_{ik}(\tau)\Big)_{i,j=1}^n,$$

$$\mathcal{B}(G,X)(t) = G(t)X(t) - G(a)X(a) - \int_0^t dG(\tau)dG(\tau)\cdot X(\tau),$$

$$\mathcal{L}(G,X)(t) = \int_0^t d[G(\tau) + B(G,X)(\tau)]\cdot G^{-1}(\tau).$$

A vector-function $x \in BV_{\rm loc}(R_+; R^n)$ is called a solution of the system (1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) \quad \text{for} \quad 0 \le s \le t < +\infty.$$

A matrix-function $G \in BV_{\rm loc}(R_+; R^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition if

$$\int_0^t dG(s)\cdot G(s) = \int_0^t G(s)\cdot dG(s) \quad \text{for} \quad t \in R_+.$$

We will assume that $A \in BV_{\rm loc}(R_+; R^{n\times n})$ is such that

$$\det(I_n + (-1)^j d_jA(t)) \ne 0 \quad \text{for} \quad t \in R_+ \quad (j=1,2) \tag{2}$$

and

$$\lim_{t\to+\infty} \vee_0^t(A) = +\infty.$$

Let $x_0 \in BV_{\rm loc}(R_+; R^n)$ be a solution of the system (1).

Definition 1. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty. \tag{3}$$

The solution x_0 of the system (1) is called ξ-exponentially asymptotically stable, if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of the system (1), satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in R_+$, admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp(-\eta(\xi(t) - \xi(t_0))) \quad \text{for} \quad t \geq t_0.$$

Stabililty, uniformly stability and asymptotically stability of the solution x_0 are defined analoguosly as for systems of ordinary differential equations (see [1] or [2] for example), i.e. in case when $A(t)$ is the diagonal matrix-function with diagonal elements equal to t. Note the exponentially asymptotically stability ([1], [2]) is a particular case of ξ-exponentially asymptotically stability ($\xi(t) \equiv t$).

Definition 2. The system (1) is called stable (uniformly stable, asymptotically stable, ξ-exponentially asymptotically stable) if every solution of this system is stable (uniformly stable, asymptotically stable, ξ-asymptotically stable).

Theorem 1. *The system* (1) *is stable if and only if there exists a matrix-function* $H \in BV_{\text{loc}}(R_+; R^{n\times n})$ *such that the conditions*

$$\det(H(t)) \neq 0 \quad \textit{for} \quad t \in R_+, \qquad (4)$$
$$\sup\{\|H^{-1}(t)\| : t \in R_+\} < +\infty$$

and

$$\vee_0^{+\infty}(H + B(H, A)) < +\infty \qquad (5)$$

hold.

Theorem 2. *The system* (1) *is uniformly stable if and only if there exists a matrix-function* $H \in BV_{\text{loc}}(R_+; R^{n\times n})$ *such that the conditions* (4), (5) *and*

$$\sup\{\|H^{-1}(t)H(S)\| : t \geq s \geq 0\} < +\infty$$

hold.

Theorem 3. *The system* (1) *is asymptotically stable if and only if there exists a matrix-function* $H \in BV_{\text{loc}}(R_+; R^{n\times n})$ *such that the conditions* (4), (5) *and*

$$\lim_{t\to+\infty} \|H^{-1}(t)\| = 0$$

hold.

Theorem 4. *The system* (1) *is* ξ*-exponentially asymptotically stable if and only if there exist a matrix-function* $H \in BV_{\text{loc}}(R_+; R^{n\times n})$ *and positive constants* η *and* ρ *such that the conditions* (4),

$$\|H^{-1}(t)H(s)\| \leq \rho e^{-\eta(\xi(t)-\xi(s))} \quad \textit{for} \quad t \geq s \geq 0$$

and

$$\int_0^{+\infty} e^{-\eta\xi(\tau)} d\|V(H + B(H, A))(\tau)\| < +\infty$$

hold, where $\xi : R_+ \to R_+$ *is a nondecreasing function satisfying the condition* (3).

Corollary 1. *Let there exist a matrix-function* $Q \in BV_{\mathrm{loc}}(R_+; R^{n\times n})$ *such that the system*

$$dz(t) = dQ(t) \cdot z(t) \tag{6}$$

is stable (*uniformly stable, asymptotically stable,* ξ*-exponentially asymptotically stable*) *and the conditions*

$$\det(I_n + (-1)^j d_j Q(t)) \neq 0 \quad \textit{for} \quad t \in R_+ \ (j = 1, 2) \tag{7}$$

and

$$\vee_0^{+\infty} \mathcal{B}(Z^{-1}, A - Q) < +\infty$$

hold, where $Z(Z(0) = I_n)$ *is a fundamental matrix of the system* (6)*. Then* (1) *is stable* (*uniformly stable, asymptotically stable,* ξ*-exponentially asymptotically stable*).

Corollary 2. *Let the system* (6) *be stable* (*uniformly stable, asymptotically stable,* ξ*-exponentially asymptotically stable*)*, where* Q *is a continuous locally bounded variation matrix-function satisfying Lappo-Danilevskiĭ condition. Let, moreover, the condition*

$$\Big\| \int_0^{+\infty} |e^{-Q(\tau)}| \, dV(A - Q)(\tau) \Big\| < +\infty$$

hold. Then (1) *is stable* (*uniformly stable, asymptotically stable,* ξ*-exponentially asymptotically stable*).

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Authors' addresses:

M. Ashordia
I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 380043
Georgia

N. Kekelia and T. Chilachava
Sukhumi Branch of
Tbilisi State University
12, Djikia St., Tbilisi 380086
Georgia