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NECESSARY CONDITIONS OF OPTIMALITY IN NEUTRAL TYPE OPTIMAL PROBLEMS WITH NON-FIXED INITIAL MOMENT

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Let $J=[a,b]$ be a finite interval; $O\subset\mathbb{R}^n$, $G\subset\mathbb{R}^r$ be open sets; $\tau:\mathbb{R}^1\to\mathbb{R}^1$, $\eta:\mathbb{R}^1\to\mathbb{R}^1$ be absolutely continuous and continuously differentiable functions, respectively, satisfying the conditions: $\tau(t)\le t$, $\dot\tau(t)>0$, $\eta(t)<t$, $\dot\eta(t)>0$; $\gamma(t)=\tau^{-1}(t)$, $\sigma(t)=\eta^{-1}(t)$; $q^i: J^2\times O^2\to\mathbb{R}^1$, $i=0,\dots,l$, be continuously differentiable functions; $\Delta=\Delta(J_1,M)$ be the set of continuously differentiable functions $\varphi:J_1\to M$, $J_1=[\rho(a),b]$, $\rho(t)=\min\{\eta(t),\tau(t)\}$, $t\in J$, $\|\varphi\|=\sup\{|\varphi(a)|+|\dot\varphi(t)|: t\in J_1\}$, $M\subset O$ be a convex bounded set; Ω_1 be the set of measurable functions $u:J\to U$ such that $cl\{u(t): t\in J\}\subset G$ is compact, $U\subset G$ be an arbitrary set; Ω_2 be a set of measurable functions $v:J\to V$, $V\subset G$ be a convex bounded set; $A(t,v)$ the an $n\times n$ dimensional matrix function, continuous on $J\times V$ and continuously differentiable with respect to $v\in V$.

Next, let the function $f:J\times O^2\times G\to\mathbb{R}^n$ satisfy the following conditions:

1) for a fixed $t\in J$ the function $f(t,x_1,x_2,u)$ is continuous with respect to $(x_1,x_2,u)\in O^2\times G$ and continuously differentiable with respect to $(x_1,x_2)\in O^2$;

2) for a fixed $(x_1,x_2,u)\in O^2\times G$ the functions f, f_{x_i}, $i=1,2$, are measurable with respect to t; for arbitrary compacts $K\subset O$, $W\subset G$ there exists a function $m_{K,W}(\cdot)\in L_1(J,\mathbb{R}_0^+)$, $\mathbb{R}_0^+=[0,\infty)$, such that

$$|f(t,x_1,x_2,u)|+\sum_{i=1}^2|f_{x_i}(\cdot)|\le m_{K,W}(t),\quad \forall(t,x_1,x_2,u)\in J\times K^2\times W.$$

To every element $\mu=(t_0,t_1,x_0,\varphi,u,v)\in B=J^2\times O\times\Delta\times\Omega_1\times\Omega_2$, $t_0<t_1$, there corresponds the differential equation

$$\dot x(t)=A(t,v(t))\dot x(\eta(t))+f(t,x(t),x(\tau(t)),u(t)),\quad t\in[t_0,t_1],\tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\rho(t_0),t_0),\quad x(t_0)=x_0.\tag{2}$$

Definition 1. The function $x(t)=x(t,\mu)\in O$, $t\in[\rho(t_0),t_1]$, said to be a solution corresponding to the element $\mu\in B$, if on $[\rho(t_0),t_0]$ it satisfies the condition (2), while on the interval $[t_0,t_1]$ is absolutely continuous and satisfies the equation (1) almost everywhere.

Definition 2. The element $\mu\in B$ is said to be admissible, if the corresponding solution $x(t)$ satisfies the conditions

$$q^i(t_0,t_1,x_0,x(t_1))=0,\quad i=1,\dots,l\,.$$

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Definition 3. The element $\tilde{\mu} = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{\varphi}, \tilde{u}, \tilde{v}) \in B_0$ is said to be locally optimal, if there exist a number $\delta > 0$ and a compact set $X \subset O$ such that for an arbitrary element $\mu \in B_0$ satisfying

$$|\tilde{t}_0 - t_0| + |\tilde{t}_1 - t_1| + |\tilde{x}_0 - x_0| + \|\tilde{\varphi} - \varphi\| + \|\tilde{f} - f\|_X + \sup_{t \in J} |\tilde{v}(t) - v(t)| \le \delta,$$

the inequality

$$q^0(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{x}(\tilde{t}_1)) \le q^0(t_0, t_1, x_0, x(t_1))$$

is fulfilled.

Here

$$\|\tilde{f} - f\|_X = \int_J H(t; f, X),$$

$$H(t; f, X) = \sup\left\{ |\tilde{f}(t, x_1, x_2) - f(t, x_1, x_2)| + \sum_{i=1}^{2} |\tilde{f}_{x_i}(\cdot) - f_{x_i}(\cdot)| : (x_1, x_2) \in X^2 \right\};$$

$\tilde{f}(t, x_1, x_2) = f(t, x_1, x_2, \tilde{u}(t)),\ f(t, x_1, x_2) = f(t, x_1, x_2, u(t)),\ \tilde{x}(t) = x(t, \tilde{\mu})$.

The problem of optimal control consists in finding a locally optimal element.

Theorem 1. *Let $\tilde{\mu} \in B_0,\ \tilde{t}_i \in (a, b),\ i = 0, 1,$ be a locally optimal element, $\tilde{v}(t)$ be a piecewise continuous function; $\gamma_0 = \gamma(\tilde{t}_0) \in (\tilde{t}_0, \tilde{t}_1),\ \sigma_0 = \sigma(\tilde{t}_0) \in (\tilde{t}_0, \tilde{t}_1)$, there exist integer numbers $m_i \ge 0,\ i = 1, 2,$ such that $\gamma_0 \in (\eta^{m_1+1}(\tilde{t}_1), \eta^{m_1}(\tilde{t}_1)),\ \sigma_0 \in (\eta^{m_2+1}(\tilde{t}_1), \eta^{m_2}(\tilde{t}_1))$ $(\eta^i(t) = \eta(\eta^{i-1}(t)),\ \eta^0(t) = t)$ and there exist the finite limits:*

$$\lim_{\omega \to \nu_0} \tilde{f}(\omega) = f_0^-,\ \omega = (t, x_1, x_2) \in \mathbb{R}^-_{\tilde{t}_0} \times O^2,\ \mathbb{R}^-_{\tilde{t}_0} = (-\infty, \tilde{t}_0],\ \nu_0 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0)));$$

$$\lim_{(\omega_1, \omega_2) \to (\nu_1, \nu_2)} [\tilde{f}(\omega_1) - \tilde{f}(\omega_2)] = f_1^-,\ \omega_i \in \mathbb{R}^-_{\gamma_0} \times O^2,\ i = 1, 2, \quad \nu_1 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{x}_0);$$

$$\nu_2 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(\tilde{t}_0)), \quad \lim_{t \to \tilde{t}_0^-} \dot{\gamma}(t) = \dot{\gamma}^-;$$

$$\lim_{\omega \to \nu_3} \tilde{f}(\omega) = f_2^-,\ \omega \in \mathbb{R}^-_{\tilde{t}_1} \times O^2,\ \nu_3 = (\tilde{t}_1, \tilde{x}(\tilde{t}_1), \tilde{x}(\tau(\tilde{t}_1)));$$

$$\lim_{t \to \tilde{t}_i^-} \tilde{A}(t) = A^-_{\tilde{t}_i}, \quad i = 0, 1, \quad \tilde{A}(t) = A(t, \tilde{v}(t));$$

$$\lim_{t \to \sigma^i(\gamma_0)} \tilde{A}(t) = A^-_{\sigma^i(\gamma_0)}, \quad t \in \mathbb{R}^-_{\sigma^i(\gamma_0)}, \quad i = 1, \dots, m_1;$$

$$\lim_{t \to \sigma^i(\sigma_0)} \tilde{A}(t) = A^-_{\sigma^i(\sigma_0)}, \quad t \in \mathbb{R}^-_{\sigma^i(\sigma_0)}, \quad i = 0, \dots, m_2;$$

Then there exists a non-zero vector $\pi = (\pi_0, \dots, \pi_l),\ \pi_0 \le 0$, and solutions $\psi(t),\ \chi(t)$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t)\tilde{f}_{x_1}[t] - \psi(\gamma(t))\tilde{f}_{x_2}[\gamma(t)]\dot{\gamma}(t), \\ \psi(t) = \chi(t) + \psi(\sigma(t))\tilde{A}(\sigma(t))\dot{\sigma}(t), \quad t \in [\tilde{t}_0, \tilde{t}_1], \quad \psi(t) = 0, \quad t > \tilde{t}_1, \end{cases} \tag{3}$$

such that the following conditions are fulfilled:

$$\int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma(t))\tilde{f}_{x_2}[\gamma(t)]\dot{\gamma}(t)\dot{\tilde{\varphi}}(t)dt + \int_{\eta(\tilde{t}_0)}^{\tilde{t}_0} \psi(\sigma(t))\tilde{A}(\sigma(t))\dot{\sigma}(t)\dot{\tilde{\varphi}}(t)dt \ge$$

$$\geq \int\limits_{\tau(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma(t))\tilde{f}_{x_2}[\gamma(t)]\dot{\gamma}(t)\varphi(t)dt + \int\limits_{\eta(\tilde{t}_0)}^{\tilde{t}_0} \psi(\sigma(t))\tilde{A}(\sigma(t))\dot{\sigma}(t)\dot{\varphi}(t)dt, \ \ \forall \varphi \in \Delta, \tag{4}$$

$$\int\limits_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)\tilde{f}[t]dt \geq \int\limits_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)f(t,\tilde{x}(t),\tilde{x}(\tau(t)),u(t))dt, \ \ \forall u \in \Omega_1; \tag{5}$$

$$\int\limits_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)\tilde{A}_v(t)\dot{\tilde{x}}(\eta(t))\tilde{v}(t)dt \geq \int\limits_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)\tilde{A}_v(t)\dot{\tilde{x}}(\eta(t))v(t)dt, \ \ \forall v \in \Omega_2; \tag{6}$$

$$\pi\tilde{Q}_{x_0} = -\chi(\tilde{t}_0), \quad \pi\tilde{Q}_{x_1} = \chi(\tilde{t}_1), \tag{7}$$

$$\pi\tilde{Q}_{t_0} \geq \chi(\tilde{t}_0)[A^-_{\tilde{t}_0}\dot{\varphi}(\eta(\tilde{t}_0)) +$$

$$+f^-_0] + \psi(\sigma^-_0)A^-_{\sigma_0}[A^-_{\tilde{t}_0}\dot{\varphi}(\eta(\tilde{t}_0)) + f^-_0 - \dot{\varphi}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0) + \psi(\gamma^-_0)f^-_1\dot{\gamma}^-,$$

$$\pi\tilde{Q}_{t_1} \geq -\psi(\tilde{t}_1)[A^-_{\tilde{t}_1}\dot{\tilde{x}}(\eta(\tilde{t}^-_1)) + f^-_2].$$

Here $Q = (q^0, \ldots, q^l)^T$, the tilde over Q means that the corresponding gradient is calculated at the point $(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{x}(\tilde{t}_1))$; $\tilde{f}_{x_i}[t] = \tilde{f}_{x_i}(t,\tilde{x}(t),\tilde{x}(\tau(t)))$, $\tilde{f}[t] = \tilde{f}(t,\tilde{x}(t),\tilde{x}(\tau(t)))$.

Theorem 2. *Let $\tilde{\mu} \in B_0$, $\tilde{t}_i \in (a,b)$, $i = 0,1$, be a locally optimal element, $\tilde{v}(t)$ be a piecewise continuous function; $\gamma_0 \in (\tilde{t}_0, \tilde{t}_1)$, $\sigma_0 \in (\tilde{t}_0, \tilde{t}_1)$, there exist integer numbers $m_i \geq 0$, $i = 1,2$, such that $\gamma_0 \in (\eta^{m_1+1}(\tilde{t}_1), \eta^{m_1}(\tilde{t}_1))$, $\sigma_0 \in (\eta^{m_2+1}(\tilde{t}_1), \eta^{m_2}(\tilde{t}_1))$ and there exist the finite limits:*

$$\lim_{\omega\to\nu_0} \tilde{f}(\omega) = f^+_0, \ \omega \in \mathbb{R}^+_{\tilde{t}_0} \times O^2, \ \ \lim_{t\to\tilde{t}^+_i} \tilde{A}(t) = A^+_{\tilde{t}_i}, \ \ i = 0,1, \ \ \lim_{t\to\tilde{t}^+_0} \dot{\gamma}(t) = \dot{\gamma}^+;$$

$$\lim_{(\omega_1,\omega_2)\to(\nu_1,\nu_2)} [\tilde{f}(\omega_1) - \tilde{f}(\omega_2)] = f^+_1, \ \omega_i \in \mathbb{R}^+_{\gamma_0} \times O^2, \ i = 1,2,$$

$$\lim_{\omega\to\nu_3} \tilde{f}(\omega) = f^+_2, \ \omega \in \mathbb{R}^+_{\tilde{t}_1} \times O^2;$$

$$\lim_{t\to\sigma^i(\gamma_0)} \tilde{A}(t) = A^+_{\sigma^i(\gamma_0)}, \ \ t \in \mathbb{R}^+_{\sigma^i(\gamma_0)}, \ \ i = 1,\ldots,m_1;$$

$$\lim_{t\to\sigma^i(\sigma_0)} \tilde{A}(t) = A^+_{\sigma^i(\sigma_0)}, \ \ t \in \mathbb{R}^+_{\sigma^i(\sigma_0)}, \ \ i = 0,\ldots,m_2.$$

Then there exists a non-zero vector $\pi = (\pi_0, \ldots, \pi_l)$, $\pi_0 \leq 0$, and solutions $\psi(t)$, $\chi(t)$ of the system (3) such that the conditions (4) – (7) are fulfilled. Moreover,

$$\pi\tilde{Q}_{t_0} \leq \chi(\tilde{t}_0)[A^+_{\tilde{t}_0}\dot{\varphi}(\eta(\tilde{t}_0)) + f^+_0] + \psi(\sigma^+_0)A^+_{\sigma_0}[A^+_{\tilde{t}_0}\dot{\varphi}(\eta(\tilde{t}_0)) + f^+_0 - \dot{\varphi}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0) + \psi(\gamma^+_0)f^+_1\dot{\gamma}^+,$$

$$\pi\tilde{Q}_{t_1} \leq -\psi(\tilde{t}_1)[A^+_{\tilde{t}_1}\dot{\tilde{x}}(\eta(\tilde{t}^+_1)) + f^+_2].$$

Theorem 3. *Let $\tilde{\mu} \in B_0$, $\tilde{t}_i \in (a,b)$, $i = 0,1$, be a locally optimal element, $\tilde{v}(t)$ be a piecewise continuous function; $\gamma_0 \in (\tilde{t}_0, \tilde{t}_1)$, $\sigma_0 \in (\tilde{t}_0, \tilde{t}_1)$, there exist integer numbers $m_i \geq 0$, $i = 1,2$, such that $\gamma_0 \in (\eta^{m_1+1}(\tilde{t}_1), \eta^{m_1}(\tilde{t}_1))$, $\sigma_0 \in (\eta^{m_2+1}(\tilde{t}_1), \eta^{m_2}(\tilde{t}_1))$ the function $\dot{\tau}(t)$ be continuous at the point $\tilde{t}_0$, the function $\tilde{f}(\omega)$ be continuous at the points ν_i, $i = 0,1,2,3$, the function $\tilde{A}(t)$ be continuous at the points $\tilde{t}_0$, $\tilde{t}_1$, $\sigma^i(\gamma_0)$, $i = 1,\ldots,m_1$, $\sigma^i(\sigma_0)$, $i = 0,\ldots,m_2$, the function $\dot{\tilde{x}}(\eta(t))$ be continuous at the point $\tilde{t}_1$.*

Then there exists a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$, *and solutions* $\psi(t)$, $\chi(t)$ *of the system* (3) *such that the conditions* (4)–(7) *are fulfilled. Moreover,*

$$\pi\tilde{Q}_{t_0} = \chi(\tilde{t}_0)[\tilde{A}(\tilde{t}_0)\dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + \tilde{f}(\nu_0)] + \psi(\sigma_0)\tilde{A}(\sigma_0)[\tilde{A}(\tilde{t}_0)\dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + f(\nu_0) -$$
$$-\dot{\tilde{\varphi}}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0) + \psi(\gamma_0)[\tilde{f}(\nu_1) - \tilde{f}(\nu_2)]\dot{\gamma}(\tilde{t}_0),$$
$$\pi\tilde{Q}_{t_1} = -\psi(\tilde{t}_1)[A(\tilde{t}_1)\dot{\tilde{x}}(\eta(\tilde{t}_1)) + \tilde{f}(\nu_3)].$$

Finally we note that the theorems formulated above are analogues of the theorems given in [1]. These theorems are proved using formulas for the differential of the solution with respect to the initial data and the right-hand side given in [2], by the scheme described in [3].

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