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**ON EXISTENCE OF SOLUTIONS OF THE PERIODIC
BOUNDARY VALUE PROBLEM FOR NONLINEAR SYSTEM
OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS**

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Consider the ω -periodic problem for the nonlinear system

$$dx(t) = dA(t) \cdot f(t, x(t)), \quad (1)$$

where ω is a positive number, $A(t) = A^{(1)}(t) - A^{(2)}(t)$, $A(t) = (a_{ik}(t))_{i,k=1}^n$, $A^{(m)} = (a_{ik}^{(m)})_{i,k=1}^n \in \text{BV}_\omega(\mathbb{R}, \mathbb{R}^{n \times n})$ ($m=1, 2$) are nondecreasing on $[0, \omega]$, $f(t, x) = (f_k(t, x))_{k=1}^n$, the functions $f_k : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ($k = 1, \dots, n$) are periodic with respect to the first variable by period ω , i.e., $f_k(t + \omega, x) \equiv f_k(t, x)$ ($k = 1, \dots, n$), and their restriction on $[0, \omega]$ belong to $K([0, \omega] \times \mathbb{R}^n, \mathbb{R}; a_{ik})$ for every $i \in \{1, \dots, n\}$.

The following notation and definitions will be used: $\mathbb{R} =] - \infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ ($a, b \in \mathbb{R}$) is a closed segment, $\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ -matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm

$$\|X\| = \max_{k=1, \dots, m} \sum_{i=1}^n |x_{ik}|;$$

$$\mathbb{R}_+^{n \times m} = \{(x_{ik})_{i,k=1}^{n,m} : x_{ik} \geq 0 \ (i = 1, \dots, n; k = 1, \dots, m)\};$$

$\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column n -vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.

$\text{BV}([a, b], D)$, where $D \subset \mathbb{R}^{n \times m}$, is the set of all matrix-functions $X = (x_{ik})_{i,k=1}^{n,m} : [a, b] \rightarrow \mathbb{R}^{n \times m}$ such that every its component x_{ik} has bounded total variation on $[a, b]$.

$s_k : \text{BV}([a, b], \mathbb{R}) \rightarrow \text{BV}([a, b], \mathbb{R})$ ($k = 0, 1, 2$) are the operators defined by

$$s_1(x)(a) = s_2(x)(a) = 0, \quad s_1(x)(t) = \sum_{a < \tau \leq t} d_1 x(\tau),$$

$$s_2(x)(t) = \sum_{a \leq \tau < t} d_2 x(\tau) \quad \text{for } t \in]a, b]; \quad s_0(x)(t) \equiv x(t) - s_1(x)(t) - s_2(x)(t).$$

$\text{BV}_\omega(\mathbb{R}, \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ such that $X(t + \omega) = X(t) + X(\omega)$ for $t \in \mathbb{R}$, and its restriction on $[0, \omega]$ belongs to $\text{BV}([0, \omega], \mathbb{R}^{n \times m})$; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point $t \in \mathbb{R}$; $d_1 X(t) = X(t) - X(t-)$, $d_2 X(t) = X(t+) - X(t)$.

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If $x \in \text{BV}([0, \omega], \mathbb{R})$, $1 + (-1)^j d_j x(t) \neq 0$ for $t \in [0, \omega]$ ($j = 1, 2$) and $\lambda(x)(\omega) \neq 1$, where

$$\lambda(x)(t) = \exp\left(s_0(x)(t)\right) \prod_{0 \leq \tau < t} (1 + d_2 x(\tau)) / \prod_{0 < \tau \leq t} (1 - d_1 x(\tau)), \quad (2)$$

then

$$g_0(x)(t, \tau) = \begin{cases} (1 - \lambda(x)(\omega))^{-1} \lambda(x)(\omega) \cdot \lambda(x)(t) [\lambda(x)(\tau)]^{-1} & \text{for } 0 \leq t \leq \tau \leq \omega, \\ (1 - \lambda(x)(\omega))^{-1} \cdot \lambda(x)(t) [\lambda(x)(\tau)]^{-1} & \text{for } 0 \leq \tau < t \leq \omega; \end{cases} \quad (3)$$

$$g_j(x)(t, \tau) = (1 + (-1)^j d_j x(\tau))^{-1} g_0(x)(t, \tau) \text{ for } t \neq \tau, \quad t, \tau \in [0, \omega] \quad (j = 1, 2) \quad (4)$$

and

$$g_j(x)(t, t) = (1 + (-1)^j d_j x(t))^{-1} \cdot \lambda^{j-2}(x)(\omega) g_0(x)(t, t) \text{ for } t \in [0, \omega] \quad (j = 1, 2). \quad (5)$$

If $g : [a, b] \rightarrow \mathbb{R}$ is a nondecreasing function, $x : [a, b] \rightarrow \mathbb{R}$ and $a \leq s < t \leq b$, then

$$\int_s^t x(\tau) dg(\tau) = \int_{]s, t[} x(\tau) dg_1(\tau) - \int_{]s, t[} x(\tau) dg_2(\tau) + \sum_{s < \tau \leq t} x(\tau) d_1 g(\tau) - \sum_{s \leq \tau < t} x(\tau) d_2 g(\tau),$$

where $g_k : [a, b] \rightarrow \mathbb{R}$ ($k = 1, 2$) are nondecreasing functions such that

$$g_1(t) - g_2(t) \equiv s_0(g)(t)$$

and $\int_{]s, t[} x(\tau) dg_k(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s, t[$ with respect to the measure $\mu(g_k)$, corresponding to the function g_k (if $s = t$, then $\int_s^t x(\tau) dg(\tau) = 0$); $L([a, b], \mathbb{R}; g)$ is the set of all $\mu(g_k)$ -measurable (i.e., measurable with respect to the measures $\mu(g_1)$ and $\mu(g_2)$) functions $x : [a, b] \rightarrow \mathbb{R}$ such that $\int_a^b |x(t)| dg(t) < +\infty$.

A matrix-function is said to be nondecreasing if every its components are such.

If $G = (g_{ik})_{i,k=1}^{l,n} : [a, b] \rightarrow \mathbb{R}^{l \times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n \times m}$, then $L([a, b], D; G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a, b] \rightarrow D$ such that $x_{kj} \in L([a, b], \mathbb{R}; g_{ik})$ ($i = 1, \dots, l; k = 1, \dots, n; j = 1, \dots, m$);

$$\int_s^t dG(\tau) \cdot X(\tau) = \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau) dg_{ik}(\tau) \right)_{i,j=1}^{l,m} \quad \text{for } a \leq s \leq t \leq b.$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n \times m}$, then $K([a, b] \times D_1, D_2; G)$ is the Caratheodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a, b] \times D_1 \rightarrow D_2$ such that for each $i \in \{1, \dots, l\}$, $j \in \{1, \dots, m\}$, and $k \in \{1, \dots, n\}$: (a) the function $f_{kj}(\cdot, x) : [a, b] \rightarrow D_2$ is $\mu(g_{ik})$ -measurable for every $x \in D_1$; (b) the function $f_{kj}(t, \cdot) : D_1 \rightarrow D_2$ is continuous for $\mu(g_{ik})$ -almost all $t \in [a, b]$, and

$$\sup \{ |f_{kj}(\cdot, x)| : x \in D_0 \} \in L([a, b], \mathbb{R}; g_{ik})$$

for every compact $D_0 \subset D_1$.

If $G_j : [a, b] \rightarrow \mathbb{R}^{l \times n}$ ($j = 1, 2$) are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$, then

$$\int_s^t dG(\tau) \cdot X(\tau) = \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \text{ for } a \leq s \leq t \leq b.$$

Inequalities between both vectors and matrices are understood componentwise.

A vector-function $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be a solution of the system (1) (of the system or generalized ordinary differential inequalities $dx(t) \leq dA(t) \cdot f(t, x(t))$) if its restriction on $[s, t]$ belongs to $BV([s, t], \mathbb{R}^n)$ and

$$x(t) - x(s) - \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) = 0 \quad (\leq 0) \quad \text{for } s < t \quad (s, t \in \mathbb{R}).$$

If $\sigma_i \in \{-1, 1\}$, then $t_i = \frac{1-\sigma_i}{2}\omega$, $j_i = \frac{3+\sigma_i}{2}$, and $I_\omega^{\sigma_i} = [0, \omega] \setminus \{t_i\}$.

Definition 1. Let $\sigma_i \in \{-1, 1\}$ ($i = 1, \dots, n$). We shall say that a matrix-function $C = (c_{il})_{i,l=1}^n : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ belongs to the set $U_\omega^{\sigma_1, \dots, \sigma_n}$ if $c_{il} \in BV_\omega(\mathbb{R}, \mathbb{R})$ ($i, l = 1, \dots, n$), the functions c_{il} ($i \neq l; i, l = 1, \dots, n$) are nondecreasing on $[0, \omega]$ and continuous at the point t_i ,

$$d_j c_{ii}(t_i) \geq 0 \quad (j = 1, 2; i = 1, \dots, n), \tag{6}$$

$$\|d_j C(t)\| < 1 \quad \text{for } t \in [0, \omega] \quad (j = 1, 2) \tag{7}$$

and the system

$$\sigma_i dy_i(t) \leq \sum_{l=1}^n y_l(t) dc_{il}(t) \quad (i = 1, \dots, n)$$

has no nontrivial nonnegative ω -periodic solution.

Theorem 1. *Let the conditions*

$$(-1)^{m+1} \sigma_i f_k(t, x_1, \dots, x_n) \text{ sign } x_i \leq \sum_{l=1}^n p_{mikl}(t) |x_l| + q_k(t)$$

for $\mu(a_{ik}^{(m)})$ -almost all $t \in I_\omega^{\sigma_i}$ ($i, k = 1, \dots, n$) (8)

and

$$\left[(-1)^{m+1} \sigma_i f_k(t_i, x_1, \dots, x_n) \text{ sign } x_i - \sum_{l=1}^n \alpha_{mikjl} |x_l| - q_k(t_i) \right] d_j a_{ik}^{(m)}(t_i) \leq 0$$

($j = 1, 2; i, k = 1, \dots, n$) (9)

be fulfilled on \mathbb{R}^n for every $m \in \{1, 2\}$, where $\sigma_i \in \{-1, 1\}$, $\alpha_{mikjl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0, \omega], \mathbb{R}^{n \times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0, \omega], \mathbb{R}_+^n; A)$. Let, moreover,

$$\sum_{m=1}^2 \sum_{k=1}^n \int_s^t p_{mikl}(\tau) da_{ik}^{(m)}(\tau) \leq c_{il}(t) - c_{il}(s) \quad \text{for } s < t; s, t \in I_\omega^{\sigma_i} \quad (i, l = 1, \dots, n) \tag{10}$$

and

$$\sum_{m=1}^2 \sum_{k=1}^n \alpha_{mikjl} d_j a_{ik}^{(m)}(t_i) \leq \delta_{il} d_j c_{ii}(t_i) \quad (i, l = 1, \dots, n), \tag{11}$$

where $C = (c_{il})_{i,l=1}^n \in U_\omega^{\sigma_1, \dots, \sigma_n}$. Then the system (1) has at least one ω -periodic solution.

Corollary 1. *Let the conditions (6)–(11) and*

$$\sigma_i \lambda(\sigma_i c_{ii})(\omega) < 1 \quad (i = 1, \dots, n)$$

be fulfilled on \mathbb{R}^n for every $m \in \{1, 2\}$, where $\sigma_i \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0, \omega], \mathbb{R}^{n \times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0, \omega], \mathbb{R}_+^n; A)$, $C = (c_{il})_{i,l=1}^n \in BV_\omega(\mathbb{R}, \mathbb{R}^{n \times n})$, the functions c_{il} ($i \neq l$) are nondecreasing on $[0, \omega]$ and continuous at the point t_i , the function $\lambda(\sigma_i c_{ii})$ is defined by (2). Let, moreover, the module of every characteristic value of the matrix $S = (s_{il})_{i,l=1}^n$ be less than 1, where

$$s_{ii} = 0, \quad s_{il} = \sup \left\{ \sum_{j=0}^2 \int_0^\omega \sigma_i g_j(\sigma_i c_{ii})(t, \tau) ds_j(c_{il})(\tau) : t \in [0, \omega] \right\}$$

$$(i \neq l; i, l = 1, \dots, n),$$

g_j ($j = 0, 1, 2$) are operators defined by (3)–(5). Then the conclusion of Theorem 1 is true.

Corollary 2. *Let the conditions (8)–(11),*

$$c_{il}(t) = \eta_{il} \alpha_i(t) \quad \text{for } t \in \mathbb{R} \quad (i, l = 1, \dots, n)$$

and

$$\left(|\eta_{ii}| + \sum_{l \neq i, l=1}^n \eta_{il} \right) d_j \alpha_i(t) < 1 \quad \text{for } t \in [0, \omega] \quad (j = 1, 2; i = 1, \dots, n)$$

be fulfilled on \mathbb{R}^n for every $m \in \{1, 2\}$, where $\sigma_i \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0, \omega], \mathbb{R}^{n \times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0, \omega], \mathbb{R}_+^n; A)$, $\eta_{ii} \in \mathbb{R}$, $\eta_{il} \in \mathbb{R}_+$ ($i \neq l$), $\alpha_i \in BV_\omega(\mathbb{R}, \mathbb{R})$ is nondecreasing on $[0, \omega]$ and continuous at the point t_i , $\alpha_i(\omega) \neq 0$. Let, moreover, the real part of every characteristic value of the matrix $\mathcal{H} = (\eta_{il})_{i,l=1}^n$ be negative. Then the conclusion of Theorem 1 is true.

Corollary 3. *Let the conditions (6), (7), and (8)–(11) be fulfilled on \mathbb{R}^n for every $m \in \{1, 2\}$, where $\sigma_1 = \dots = \sigma_n = \sigma_0$, $\sigma_0 \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0, \omega], \mathbb{R}^{n \times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0, \omega], \mathbb{R}_+^n; A)$, $C = (c_{il})_{i,l=1}^n \in BV_\omega(\mathbb{R}, \mathbb{R}^{n \times n})$, the functions c_{il} ($i \neq l$) are nondecreasing on $[0, \omega]$ and continuous at the point t_i . Let, moreover, the module of every multiplier of the system $dy(t) = dC_{\sigma_0}(t) \cdot y(t)$, where $C_{\sigma_0}(t) = \sigma_0 C(\sigma_0 t + \frac{1-\sigma_0}{2} \omega)$, be less than 1. Then the conclusion of Theorem 1 is true.*

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REFERENCES

1. I. T. KIGURADZE, Boundary value problems for systems of ordinary differential equations. (Russian) *Current Problems in Mathematics. Newest results*, vol. 30, 3–103; *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuzn. Inst. Nauchn. i Tekhn. Inform.*, Moscow, 1987.

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