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ON MONOTONE SOLUTIONS OF LINEAR ADVANCED EQUATIONS

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In the present note we consider the equation

$$u^{(n)}(t) = p(t)u(\tau(t)), \tag{1}$$

where $p : R_+ \to R_+$ is locally integrable and $\tau : R_+ \to R_+$ is measurable with $\tau(t) \geq t$ for $t \geq 0$.

A solution u of (1) is said to be *proper* if it is defined in a neighbourhood $[t_0, +\infty[$ of $+\infty$ and

$$\sup\{|u(s)| \ : \ s \geq t\} > 0 \quad \text{for} \quad t \geq t_0.$$

We will treat the problem of existence of a proper solution $u : [t_0, +\infty[\to R$ satisfying

$$u^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_0 \ \ (i = 0, \dots, n-1). \tag{2}$$

The problem of this kind is specific for advanced equations. It does not arise for ordinary and delay equations since in that case every solution of (1) with positive initial conditions is continuable up to infinity and satisfies (2).

The following two cases will be considered:

$$(1) \qquad \overline{\lim_{t\to+\infty}} \big(\tau(t) - t\big) = \Delta < +\infty \quad \text{with} \quad \Delta \geq 0; \tag{3}$$

$$(2) \qquad \overline{\lim_{t\to+\infty}} \frac{\tau(t)}{t} = \alpha < +\infty \quad \text{with} \quad \alpha \geq 1. \tag{4}$$

R. Koplatadze in [1] has established sufficient conditions for (1) not to have a solution satisfying (2). From Theorems 11.4 and 11.8 of [1] there follow the following results, respectively.

1) if $\tau \equiv t + \Delta$ with $\Delta \in R_+$ and for any $\lambda \in]0, +\infty[$

$$\liminf_{t\to+\infty} e^{-\lambda t} \int_0^t e^{\lambda s} p(s)\, ds > \lambda^{n-1} e^{-\lambda\Delta},$$

then the equation (1) has no solution satisfying (2).

2) if $\tau \equiv \alpha t$ with $\alpha \geq 1$ and for any $\lambda \in]n-1, +\infty[$

$$\liminf_{t\to+\infty} t^{n-\lambda-1} \int_0^t s^\lambda p(s)\, ds < \alpha^{-\lambda} \prod_{i=0}^{n-2} (\lambda - i),$$

then the equation (1) has no solution satisfying (2).

Theorems 1 and 2 below invert in a sense these results.

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Theorem 1. *Let* (3) *be fulfilled and there exist* $\lambda_0 \in]0,+\infty[$ *such that*

$$\limsup_{t\to+\infty} e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s}p(s)\,ds < \lambda_0^{n-1}e^{-\lambda_0\Delta}.$$

Then the equation (1) *has a solution satisfying* (2).

Theorem 2. *Let* (4) *be fulfilled and there exist* $\lambda_0 \in]n-1,+\infty[$ *such that*

$$\limsup_{t\to+\infty} t^{n-\lambda_0-1}\int_0^t s^{\lambda_0}p(s)\,ds < \alpha^{-\lambda_0}\prod_{i=0}^{n-2}(\lambda_0-i).$$

Then the equation (1) *has a solution satisfying* (2).

The following below comparison theorem is a basic tool in establishing Theorems 1 and 2.

Along with (1), consider the equation

$$v^{(n)} + q(t)v(\sigma(t)) = 0, \tag{5}$$

where $q : R_+ \to R_+$ is locally integrable and $\sigma : R_+ \to R_+$ is measurable with $\sigma(t) \geq t$ for $t \in R_+$.

It will be assumed that

$$\tau(t) \leq \sigma(t) \quad \text{for} \quad t \in R_+. \tag{6}$$

For the equation (5) the problem on existence of a solution v satisfying

$$v^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_0 \tag{7}$$

will be likewise considered.

Theorem 3. *Let* (6) *be fulfilled and the equation* (5) *have a proper solution* $v : [t_0,+\infty[\to R$ *satisfying* (7). *Let, moreover, there exist a continuous function* $\varphi : [t_0,+\infty[\to]0,+\infty[$ *such that* v/φ *is nonincreasing and*

$$\int_{t_0}^t \varphi(\sigma(s))p(s)\,ds \leq \int_{t_0}^t \varphi(\sigma(s))q(s)\,ds \quad \textit{for} \quad t \geq t_0.$$

Then the equation (1) *has a solution* $u : [t_0,+\infty[\to R$ *satisfying* (2).

Comparing, according to this theorem, the equation (6) with the equations

$$v^{(n)} + q_1 v(t+\Delta) = 0 \quad \text{and} \quad v^{(n)} + \frac{q_2}{t^n}\,v(\alpha t) = 0,$$

where q_1 and q_2 are appropriately chosen, we derive Theorems 1 and 2, respectively.

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References

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