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Sergo Kharibegashvili

ON THE SOLVABILITY OF A MULTIDIMENSIONAL VERSION OF THE GOURSAT PROBLEM FOR A SECOND ORDER HYPERBOLIC EQUATION WITH CHARACTERISTIC DEGENERATION

Abstract. A multidimensional version of the Goursat problem is considered for a second order hyperbolic equation with characteristic degeneration. Using the technique of functional spaces with a negative norm the correct formulation of this problem in the Sobolev weighted space is given.

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რეზიუმე. განხილულია გურსას ამოცანის მრავალგანზომილებიანი ვერსია მეორე რიგის ჰიპერბოლური განტოლებისათვის მახასიათებელი გადაგვარებით. უარყოფით ნორმიანი ფუნქციონალური სივრცეების ტექნიკის გამოყენებით მოცემულია ამ ამოცანის კორექტული ფორმულირება სობოლევის წონიან სივრცეში.

In the space of variables x_1, x_2, t we shall consider a second order degenerating hyperbolic equation of the form

$$Lu \equiv u_{tt} - u_{x_1x_1} - (|x_2|^m u_{x_2})_{x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \tag{1}$$

where a_i, $i = 1, \dots, 4$, F are the given real functions and u is the desired real function, $1 \le m = const < 2$.

Denote by

$$D: \frac{2}{2-m} x_2^{\frac{2-m}{2}} < t < 1 - \frac{2}{2-m} x_2^{\frac{2-m}{2}}, \quad 0 < x_2 < \Big(\frac{2-m}{4}\Big)^{\frac{2}{2-m}}$$

the unbounded domain lying in a half-space $x_2 > 0$ bounded by the characteristic surfaces

$$S_1 : t - \frac{2}{2-m} x_2^{\frac{2-m}{2}} = 0, \quad 0 < x_2 < \Big(\frac{2-m}{4}\Big)^{\frac{2}{2-m}},$$
$$S_2 : t + \frac{2}{2-m} x_2^{\frac{2-m}{2}} = 1, \quad 0 < x_2 < \Big(\frac{2-m}{4}\Big)^{\frac{2}{2-m}}$$

of equation (1) and by the two-dimensional surface $S_0 : x_2 = 0, 0 < t < 1$ on which this equation has characteristic degeneration. It will be assumed below that in the domain D the coefficients a_i, $i = 1, \dots, 4$, of equation (1) are the bounded functions from the class $C^2(\overline{D})$.

For equation (1) we shall consider a multidimensional version of the Goursat problem formulated as follows: in the domain D find a solution $u(x_1, x_2, t)$ of equation (1) satisfying the boundary condition

$$u\,|_{S_1} = 0. \tag{2}$$

In a similar manner we formulate the problem for the equation

$$\begin{aligned} L^* v \equiv v_{tt} - v_{x_1x_1} - (|x_2|^m v_{x_2})_{x_2} - (a_1 v)_{x_1} - \\ - (a_2 v)_{x_2} - (a_3 v)_t + a_4 v = F \end{aligned} \tag{3}$$

in the domain D using the boundary condition

$$v\,|_{S_2} = 0, \tag{4}$$

where L^* is the formal conjugate operator of L.

Similar problems, in which, along with condition (2), it is required that the condition $u\,|_{S_0} = 0$ or $\frac{\partial u}{\partial n}\,|_{S_0} = 0$ be fulfilled on the section S_0 of the boundary ∂D of the domain D, are investigated in [1–6] for $m = 0$ when equation (1) is not the degenerating one and has, in its principal part, a wave operator. As will be shown below, by virtue of the degeneration character of equation (1), where $1 \le m < 2$, we can get rid of the fulfillment of any boundary condition on the section S_0 of the boundary ∂D of the domain D, since problem (1), (2) will turn out to be correctly formulated. In the case of a second order hyperbolic equation with noncharacteristic degeneration of the form

$$u_{tt} - |x_2|^m u_{x_1x_1} - u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F$$

a multidimensional variant of the first Darboux problem is studied in [7]. Other variants of the multidimensional Goursat and Darboux problems are treated in [8–10].

Denote by E and E^* the classes of functions from the Sobolev space $W_2^2(D)$ satisfying the boundary condition (2) or (4), respectively. Let $W_+(W_+^*)$ be the Hilbert space with weight obtained by the closure of the space $E(E^*)$ with respect to the norm

$$\|u\|_{1,+}^2 = \int\limits_D (u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2 + u^2) dD.$$

Remark 1. Since $m \geq 1$, by virtue of the familiar embedding theorems for Sobolev weighted spaces [11] the class of functions $E_0(E_0^*)$ belonging to the space $C^\infty(\overline{D})$, having the bounded carriers (i.e., diam supp $u < +\infty$), satisfying the boundary condition (2) ((4)) and vanishing in some neighborhood (each function has its own neighborhood) of the surface S_0, is a dense subspace of the weighed space $W_+(W_+^*)$. Therefore, below it will be sometimes convenient for us to use, instead of the spaces E and E^*, the spaces E_0 and E_0^*.

Denote by $W_-(W_-^*)$ the space with negative norm constructed with respect to $L_2(D)$ and $W_+(W_+^*)$ [12].

Consider the condition

$$M = \sup_{\overline{D}} \left| x_2^{-\frac{m}{2}} a_2(x_1, x_2, t) \right| < +\infty \tag{5}$$

on the lower coefficient a_2 in equation (1).

The uniqueness theorem for solutions of problem (1), (2) belonging to the Sobolev space $W_2^2(D)$ is provided by

Lemma 1. *Let condition* (5) *be fulfilled. Then for any* $u \in W_2^2(D)$ *satisfying the condition*

$$\int\limits_{S_1} \Big[u^2 + x_2^{\frac{m}{2}} u_{x_1}^2 + x_2^{-\frac{m}{2}} \Big(\frac{\partial u}{\partial N} \Big)^2 \Big] ds < +\infty \tag{6}$$

there holds the following a priori estimate

$$\|u\|_{1,+} \leq c\big(\|f\|_{1,*} + \|F\|_{L_2(D)} \big), \tag{7}$$

where the positive constant c *does not depend on* u*;* $f = u\,|_{S_1}$, $F = Lu$,

$$\|f\|_{1,*}^2 = \int\limits_{S_1} \Big[f^2 + x_2^{\frac{m}{2}} f_{x_1}^2 + x_2^{-\frac{m}{2}} \Big(\frac{\partial f}{\partial N} \Big)^2 \Big] ds,$$

$\frac{\partial}{\partial N}\,|_{S_1} = -(1 + x_2^{-m})^{-\frac{1}{2}} \big[\frac{\partial}{\partial t} + x_2^m \frac{\partial}{\partial x_2} \big]$ *is the derivative with respect to the conormal which is the internal differential operator on the characteristic surface* S_1.

Proof. Let $n = (\nu_1, \nu_2, \nu_0)$ be the unit vector of the external normal to ∂D, i.e., $\nu_1 = cos(\widehat{n, x_1})$, $\nu_2 = cos(\widehat{n, x_2})$, $\nu_0 = cos(\widehat{n, t})$. By definition, the derivative with respect to the conormal on the boundary ∂D of the domain D for the operator L is calculated by the formula

$$\frac{\partial}{\partial N} = \nu_0 \frac{\partial}{\partial t} - \nu_1 \frac{\partial}{\partial x_1} - x_2^m \nu_2 \frac{\partial}{\partial x_2}.$$

Applying integration by parts, we have for $u \in W_2^2(D)$ and $\lambda = const > 0$:

$$2\int\limits_D e^{-\lambda t} u_{tt} u_t dD = \int\limits_{\partial D} e^{-\lambda t} u_t^2 \nu_0 ds + \int\limits_D \lambda e^{-\lambda t} u_t^2 dD, \tag{8}$$

$$-2\int\limits_D e^{-\lambda t}\big[u_{x_1x_1} u_t + (x_2^m u_{x_2})_{x_2} u_t\big] dD = -2\int\limits_{\partial D} e^{-\lambda t}(u_{x_1} u_t \nu_1 +$$

$$+ x_2^m u_{x_2} u_t \nu_2) ds + 2\int\limits_D e^{-\lambda t}(u_{x_1} u_{x_1 t} + x_2^m u_{x_2} u_{x_2 t}) dD =$$

$$= -2\int\limits_{\partial D} e^{-\lambda t}(u_{x_1} u_t \nu_1 + x_2^m u_{x_2} u_t \nu_2) ds + \int\limits_D e^{-\lambda t} \frac{\partial}{\partial t}(u_{x_1}^2 + x_2^m u_{x_2}^2) dD =$$

$$= -2\int\limits_{\partial D} e^{-\lambda t}(u_{x_1} u_t \nu_1 + x_2^m u_{x_2} u_t \nu_2) ds + \int\limits_{\partial D} e^{-\lambda t}(u_{x_1}^2 + x_2^m u_{x_2}^2)\nu_0 ds +$$

$$+ \int\limits_D e^{-\lambda t} \lambda (u_{x_1}^2 + x_2^m u_{x_2}^2) dD. \tag{9}$$

It is easy to verify that

$$\begin{gathered} \nu_0\big|_{S_0} = \nu_1\big|_{S_0} = 0, \quad \frac{\partial u}{\partial N}\Big|_{S_0} = 0, \\ n\big|_{S_1} = \big(0, (1 + x_2^{-m})^{-\frac{1}{2}} x_2^{-\frac{m}{2}}, -(1 + x_2^{-m})^{-\frac{1}{2}}\big), \\ \nu_0\big|_{S_2} \geq 0, \quad (\nu_0^2 - \nu_1^2 - x_2^m \nu_2^2)\big|_{S_1 \cup S_2} = 0. \end{gathered} \tag{10}$$

On multiplying both parts of equation (1) by $2e^{-\lambda t} u_t$, where $F = Lu$, and integrating the resulting expression with respect to the domain D we obtain by virtue of (6) and (8)–(10)

$$2(Lu, e^{-\lambda t} u_t)_{L_2(D)} = \int\limits_{S_1 \cup S_2} e^{-\lambda t}\big[(u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2)\nu_0 -$$

$$-2(u_{x_1} u_t \nu_1 + x_2^m u_{x_2} u_t \nu_2)\big] ds + 2\int\limits_D e^{-\lambda t}\big[a_1 u_{x_1} + a_2 u_{x_2} +$$

$$+ a_3 u_t + a_4 u\big] u_t dD + \int\limits_D e^{-\lambda t} \lambda \big[u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2\big] dD =$$

$$= \int\limits_{S_1 \cup S_2} e^{-\lambda t} \nu_0^{-1}\big[(\nu_0 u_{x_1} - \nu_1 u_t)^2 + x_2^m(\nu_0 u_{x_2} - \nu_2 u_t)^2 +$$

$$+ (\nu_0^2 - \nu_1^2 - x_2^m \nu_2^2) u_t^2\big] ds + 2\int\limits_D e^{-\lambda t}\big[\lambda(u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2) +$$

$$+ 2(a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u) u_t\big] dD \geq$$

$$\geq 2\int\limits_D e^{-\lambda t}\big[\lambda(u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2) + 2(a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u) u_t\big] dD -$$

$$-\int_{S_1} e^{-\lambda t}\Big[(1+x_2^{-m})^{-\frac{1}{2}}u_{x_1}^2+(1+x_2^{-m})^{\frac{1}{2}}\Big(\frac{\partial u}{\partial N}\Big)^2\Big]ds \geq$$

$$\geq 2\int_D e^{-\lambda t}\big[\lambda(u_t^2+u_{x_1}^2+x_2^m u_{x_2}^2)+2(a_1u_{x_1}+a_2u_{x_2}+a_3u_t+a_4u)u_t\big]dD-$$

$$-2\int_{S_1}\Big[x_2^{\frac{m}{2}}u_{x_1}^2+x_2^{-\frac{m}{2}}\Big(\frac{\partial u}{\partial N}\Big)^2\Big]ds. \tag{11}$$

In deriving inequality (11), we used the fact that

$$\Big(\frac{\partial u}{\partial N}\Big)^2\Big|_{S_1}=x_2^m(\nu_0u_{x_2}-\nu_2u_t)^2\big|_{S_1}.$$

The structure of the domain D allows one to easily verify the validity of the inequality

$$\int_D u^2dD\leq c_0\Big[\int_{S_1}u^2ds+\int_D u_t^2dD\Big] \tag{12}$$

for some $c_0=const>0$ not depending on $u\in W_2^2(D)$.

By inequality (5) we readily obtain

$$|2a_2u_xu_t|\leq 2M(x_2^{\frac{m}{2}}u_{x_2})u_t\leq M(x_2^mu_{x_2}^2+u_t^2). \tag{13}$$

By virtue of (12) and (13), inequality (11) implies for sufficiently large λ that

$$2(Lu,e^{-\lambda t}u_t)_{L_2(D)}\geq c_1\int_D(u_t^2+u_{x_1}^2+x_2^mu_{x_2}^2+u^2)dD-$$

$$-c_2\int_{S_1}\Big[u^2+x_2^{\frac{m}{2}}u_{x_1}^2+x_2^{-\frac{m}{2}}\Big(\frac{\partial u}{\partial N}\Big)^2\Big]ds, \tag{14}$$

where the positive constants c_1 and c_2 do not depend on u and the constant c_1 can be chosen arbitrarily large depending on λ. Therefore (14) obviously implies estimate (7). $\blacksquare$

Remark 2. Since for the operator L the derivative with respect to the conormal $\frac{\partial}{\partial N}$ is the internal differential operator on the characteristic surfaces of equation (1), by virtue of (2) and (4) we find for the functions $u\in E$ and $v\in E^*$ that

$$\frac{\partial u}{\partial N}\Big|_{S_1}=0,\quad \frac{\partial v}{\partial N}\Big|_{S_2}=0. \tag{15}$$

Lemma 2. *Let condition* (5) *be fulfilled. Then for all* $u\in E$, $v\in E^*$ *we have the inequalities*

$$\|Lu\|_{W_-^*}\leq c_1\|u\|_{W_+}, \tag{16}$$

$$\|L^*v\|_{W_-}\leq c_2\|v\|_{W_+^*}, \tag{17}$$

where the positive constants c_1 *and* c_2 *do not depend on* u *and* v, *respectively,* $\|\cdot\|_{W_+}=\|\cdot\|_{W_+^*}=\|\cdot\|_{1,+}$.

Proof. By the definition of a negative norm for $u \in E$ and by equalities (2), (4), (10), (15) we have

$$\|Lu\|_{W_-^*} = \sup_{v\in W_+^*} \|v\|_{W_+^*}^{-1}(Lu,v)_{L_2(D)} = \sup_{v\in E^*} \|v\|_{W_+^*}^{-1}(Lu,v)_{L_2(D)} =$$

$$= \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \int\limits_D \big[u_{tt}v - u_{x_1x_1}v - (x_2^m u_{x_2})_{x_2}v + a_1u_{x_1}v + a_2u_{x_2}v +$$

$$+a_3u_tv + a_4uv\big]dD = \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \int\limits_{\partial D} [u_tv\nu_0 - u_{x_1}v\nu_1 - x_2^m u_{x_2}v\nu_2]ds +$$

$$+ \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \int\limits_D [-u_tv_t + u_{x_1}v_{x_1} + x_2^m u_{x_2}v_{x_2} + a_1u_{x_1}v + a_2u_{x_2}v +$$

$$+a_3u_tv + a_4uv]dD = \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \int\limits_{S_1\cup S_2} \frac{\partial u}{\partial N} v\, ds +$$

$$+ \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \int\limits_D [-u_tv_t + u_{x_1}v_{x_1} + x_2^m u_{x_2}v_{x_2} + a_1u_{x_1}v + a_2u_{x_2}v +$$

$$+a_3u_tv + a_4uv]dD = \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \int\limits_D [-u_tv_t + u_{x_1}v_{x_1} + x_2^m u_{x_2}v_{x_2} +$$

$$+a_1u_{x_1}v + a_2u_{x_2}v + a_3u_tv + a_4uv]dD. \tag{18}$$

In view of condition (5) and the Schwartz inequality we obtain

$$\Big|\int\limits_D [-u_tv_t + u_{x_1}v_{x_1} + x_2^m u_{x_2}v_{x_2}]dD\Big| \le 3\Big[\int\limits_D (u_t^2 + u_{x_1}^2 +$$

$$+x_2^m u_{x_2}^2)dD\Big]^{\frac12}\Big[\int\limits_D (v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2)dD\Big]^{\frac12} \le 3\|u\|_{W_+}\|v\|_{W_+^*}, \tag{19}$$

$$\Big|\int\limits_D [a_1u_{x_1}v + a_2u_{x_2}v + a_3u_tv + a_4uv]dD\Big| \le$$

$$\le \sup_D |a_1|\ \|u_{x_2}\|_{L_2(D)}\|v\|_{L_2(D)} + M\Big(\int\limits_D x_2^m u_{x_2}^2 dD\Big)^{\frac12}\|v\|_{L_2(D)} +$$

$$+\sup_D |a_3|\ \|u_t\|_{L_2(D)}\|v\|_{L_2(D)} + \sup_D |a_4|\ \|u\|_{L_2(D)}\|v\|_{L_2(D)} \le$$

$$\le \Big(M + \sum_{i=1,i\ne 2}^{4} \sup_D |a_i|\Big)\|u\|_{W_+}\|v\|_{W_+^*} = \widetilde{c}\|u\|_{W_+}\ \|v\|_{W_+^*}. \tag{20}$$

From (18)–(20) it follows that

$$\|Lu\|_{W_-^*} \le (3+\widetilde{c}) \sup_{v\in E^*} \|v\|_{W_+^*}^{-1}\|u\|_{W_+}\|v\|_{W_+^*} = c_1\|u\|_{W_+},$$

which proves inequality (16). Since the proof of inequality (17) is quite similar to that of inequality (16), Lemma 2 is thereby completely proved. ■

Remark 3. By virtue of inequality (16) ((17)) the operator $L : W_+ \to W_-^*$ ($L : W_+^* \to W_-$) with the dense definition domain of $E(E^*)$ admits a closure which is a continuous operator from the space $W_+(W_+^*)$ into the space $W_-^*(W_-)$. If we denote this closure as previously by $L(L^*)$, it will be defined throughout the Hilbert space $W_+(W_+^*)$.

Lemma 3. *Problems* (1), (2) *and* (3), (4) *are mutually conjugate, i.e., the equality*

$$(Lu, v) = (u, L^*v). \tag{21}$$

holds for any $u \in W_+$ *and* $v \in W_+^*$.

Proof. By Remark 3 it is enough to prove equality (21) when $u \in E$ and $v \in E^*$. In that case it is obvious that $(Lu, v) = (Lu, v)_{L_2(D)}$. therefore we have

$$\begin{aligned}(Lu, v) = (Lu, v)_{L_2(D)} = \int\limits_{\partial D} [u_t v\nu_0 - u_{x_1} v\nu_1 - x_2^m u_{x_2} v\nu_2] ds + \\ + \int\limits_{\partial D} [a_1\nu_1 + a_2\nu_2 + a_3\nu_0] uv\, ds + \int\limits_{D} \big[- u_t v_t + u_{x_1} v_{x_1} + \\ + x_2^m u_{x_2} v_{x_2} - u(a_1 v)_{x_1} - u(a_2 v)_{x_2} - u(a_3 v)_t + a_4 uv \big] dD = \\ = \int\limits_{\partial D} [u_t v\nu_0 - u_{x_1} v\nu_1 - x_2^m u_{x_2} v\nu_2] ds + \int\limits_{\partial D} [a_1\nu_1 + a_2\nu_2 + \\ + a_3\nu_0] uv\, ds - \int\limits_{\partial D} [u v_t \nu_0 - u v_{x_1} \nu_1 - x_2^m u v_{x_2} \nu_2] ds + \\ + \int\limits_{D} \big[u v_{tt} - u v_{x_1 x_1} - u(x_2^m v_{x_2})_{x_2} - u(a_1 v)_{x_1} - \\ - u(a_2 v)_{x_2} - u(a_3 v)_t + a_4 uv \big] dD = \int\limits_{\partial D} \Big[\Big(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} \Big) + \\ + (a_1\nu_1 + a_2\nu_2 + a_3\nu_0) uv \Big] ds + (u, L^* v)_{L_2(D)}.\end{aligned} \tag{22}$$

Since condition (5) implies $a_2\big|_{S_0} = 0$, by virtue of (2), (4), (10) and (15) we readily obtain equality (21) from (22), which proves Lemma 3. ∎

Consider the conditions

$$\Omega\big|_{S_1} \le 0, \quad (\lambda\Omega + \Omega_t)\big|_D \le 0, \tag{23}$$

where the second inequality is fulfilled for sufficiently large λ, $\Omega = a_{1x_1} + a_{2x_2} + a_{3t} - a_4$.

Lemma 4. *Let conditions* (5) *and* (23) *be fulfilled. Then for any* $u \in W_+$ *we have the inequality*

$$c\|u\|_{L_2(D)} \le \|Lu\|_{W_-^*} \tag{24}$$

where the positive constant c *does not depend on* u.

Proof. By Remarks 1 and 3 it is enough to show that inequality (24) is fulfilled when $u \in E_0$. If $u \in E_0$ and thus vanishes in some neighborhood of the surface S_0, then one can easily verify that the function

$$v(x_1, x_2, t) = \int\limits_t^{\varphi_2(x_1,x_2)} e^{-\lambda\tau} u(x_1, x_2, \tau) d\tau, \quad \lambda = const > 0,$$

where $t = \varphi_2(x_1, x_2)$ is an equation of the characteristic surface S_2, belongs to the space E_0^* and the equalities

$$v_t(x_1, x_2, t) = -e^{-\lambda t} u(x_1, x_2, t), \quad u(x_1, x_2, t) = -e^{\lambda t} v_t(x_1, x_2, t). \tag{25}$$

are fulfilled.

In view of (10), (15) and (25) we have

$$(Lu, v)_{L_2(D)} = \int\limits_{\partial D} \Big[v \frac{\partial u}{\partial N} + (a_1\nu_1 + a_2\nu_2 + a_3\nu_0) uv \Big] ds +$$
$$+ \int\limits_D [-u_t v_t + u_{x_1} v_{x_1} + x_2^m u_{x_2} v_{x_2} - u a_{1x_1} v - u a_1 v_{x_1} - u a_{2x_2} v -$$
$$-u a_2 v_{x_2} - u a_{3t} v - u a_3 v_t + a_4 uv] dD = \int\limits_D e^{-\lambda t} u_t u \; dD +$$
$$+ \int\limits_D e^{\lambda t} [-v_{x_1 t} v_{x_1} - x_2^m v_{x_2 t} v_{x_2} + a_{1x_1} v_t v + a_1 v_t v_{x_1} + a_{2x_2} v_t v +$$
$$+ a_2 v_t v_{x_2} + a_{3t} v_t v + a_3 v_t^2 - a_4 v_t v] dD. \tag{26}$$

By (2) we obtain similarly to (8) and (9)

$$\int\limits_D e^{-\lambda t} u_t u \; dD = \frac{1}{2} \int\limits_{\partial D} e^{-\lambda t} u^2 \nu_0 \; ds + \frac{1}{2} \int\limits_D e^{-\lambda t} \lambda u^2 \; dD =$$
$$= \frac{1}{2} \int\limits_{S_2} e^{-\lambda t} u^2 \nu_0 \; ds + \frac{1}{2} \int\limits_D e^{\lambda t} \lambda v_t^2 \; dD =$$
$$= \frac{1}{2} \int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0 \; ds + \frac{1}{2} \int\limits_D e^{\lambda t} \lambda v_t^2 \; dD, \tag{27}$$

$$\int\limits_D e^{\lambda t} [-v_{x_1 t} v_{x_1} - x_2^m v_{x_2 t} v_{x_2}] dD = -\frac{1}{2} \int\limits_{\partial D} e^{\lambda t} [v_{x_1}^2 + x_2^m v_{x_2}^2] \nu_0 \; ds +$$
$$+ \frac{1}{2} \int\limits_D e^{\lambda t} \lambda [v_{x_1}^2 + x_2^m v_{x_2}^2] dD. \tag{28}$$

Since $v\big|_{S_2} = 0$, for some α we have $v_t = \alpha\nu_0$, $v_{x_1} = \alpha\nu_1$, $v_{x_2} = \alpha\nu_2$ on S_2. Therefore, recalling that the surface S_2 is characteristic, we obtain

$$(v_t^2 - v_{x_1}^2 - x_2^m v_{x_2}^2)\big|_{S_2} = \alpha^2 (\nu_0^2 - \nu_1^2 - x_2^m \nu_2^2)\big|_{S_2} = 0. \tag{29}$$

By virtue of $\nu_0\big|_{S_0} = 0$, $\nu_0\big|_{S_1} \le 0$, and equalities (4), (29) we find that

$$\frac{1}{2}\int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0\, ds - \frac{1}{2}\int\limits_{\partial D} e^{\lambda t}[v_{x_1}^2 + x_2^m v_{x_2}^2]\nu_0\, ds =$$

$$= \frac{1}{2}\int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0\, ds - \frac{1}{2}\int\limits_{S_1} e^{\lambda t}[v_{x_1}^2 + x_2^m v_{x_2}^2]\nu_0\, ds -$$

$$-\frac{1}{2}\int\limits_{S_2} e^{\lambda t}[v_{x_1}^2 + x_2^m v_{x_2}^2]\nu_0\, ds \ge \frac{1}{2}\int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0\, ds - \frac{1}{2}\int\limits_{S_2} e^{\lambda t}[v_{x_1}^2 +$$

$$+ x_2^m v_{x_2}^2]\nu_0\, ds = \frac{1}{2}\int\limits_{S_2} e^{\lambda t}[v_t^2 - v_{x_1}^2 - x_2^m v_{x_2}^2]\nu_0\, ds = 0. \tag{30}$$

Taking into account (27), (28) and (30), we obtain from (26)

$$(Lu, v)_{L_2(D)} = \frac{1}{2}\int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0\, ds + \frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda v_t^2\, dD -$$

$$-\frac{1}{2}\int\limits_{\partial D} e^{\lambda t}[v_{x_1}^2 + x_2^m v_{x_2}^2]\nu_0\, ds + \frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda[v_{x_1}^2 + x_2^m v_{x_2}^2]dD +$$

$$+\int\limits_{D} e^{\lambda t}\big[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2 + (a_{1x_1} + a_{2x_2} + a_{3t} - a_4)v_t v\big]\, dD \ge$$

$$\ge \frac{\lambda}{2}\int\limits_{D} e^{\lambda t}[v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD + \int\limits_{D} e^{\lambda t}[a_1 v_t v_{x_1} +$$

$$+ a_2 v_t v_{x_2} + a_3 v_t^2 + (a_{1x_1} + a_{2x_2} + a_{3t} - a_4)v_t v]dD. \tag{31}$$

Using $\nu_0\big|_{S_1} \le 0$ and conditions (4), (10), (23) and performing integration by parts we derive

$$\int\limits_{D} e^{\lambda t}(a_{1x_1} + a_{2x_2} + a_{3t} - a_4)v_t v\, dD = \frac{1}{2}\int\limits_{\partial D} e^{\lambda t}(a_{1x_1} + a_{2x_2} +$$

$$+ a_{3t} - a_4)v^2\nu_0 ds - \frac{1}{2}\int\limits_{D} e^{\lambda t}\big[\lambda(a_{1x_1} + a_{2x_2} + a_{3t} - a_4) +$$

$$+ (a_{1x_1} + a_{2x_2} + a_{3t} - a_4)_t\big]v^2 dD \ge 0, \tag{32}$$

where λ is a sufficiently large positive number.

With (32) taken into account (31) implies

$$(Lu, v)_{L_2(D)} \ge \frac{\lambda}{2}\int\limits_{D} e^{\lambda t}[v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD +$$

$$+\int\limits_{D} e^{\lambda t}[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2]dD \ge \frac{\lambda}{2}\int\limits_{D} e^{\lambda t}[v_t^2 + v_{x_1}^2 +$$

$$+x_2^m v_{x_2}^2]dD - \Big|\int\limits_D e^{\lambda t}[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2]dD\Big|. \tag{33}$$

Assuming

$$\mu = \max\big(\sup_D |a_1|,\ \sup_D |a_3|\big)$$

by condition (5) we find that

$$\Big|\int\limits_D e^{\lambda t}[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2]dD\Big| \le$$

$$\le \int\limits_D e^{\lambda t}\Big[\frac{\mu}{2}(v_{x_1}^2 + v_t^2) + M\frac{1}{2}(x_2^m v_{x_2}^2 + v_t^2) + \mu v_t^2\Big]dD \le$$

$$\le \Big(\frac{1}{2}M + \frac{3}{2}\mu\Big)\int\limits_D e^{\lambda t}[v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD. \tag{34}$$

By virtue of (34) and (25) inequality (33) implies

$$(Lu, v)_{L_2(D)} \ge \Big[\frac{\lambda}{2} - \Big(\frac{1}{2}M + \frac{3}{2}\mu\Big)\Big]\int\limits_D e^{\lambda t}[v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD \ge$$

$$\ge \sigma\Big[\int\limits_D e^{\lambda t} v_t^2 dD\Big]^{\frac{1}{2}}\Big[\int\limits_D [v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD\Big]^{\frac{1}{2}} =$$

$$= \sigma\Big[\int\limits_D e^{-\lambda t} u^2 dD\Big]^{\frac{1}{2}}\Big[\int\limits_D [v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD\Big]^{\frac{1}{2}} \ge$$

$$\ge \sigma \inf_D e^{-\lambda t}\|u\|_{L_2(D)}\Big[\int\limits_D [v_t^2 + v_{x_1}^2 + x_2^m v_{x_2}^2]dD\Big]^{\frac{1}{2}}, \tag{35}$$

where $\sigma = \Big(\frac{\lambda}{2} - \Big(\frac{1}{2}M + \frac{3}{2}\mu\Big)\Big) > 0$ for sufficiently large λ, and $\inf_D e^{-\lambda t} = const > 0$ by the structure of the domain D.

Since $v\big|_{S_2} = 0$, similarly to (12) one can easily show that the inequality

$$\int\limits_D v^2 dD \le c_0 \int\limits_D v_t^2 dD$$

is valid for some $c_0 = const > 0$ not depending on v. Thus we conclude that, in the space $W_+(W_+^*)$, the norm

$$\|u\|^2_{W_+(W_+^*)} = \int\limits_D (u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2 + u^2)dD$$

is equivalent to the norm

$$\|u\|^2 = \int\limits_D (u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2)dD. \tag{36}$$

Therefore, retaining the previous notation $\|u\|_{W_+(W_+^*)}$ for norm (36), we obtain from (35)

$$(Lu, v)_{L_2(D)} \geq \sigma \inf_D e^{-\lambda t} \|u\|_{L_2(D)} \, \|v\|_{W_+^*}. \tag{37}$$

If now we apply the generalized Schwartz inequality

$$(Lu, v) \leq \|Lu\|_{W_-^*} \, \|v\|_{W_+^*}$$

to the left-hand side of (37), then after reducing by $\|v\|_{W_+^*}$, we obtain inequality (24) where $c = \sigma \inf_D e^{-\lambda t} = const > 0$. Lemma 4 is thereby completely proved. ■

Consider the conditions

$$a_4\big|_{S_2} \geq 0, \quad (\lambda a_4 - a_{4t})\big|_D \geq 0, \tag{38}$$

where the second inequality holds for sufficiently large λ.

Lemma 5. *Let conditions* (5) *and* (38) *be fulfilled. Then for any $v \in W_+^*$ the inequality*

$$c\|v\|_{L_2(D)} \leq \|L^* v\|_{W_-} \tag{39}$$

holds for a constant $c = const > 0$ which does not depend on $v \in W_+^*$.

Proof. Like in the case of Lemma 4, by Remarks 1 and 3 it is enough to show that inequality (39) is valid for $v \in E_0^*$. Assume that $v \in E_0^*$ and introduce into the consideration the function

$$u(x_1, x_2, t) = \int\limits_{\varphi_1(x_1, x_2)}^{t} e^{\lambda \tau} v(x_1, x_2, \tau) d\tau, \quad \lambda = const > 0,$$

where $t = \varphi_1(x_1, x_2)$ is an equation of the characteristic surface S_1. It is easy to verify that the function $u(x_1, x_2, t)$ belongs to the class E_0 and the following equalities are fulfilled:

$$u_t(x_1, x_2, t) = e^{\lambda t} v(x_1, x_2, t), \quad v(x_1, x_2, t) = e^{-\lambda t} u_t(x_1, x_2, t). \tag{40}$$

From (10), (15) and (40) we have

$$(L^* v, u)_{L_2(D)} = \int\limits_{\partial D} \Big[u \frac{\partial v}{\partial N} - (a_1 \nu_1 + a_2 \nu_2 + a_3 \nu_0) v u\Big] ds +$$

$$+ \int\limits_D [-v_t u_t + v_{x_1} u_{x_1} + x_2^m v_{x_2} u_{x_2} + a_1 v u_{x_1} + a_2 v u_{x_2} + a_3 v u_t +$$

$$+ a_4 u v] dD = - \int\limits_D e^{\lambda t} v_t v \, dD + \int\limits_D e^{-\lambda t} [u_{x_1 t} u_{x_1} + x_2^m u_{x_2 t} u_{x_2}] dD +$$

$$+ \int\limits_D e^{-\lambda t} [a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u] u_t \, dD. \tag{41}$$

Similarly to (27)–(30), we can prove the equalities

$$-\int\limits_D e^{\lambda t} v_t v \, dD = -\frac{1}{2}\int\limits_{\partial D} e^{\lambda t} v^2 \nu_0 \, ds + \frac{1}{2}\int\limits_D e^{\lambda t}\lambda v^2 \, dD =$$

$$= -\frac{1}{2}\int\limits_{S_1} e^{\lambda t} v^2 \nu_0 \, ds + \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda u_t^2 \, dD =$$

$$= -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 \, ds + \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda u_t^2 \, dD, \tag{42}$$

$$\int\limits_D e^{-\lambda t}[u_{x_1 t}u_{x_1} + x_2^m u_{x_2 t}u_{x_2}]dD = \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}[u_{x_1}^2 + x_2^m u_{x_2}^2]\nu_0 \, ds +$$

$$+\frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}\lambda[u_{x_1}^2 + x_2^m u_{x_2}^2]dD, \tag{43}$$

$$(u_t^2 - u_{x_1}^2 - x_2^m u_{x_2}^2)\big|_{S_1} = 0, \tag{44}$$

as well as the inequality

$$-\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 \, ds + \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}[u_{x_1}^2 + x_2^m u_{x_2}^2]\nu_0 \, ds =$$

$$= -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 \, ds + \frac{1}{2}\int\limits_{S_1} e^{-\lambda t}[u_{x_1}^2 + x_2^m u_{x_2}^2]\nu_0 \, ds +$$

$$+\frac{1}{2}\int\limits_{S_2} e^{-\lambda t}[u_{x_1}^2 + x_2^m u_{x_2}^2]\nu_0 \, ds \geq$$

$$\geq -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t}[u_t^2 - u_{x_1}^2 - x_2^m u_{x_2}^2]\nu_0 \, ds = 0. \tag{45}$$

In deriving (45), we used the fact that $\nu_0\big|_{S_2} \geq 0$.

By virtue of (42)–(45) equality (41) implies

$$(L^* v, u)_{L_2(D)} \geq \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda[u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2]dD +$$

$$+\int\limits_D e^{-\lambda t}[a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u]u_t \, dD. \tag{46}$$

Using the fact that $\nu_0\big|_{S_2} \geq 0$ and conditions (2), (10), (38) and performing integration by parts, we obtain

$$\int\limits_D e^{-\lambda t} a_4 u u_t dD = \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t} a_4 u^2 \nu_0 ds +$$

$$+\frac{1}{2}\int\limits_D e^{-\lambda t}(\lambda a_4 - a_{4t})u^2 dD \geq 0. \tag{47}$$

By (47) we find from (46) that

$$(L^*v,u)_{L_2(D)} \geq \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda[u_t^2+u_{x_1}^2+x_2^m u_{x_2}^2]dD +$$
$$+\int\limits_D e^{-\lambda t}[a_1u_{x_1}+a_2u_{x_2}+a_3u_t]u_t\,dD \geq \frac{\lambda}{2}\int\limits_D e^{-\lambda t}[u_t^2+u_{x_1}^2+$$
$$+x_2^m u_{x_2}^2]dD - \Big|\int\limits_D e^{-\lambda t}[a_1u_{x_1}+a_2u_{x_2}+a_3u_t+a_4u]u_t\,dD\Big|.$$

Hence, like in deriving inequality (35), from (33) we obtain

$$(L^*v,u)_{L_2(D)} \geq \Big[\frac{\lambda}{2}-\Big(\frac{1}{2}M+\frac{3}{2}\mu\Big)\Big]\inf_D e^{-\lambda t}\|v\|_{L_2(D)}\,\|u\|_{W_+}. \tag{48}$$

For sufficiently large λ the latter inequality immediately implies (39). This proves Lemma 5. ■

Definition 1. For $F\in L_2(D)$ the function u will be called a strongly generalized solution of problem (1), (2) from the class W_+ provided that $u\in W_+$ and there exists a sequence of functions $u_n\in E_0$ such that $u_n\to u$ in the space W_+ and $Lu_n\to F$ in the space W_-^*, i.e.,

$$\lim_{n\to\infty}\|u_n-u\|_{W_+}=0,\quad \lim_{n\to\infty}\|Lu_n-F\|_{W_-^*}=0.$$

Definition 2. For $F\in W_-^*$ the function u will be called a strongly generalized solution of problem (1), (2) from the class L_2 provided that $u\in L_2(D)$ and there exists a sequence of functions $u_n\in E_0$ such that $u_n\to u$ in the space $L_2(D)$ and $Lu_n\to F$, $n\to\infty$, in the space W_-^*, i.e.,

$$\lim_{n\to\infty}\|u_n-u\|_{L_2(D)}=0.\quad \lim_{n\to\infty}\|Lu_n-F\|_{W_-^*}=0.$$

By the results of [13] Lemmas 2–5 give rise to the following theorems.

Theorem 1. *Let conditions* (5), (23) *and* (38) *be fulfilled. Then for any $F\in W_-^*$ there exists a unique strongly generalized solution u of problem* (1), (2) *from the class L_2, for which the estimate*

$$\|u\|_{L_2(D)}\leq c\|F\|_{W_-^*} \tag{49}$$

where the positive constant c does not depend on F, is valid.

Theorem 2. *Let conditions* (5), (23) *and* (38) *be fulfilled. Then for any $F\in L_2(D)$ there exists a unique strongly generalized solution u of problem* (1), (2) *from the class W_+, for which estimate* (49) *holds.*

Similar results hold for problem (3), (4) as well.

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Author's address:
Department of Theoretical Mechanics (4)
Georgian Technical University
77, M. Kostava St., Tbilisi 380075
Georgia