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ON UNIQUE SOLVABILITY OF THE PERIODIC PROBLEM IN THE PLANE FOR LINEAR HYPERBOLIC EQUATIONS

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Consider the linear hyperbolic equation

$$\frac{\partial^2 u}{\partial x \partial y} = p_0(x,y)u + p_1(x,y)\frac{\partial u}{\partial x} + p_2(y)\frac{\partial u}{\partial y} + q(x,y), \tag{1}$$

where $p_j : \mathbb{R}^2 \to \mathbb{R}$ $(j = 0, 1)$, $p_2 : \mathbb{R} \to \mathbb{R}$ are essentially bounded measurable functions and $q : \mathbb{R}^2 \to \mathbb{R}$ is a locally summable function. Besides, let there exist constants $\omega_1 > 0$ $\omega_2 > 0$ such that

$$p_j(x+\omega_1, y) = p_j(x,y) = p_j(x, y+\omega_2) \ \ (j=0,1) \quad \text{for} \quad (x,y) \in \mathbb{R}^2,$$
$$p_2(y+\omega_2) = p_2(y) \quad \text{for} \quad y \in \mathbb{R}.$$

By a solution of the equation (1) we understand a locally absolutely continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying the equation (1) almost everywhere in $\mathbb{R}^2$. Below we formulate sufficient conditions of existence and uniqueness of a solution of the equation (1) satisfying the conditions

$$u(x+\omega_1, y) = u(x,y), \quad u(x, y+\omega_2) = u(x,y) \quad \text{for} \quad (x,y) \in \mathbb{R}^2. \tag{2}$$

First we consider the case where $p_1(x,y) \equiv 1$ and $p_2(x,y) \equiv 1$, i.e., the equation (1) has the form

$$\frac{\partial^2 u}{\partial x \partial y} = p_0(x,y)u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + q(x,y). \tag{1'}$$

Theorem 1. *Let $p_0(x,y) \not\equiv 0$ be an absolutely continuous function and let either of following two conditions*

$$p_0(x,y) + \frac{1}{2}\frac{\partial p_0(x,y)}{\partial x} \le 0 \quad \textit{for} \quad (x,y) \in \mathbb{R}^2$$

or

$$p_0(x,y) + \frac{1}{2}\frac{\partial p_0(x,y)}{\partial y} \le 0 \quad \textit{for} \quad (x,y) \in \mathbb{R}^2.$$

take place. Then the problem $(1'), (2)$ is uniquely solvable.

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Below we study the problem (1),(2) in the case where the following conditions take place:

$$\int_0^{\omega_2} p_1(x,t)\,dt > 0 \quad \text{for} \quad x \in [0,\omega_1], \quad p_2(y) > 0 \quad \text{for} \quad y \in [0,\omega_2],$$

$$\int_0^{\omega_1} p_0(s,y)\,ds \le 0, \quad \int_0^{\omega_1} p_1(s,y)\,ds \ge 0 \quad \text{for} \quad y \in [0,\omega_2]$$

and

$$\int_0^{\omega_1}\int_0^{\omega_2} p_0(s,t)\,dsdt \ne 0.$$

Introduce the following notation:

$$p_{00}(y) = \frac{1}{\omega_1}\int_0^{\omega_1} p_0(s,y)\,ds, \quad p_{01}(y) = \frac{1}{\omega_1}\int_0^{\omega_1} p_1(s,y)\,ds,$$

$$\rho_m(y) = \frac{\frac{4\pi^2}{\omega_1^2}m^2 p_{10}(y) - p_{00}(y)p_2(y)}{\frac{4\pi^2}{\omega_1^2}m^2 + p_2^2(y)}, \quad \alpha_m(y) = \frac{1}{\frac{4\pi^2}{\omega_1^2}m^2 + p_2^2(y)} \quad \text{for} \quad m \in \mathbb{Z},$$

$$\beta_{mk}(y) = \frac{k^2}{\Big(\frac{4\pi^2}{\omega_1^2}m^2 + p_2^2(y)\Big)(m-k)^2} \quad \text{for} \quad m \ne k,\ m,k \in \mathbb{Z},$$

$$\mathcal{I}_0(p_0,p_1,p_2) = \sup_{y\in[0,\omega_2]}\Bigg(\int_y^{y+\omega_2}\sum_{m\in\mathbb{Z}}\frac{\exp\Big(-2\int_t^y \rho_m(\tau)\,d\tau\Big)}{\Big(\exp\Big(\int_0^{\omega_2}\rho_m(\tau)\,d\tau\Big)-1\Big)^2}\alpha_m(t)\,dt\Bigg)^{\frac{1}{2}},$$

$$\mathcal{I}_1(p_0,p_1,p_2) = \sup_{y\in[0,\omega_2]}\Bigg(\int_y^{y+\omega_2}\sup_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z},m\ne k}\frac{\exp\Big(-2\int_t^y \rho_m(\tau)\,d\tau\Big)}{\Big(\exp\Big(\int_0^{\omega_2}\rho_m(\tau)\,d\tau\Big)-1\Big)^2}\beta_{mk}(t)\,dt\Bigg)^{\frac{1}{2}}.$$

Theorem 2. *Let p_1 be an absolutely continuous function and let*

$$\mathcal{I}_0(p_0,p_1,p_2)\Bigg(\frac{1}{\omega_1}\int_0^{\omega_1}\int_0^{\omega_2}|p_0(s,t)-p_{00}(t)|^2\,ds\,dt\Bigg)^{\frac{1}{2}} +$$

$$+\mathcal{I}_1(p_0,p_1,p_2)\Bigg(\frac{1}{\omega_1}\int_0^{\omega_1}\int_0^{\omega_2}\left|\frac{\partial p_1(s,t)}{\partial s}\right|^2 ds\,dt\Bigg)^{\frac{1}{2}} < 1.$$

Then the problem (1), (2) *is uniquely solvable.*

References

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