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BOUNDARY VALUE PROBLEM FOR PIECEWISE-HOMOGENEOUS VISCOELASTIC PLATE WITH FINITE CRACK

Abstract. A piecewise-homogeneous viscoelastic plate, weakened by a finite crack, which meets the interface at a right angle, is considered. The crack boundary is loaded with normal symmetric forces. Using the analogues of the Kolosov–Mushkelishvili formulas of viscoelasticity theory, the complex potentials are determined and a system of singular integral equations of the first kind with respect to jump of the normal displacement is obtained. The asymptotic behavior of a solution of the resulting system is investigated. In the particular case, using the methods of the theory of analytic functions, the solution to the problem is presented in explicit form. The behavior of normal contact stresses in the neighborhood of singular points is established.

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1 Introduction

The first fundamental problem for a piecewise-homogeneous plane was solved when a crack of finite length arrives at the interface of two bodies at the right angle. Various problems with mixed boundary conditions are reduced to the Wiener-Hopf equation [9]. Similar problems for a piecewise-orthotropic and piecewise-isotropic planes under the action of symmetrical normal stresses at the cracks sides (of finite or half-infinite length) are investigated. These problems are reduced to the singular integral equations with fixed singularity [2,14] and their solutions are presented explicitly. Additionally, contact problems for a piecewise-homogeneous planes with a semi-infinite or finite inclusion or stringer are investigated [3, 4, 8, 12]. In the present paper, a piecewise-homogeneous viscoelastic plate (under the condition of Volterra viscoelastic model) with a finite crack is considered. When the crack crosses the interface between two materials, the problem is reduced to the system of singular integral equations of the first kind. The asymptotic estimates of solutions of this problem are derived. In the special case, when the crack passes through the interface, the singular integral equation with fixed singularity is investigated by using the methods of analytical functions and integral transformation.

2 Statement of the problem and reduction to the system of integral equations

Suppose the body holds a complex plane $z = x + iy$, consisting of two dissimilar isotropic half-plane with viscoelastic properties. It is weakened by a finite crack the boundary of which is loaded by normal symmetric forces (see Figure 1).

The half-planes $S_1 = \{z \mid \operatorname{Re} z > 0,\ \ z \notin l_1 = (0, a)\}$ and $S_2 = \{z \mid \operatorname{Re} z < 0,\ \ z \notin l_2 = [-a,0)\}$ are connected along the $0y$ axis. Quantities and functions related to the half-planes S_k well be marked by the index k $(k = 1, 2)$, and the boundary values of the functions at the upper and lower edges of the crack will be marked by the signs $(+)$ and $(-)$, respectively. The contact conditions along the interface have the form

$$\sigma_x^{(1)} = \sigma_x^{(2)}, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y}, \quad \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y}\,. \tag{2.1}$$

On the crack boundary we have the following conditions:

$$\begin{aligned} &\sigma_y^{(k)+} = \sigma_y^{(k)-} = N_k(x,t), \quad \tau_{xy}^{(k)+} = \tau_{xy}^{(k)-} = 0, \\ &u_k^+ - u_k^- = 0, \quad v_k^+ - v_k^- = v_k(x,t), \quad x \in l_k, \ \ k = 1, 2, \end{aligned} \tag{2.2}$$

${\sigma_y}^{(k)}, \tau_{xy}^{(k)}$ and u_k, v_k are the stress and displacement components, respectively. Given continuous functions $N_k(x,t)$ on the interval, $v_k(x,t)$ are unknown functions, showing the opening of the crack at the corresponding points.

The analogues of the Kolosov–Muskhelishvili formulas of the viscoelasticity theory are presented in [8] as follows:

$$\begin{aligned} \sigma_y^{(k)} - i\tau_{xy}^{(k)} &= \Phi_k(z,t) + \overline{\Phi_k(z,t)} + z\overline{\Phi_k'(z,t)} + \overline{\Psi_k(z,t)}, \\ (I-L)\left[\varkappa_k \Phi_k(z,t) - \overline{\Phi_k(z,t)} - z\overline{\Phi_k'(z,t)} - \overline{\Psi_k(z,t)}\right] &= 2\mu_k\left(u_k' + iv_k'\right), \end{aligned} \tag{2.3}$$

where

$$(I-L)g_k(t) = g_k(t) - \int\limits_{t_0}^{t} E_k \frac{\partial}{\partial \tau} C_k(t,\tau) g_k(\tau) d\tau, \quad 2\mu_k = \frac{E_k}{1+\nu_k}\,,$$

$$C_k(t,\tau) = \varphi_k(\tau)(1 - e^{-\gamma(t-\tau)}),$$

and $\varkappa_k = 3 - 4\nu_k$ (in the case of plane strain) or $\varkappa_k = \frac{3-\nu_k}{1+\nu_k}$ (in the case of generalized plane stress), $k = 1, 2$; $C_k(t,\tau)$ and E_k are the creep measure and Jung's module of the materials, respectively.

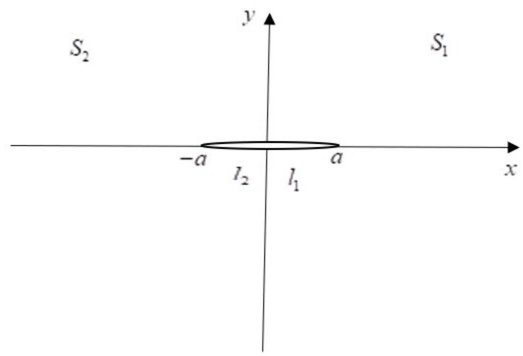

Figure 1

Here, $\varphi_k(\tau)$ is known as the ageing function, the function $(1-e^{-\gamma(t-\tau)})$ characterizes the hereditary properties of materials, and t_0 is the ageing of the material at the beginning of load. Besides, the Poisson coefficients for elastic-instant deformation $\nu_k(t)$ and creep deformation $\nu_k(t,\tau)$ are the same and constant: $\nu_k(t)=\nu_k(t,\tau)=\nu_k=const$ [1,5,6].

Based on formulas (2.2) and (2.3), we obtain the following conditions of the problems of linear conjugation:

$$\begin{aligned} &\Phi_k^+(x,t)-\Phi_k^-(x,t)=a_k(x,t), \quad x\in l_k,\\ &\Psi_k^+(x,t)-\Psi_k^-(x,t)=b_k(x,t), \quad x\in l_k, \end{aligned} \tag{2.4}$$

where

$$a_k(x,t)=if_k(x,t), \quad b_k(x,t)=-ixf_k'(x,t), \quad f_k(x,t)=\frac{2\mu_k}{\varkappa_k+1}(I-L)^{-1}\frac{\partial v_k(x,t)}{\partial x},$$

and $(I-L)^{-1}$ is an inverse operator of $(I-L)$, which can be clearly given as [8]

$$(I-L)^{-1}g_k(t)= \ C_k\,(t_0)\int\limits_{t_0}^{t}\delta_k(\tau)d\tau+\int\limits_{t_0}^{t}\delta_k(\tau)\bigg(\int\limits_{t_0}^{\tau}\frac{A_k(s)}{\delta_k(s)}ds\bigg)d\tau+\widetilde{C}_k\,(t_0)\,,$$

where

$$\delta_k(t)=\exp\bigg(-\gamma\int\limits_{t_0}^{t}\alpha_k(\tau)d\tau\bigg), \quad \alpha_k(t)=1+E_k\varphi_k(t), \quad \widetilde{C}_k\,(t_0)=E_kg_k(t_0),$$

$$A_k(t)=E_k\ddot{g}_k(t)+\gamma\dot{g}_k(t), \quad C_k\,(t_0)=E_k(\dot{g}_k(t_0)-E\varphi_k(t_0)\gamma g_k(t_0)),$$

$$k=1,2; \quad \dot{g}\equiv\frac{\partial g}{\partial t}, \quad \ddot{g}\equiv\frac{\partial^2 g}{\partial t^2}.$$

The general solutions of the obtained jump problems (2.4) can be written in the following form:

$$\begin{aligned} \Phi_k(z,t)&=\frac{1}{2\pi i}\int\limits_{l_k}\frac{a_k(x,t)dx}{x-z}+W_k(z,t)\equiv A_k(z,t)+W_k(z,t),\\ \Psi_k(z,t)&=\frac{1}{2\pi i}\int\limits_{l_k}\frac{b_k(x,t)dx}{x-z}+Q_k(z,t)\equiv B_k(z,t)+Q_k(z,t). \end{aligned} \tag{2.5}$$

where $W_k(z)$, $Q_k(z)$, $k=1,2$, are unknown analytic functions in the half-plates S_k to be determined from conditions (2.1) at the interface of two materials.

Using the methods of the theory of analytic functions (particularly, the Cauchy theorems), for the search functions [10], we obtain the following representations:

$$
\begin{aligned}
W_1(z,t) &= e_1t_1\int\limits_{l_1}\frac{xf_1'(x,t)dx}{x+z}+e_1t_1\int\limits_{l_1}\frac{xf_1(x,t)dx}{(x+z)^2}+h_1t_2\int\limits_{l_2}\frac{f_2(x,t)dx}{x-z},\\
W_2(z,t) &= -e_2t_2\int\limits_{l_2}\frac{xf_2'(x,t)dx}{x+z}-e_2t_2\int\limits_{l_2}\frac{xf_2(x,t)dx}{(x+z)^2}+h_2t_1\int\limits_{l_1}\frac{f_1(x,t)dx}{x-z},\\
Q_1(z,t) &= e_1t_1\int\limits_{l_1}\frac{xf_1'(x,t)dx}{x+z}-m_1t_1\int\limits_{l_1}\frac{f_1(x,t)dt}{x+z}-2e_1t_1z\int\limits_{l_1}\frac{(xf_1(x,t))'dx}{(x+z)^2}\\
&\quad -h_3t_2\int\limits_{l_2}\frac{xf_2'(x,t)dx}{x-z}-m_2t_2\int\limits_{l_2}\frac{xf_2(x,t)dt}{(x-z)^2},\\
Q_2(z,t) &= e_2t_2\int\limits_{l_2}\frac{xf_2'(x,t)dx}{x+z}+m_2t_2\int\limits_{l_2}\frac{f_2(x,t)dt}{x+z}+2e_2t_2z\int\limits_{l_2}\frac{(xf_2(x,t))'dx}{(x+z)^2}\\
&\quad -h_4t_2\int\limits_{l_1}\frac{xf_1'(x,t)dx}{x-z}+m_1t_1\int\limits_{l_1}\frac{xf_1(x,t)dt}{(x-z)^2},
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
&t_1=\frac{\mu_1}{1+\varkappa_1},\quad t_2=\frac{\mu_2}{1+\varkappa_2},\quad e_1=\frac{\mu_2-\mu_1}{\varkappa_1\mu_2+\mu_1},\quad e_2=\frac{\mu_2-\mu_1}{\varkappa_2\mu_1+\mu_2},\\
&m_1=(\varkappa_1+1)\,\mu_2\left[\frac{1}{\varkappa_2\mu_1+\mu_2}-\frac{1}{\varkappa_1\mu_2+\mu_1}\right]=h_2-h_4,\\
&m_2=(\varkappa_2+1)\,\mu_1\left[\frac{1}{\varkappa_2\mu_1+\mu_2}-\frac{1}{\varkappa_1\mu_2+\mu_1}\right]=h_3-h_1,\\
&h_1=\frac{(\varkappa_2+1)\,\mu_1}{\varkappa_1\mu_2+\mu_1},\quad h_2=\frac{(\varkappa_1+1)\,\mu_2}{\varkappa_2\mu_1+\mu_2},\quad h_3=\frac{(\varkappa_2+1)\,\mu_1}{\varkappa_2\mu_1+\mu_2},\quad h_4=\frac{(\varkappa_1+1)\,\mu_2}{\varkappa_1\mu_2+\mu_1}.
\end{aligned}
$$

Taking into account relations (2.5)–(2.6) in the equality

$$
\sigma_y^{(k)}(z,t)=\mathrm{Re}[\Phi_k(z,t)+\overline{\Phi_k(z,t)}+z\overline{\Phi_k'(z,t)}+\overline{\Psi_k(z,t)}],
$$

and passing the limit as $z\to x\pm i0$, from the condition

$$
\sigma_y^{(k)+}+\sigma_y^{(k)-}=2N_k(x,t),\quad x\in l_k,\ \ k=1,2,
$$

we obtain the following system of singular integral equations of the first kind:

$$
\begin{aligned}
&\frac{1}{\pi}\int\limits_0^1\left\{\frac{1}{x-s}-\frac{2e_1+m_1}{2(x+s)}-\frac{2e_1x}{(x+s)^2}+\frac{4e_1x^2}{(x+s)^3}\right\}f_1(x,t)dx\\
&\qquad+\frac{t_2}{\pi t_1}\int\limits_0^1\left\{\frac{-(h_1+h_3)}{x+s}+\frac{m_2x}{(x+s)^2}\right\}f_2(-x,t)dx=N_1(s,t),\\
&\frac{1}{\pi}\int\limits_0^1\left\{\frac{1}{x-s}-\frac{2e_2+m_2}{2(x+s)}-\frac{2e_2x}{(x+s)^2}+\frac{4e_2x^2}{(x+s)^3}\right\}f_2(-x,t)dx\\
&\qquad+\frac{t_1}{\pi t_2}\int\limits_0^1\left\{\frac{-(h_2+h_4)}{x+s}+\frac{m_1x}{(x+s)^2}\right\}f_1(x,t)dx=N_2(-s,t),\quad 0<s<1.
\end{aligned}
\tag{2.7}
$$

3 Asymptotic study of the system of singular integral equations

To find the behavior of the solution of the system of integral equations (2.7) at the singular points, we present its solution in the neighborhood of the point $s=1$ in the following form:

$$\begin{aligned} &f_1(s,t)=(1-s)^{-\alpha}g_1(s,t),\\ &\varphi(s,t)\equiv f_2(-s,t)=(1-s)^{-\delta}\varphi_1(s,t), \quad 0\le \mathrm{Re}(\alpha,\delta)<1, \end{aligned} \tag{3.1}$$

where $g_1(x,t)$ and $\varphi_1(x,t)$ are the continuous functions in the neighborhood of the point $s=1$.

The functions

$$\begin{aligned} J_1(s,t)&=\frac{1}{\pi}\int_0^1\left\{-\frac{2e_1+m_1}{2(x+s)}-\frac{2e_1x}{(x+s)^2}+\frac{4e_1x^2}{(x+s)^3}\right\}f_1(x,t)dx\\ &+\frac{t_2}{\pi t_1}\int_0^1\left\{\frac{-(h_1+h_3)}{x+s}+\frac{m_2x}{(x+s)^2}\right\}f_2(-x,t)dx,\\ J_2(s,t)&=\frac{1}{\pi}\int_0^1\left\{-\frac{2e_2+m_2}{2(x+s)}-\frac{2e_2x}{(x+s)^2}+\frac{4e_2x^2}{(x+s)^3}\right\}f_2(x,t)dx\\ &+\frac{t_1}{\pi t_2}\int_0^1\left\{\frac{-(h_2+h_4)}{x+s}+\frac{m_1x}{(x+s)^2}\right\}f_1(x,t)dx \end{aligned}$$

are regular in the neighborhood of the point $s=1$. Therefore, they can be represented as follows:

$$\begin{aligned} J_1(s,t)&=J_1(1,t)+{J_1}'(1,t)(1-s)+\frac{1}{2}{J_1}''(1,t)(1-s)^2+\cdots,\\ J_2(s,t)&=J_2(1,t)+{J_2}'(1,t)(1-s)+\frac{1}{2}{J_2}''(1,t)(1-s)^2+\cdots. \end{aligned} \tag{3.2}$$

According to the well-known theorems about the behavior of the Cauchy-type integral near the ends of the integration curve, from (2.7) and (3.2), we have (see [11])

$$\begin{gathered} -\operatorname{ctg}\pi\alpha g_1(1,t)(1-s)^{-\alpha}+O((1-s)^{-\alpha_0})+J_1(1,t)+\cdots=N_1(1,t),\\ -\operatorname{ctg}\pi\delta\varphi_1(1,t)(1-s)^{-\delta}+O((1-s)^{-\delta_0})+J_2(1,t)+\cdots=N_2(-1,t),\\ \alpha_0<\mathrm{Re}\,\alpha, \quad \delta_0<\mathrm{Re}\,\delta. \end{gathered}$$

Since the first terms have the greatest singularity, from the system of equations (2.7), we obtain

$$\operatorname{ctg}\pi\alpha=0, \quad \operatorname{ctg}\pi\delta=0, \quad \alpha=\delta=\frac{1}{2}\,. \tag{3.3}$$

Accordingly, in the neighborhood of the point $s=1$, the solutions of the system have a singularity of order $\frac{1}{2}$.

We present the solution of the system in the neighborhood of the point $s=0$ as follows:

$$f_1(s,t)=s^{-\beta}g_2(s,t), \quad \varphi(s,t)=s^{-\gamma}\varphi_2(s,t), \quad 0\le\mathrm{Re}(\beta,\gamma)<1, \tag{3.4}$$

where $g_2(x,t)$ and $\varphi_2(x,t)$ are the continuous functions in the neighborhood of the point $s=0$.

Based on the known theorems on the behavior of the Cauchy-type integral [11], we have

$$\frac{1}{\pi}\int_0^1 \frac{f_1(x,t)dx}{x-s} = \operatorname{ctg}\pi\beta g_2(0,t)s^{-\beta} + O(s^{-\beta_0}), \quad s\to 0,$$
$$\frac{1}{\pi}\int_0^1 \frac{\varphi(x,t)dx}{x-s} = \operatorname{ctg}\pi\gamma g_2(0,t)s^{-\gamma} + O(s^{-\gamma_0}), \quad s\to 0,$$
$$\beta_0 < \operatorname{Re}\beta, \quad \gamma_0 < \operatorname{Re}\gamma.$$

For the functions

$$\begin{aligned}
J_3(s,t) &= \frac{1}{\pi}\int_0^1 \left\{-\frac{2e_1x}{(x+s)^2} + \frac{4e_1x^2}{(x+s)^3}\right\} f_1(x,t)dx + \frac{t_2}{\pi t_1}\int_0^1 \frac{m_2x}{(x+s)^2} f_2(-x,t)dx \\
&= \frac{1}{\pi}\frac{d}{ds}\int_0^1 \frac{2e_1x}{x+s} f_1(x,t)dx + \frac{1}{\pi}\frac{d^2}{ds^2}\int_0^1 \frac{2e_1x^2}{x+s} f_1(x,t)dx - \frac{t_2}{\pi t_1}\frac{d}{ds}\int_0^1 \frac{m_2x}{x+s} f_2(-x,t)dx, \\
J_4(s,t) &= \frac{1}{\pi}\int_0^1 \left\{-\frac{2e_2x}{(x+s)^2} + \frac{4e_2x^2}{(x+s)^3}\right\} f_2(x,t)dx + \frac{t_1}{\pi t_2}\int_0^1 \frac{m_2x}{(x+s)^2} f_2(-x,t)dx \\
&= \frac{1}{\pi}\frac{d}{ds}\int_0^1 \frac{2e_2x}{x+s} f_2(-x,t)dx + \frac{1}{\pi}\frac{d^2}{ds^2}\int_0^1 \frac{2e_2x^2}{x+s} f_2(-x,t)dx - \frac{t_1}{\pi t_2}\frac{d}{ds}\int_0^1 \frac{m_1x}{x+s} f_1(x,t)dx,
\end{aligned}$$

we obtain

$$\begin{aligned}
J_3(s,t) &= 2e_1\left[\frac{(1-\beta)^2}{\sin\pi\beta}\frac{g_2(0,t)}{s^\beta} + O(s^{-\beta_0})\right] + \frac{t_2m_2}{t_1}\left[\frac{1-\gamma}{\sin\pi\gamma}\frac{\varphi_2(0,t)}{s^\gamma} + O(s^{-\gamma_0})\right], \\
J_4(s,t) &= 2e_2\left[\frac{(1-\gamma)^2}{\sin\pi\gamma}\frac{\varphi_2(0,t)}{s^\gamma} + O(s^{-\gamma_0})\right] + \frac{t_1m_1}{t_2}\left[\frac{1-\beta}{\sin\pi\beta}\frac{g_2(0,t)}{s^\beta} + O(s^{-\beta_0})\right], \quad s\to 0,
\end{aligned}$$

and the functions

$$L_1(s,t) = -\frac{A_1}{\pi}\int_0^1 \frac{f_1(x,t)dx}{x+s} - \frac{D_1}{\pi}\int_0^1 \frac{\varphi(x,t)dx}{x+s}, \quad L_2(s,t) = -\frac{A_2}{\pi}\int_0^1 \frac{\varphi(x,t)dx}{x+s} - \frac{D_2}{\pi}\int_0^1 \frac{f_1(x,t)dx}{x+s}$$

satisfy the following estimates:

$$\begin{aligned}
L_1(s,t) &= -A_1\left\{\frac{e^{i\beta\pi}}{\sin\pi\beta}g_2(0,t)(-s)^{-\beta} + O((-s)^{-\beta_0})\right\} - D_1\left\{\frac{e^{i\gamma\pi}}{\sin\pi\gamma}\varphi_2(0,t)(-s)^{-\gamma} + O(-s)^{-\gamma_0}\right\} \\
&= -A_1\left\{\frac{1}{\sin\pi\beta}g_2(0,t)s^{-\beta} + O(s^{-\beta_0})\right\} - D_1\left\{\frac{1}{\sin\pi\gamma}\varphi_2(0,t)s^{-\gamma} + O(s^{-\gamma_0})\right\}, \quad s\to 0, \\
L_2(s,t) &= -A_2\left\{\frac{e^{i\gamma\pi}}{\sin\pi\gamma}\varphi_2(0,t)(-s)^{-\gamma} + O((-s)^{-\gamma_0})\right\} - D_2\left\{\frac{e^{i\beta\pi}}{\sin\pi\beta}g_2(0,t)(-s)^{-\beta} + O((-s)^{-\beta_0})\right\} \\
&= -A_2\left\{\frac{1}{\sin\pi\gamma}\varphi_2(0,t)s^{-\gamma} + O(s^{-\gamma_0})\right\} - D_2\left\{\frac{1}{\sin\pi\beta}g_2(0,t)s^{-\beta} + O(s^{-\beta_0})\right\}, \quad s\to 0,
\end{aligned}$$
$$\beta_0 < \operatorname{Re}\beta, \quad \gamma_0 < \operatorname{Re}\gamma.$$

Taking into account these estimates, from the system of singular integral equations (2.7), in the neighborhood of the point $s=0$, we obtain

$$\begin{aligned}
&-\operatorname{ctg}\pi\beta g_2(0,t)s^{-\beta}-A_1\frac{g_2(0,t)}{\sin\pi\beta}s^{-\beta}-D_1\frac{\varphi_2(0,t)}{\sin\pi\gamma}s^{-\gamma}+2e_1\frac{(1-\beta)^2}{\sin\pi\beta}g_2(0,t)s^{-\beta}\\
&\quad+\frac{t_2m_2}{t_1}\frac{1-\gamma}{\sin\pi\gamma}\varphi_2(0,t)s^{-\gamma}+O(s^{-\beta_0})+O(s^{-\gamma_0})=N_1(0,t),\\
&-\operatorname{ctg}\pi\gamma\varphi_2(0,t)s^{-\gamma}-A_2\frac{\varphi_2(0,t)}{\sin\pi\gamma}s^{-\gamma}-D_2\frac{g_2(0,t)}{\sin\pi\beta}s^{-\beta}+2e_2\frac{(1-\gamma)^2}{\sin\pi\gamma}\varphi_2(0,t)s^{-\gamma}\\
&\quad+\frac{t_1m_1}{t_2}\frac{1-\beta}{\sin\pi\beta}g_2(0,t)s^{-\beta}+O(s^{-\beta_0})+O(s^{-\gamma_0})=N_2(0,t),
\end{aligned}\tag{3.5}$$

where

$$A_1=\frac{2e_1+m_1}{2},\quad D_1=\frac{(h_1+h_3)t_2}{t_1},\quad A_2=\frac{2e_2+m_2}{2},\quad D_2=\frac{(h_2+h_4)t_1}{t_2}\,.$$

It is easy to prove that $\beta=\gamma$, since otherwise ($\operatorname{Re}\beta>\operatorname{Re}\gamma$, $\operatorname{Re}\beta<\operatorname{Re}\gamma$) the last equalities are not simultaneously satisfied. Therefore, from (3.5), we obtain the following system of transcendental equations:

$$\begin{aligned}
&\cos\pi\beta+(A_1+2e_1(1-\beta)^2)+\Big(D_1-\frac{t_2m_2(1-\beta)}{t_1}\Big)C=0,\\
&\cos\pi\beta+(A_2+2e_2(1-\beta)^2)+\Big(D_2-\frac{t_1m_1(1-\beta)}{t_2}\Big)C^{-1}=0,
\end{aligned}\tag{3.6}$$

where $C=\frac{\varphi_2(0,t)}{g_2(0,t)}$ does not depend on β and t.

It can be shown that if $\varkappa_1\mu_2+\mu_1\neq\varkappa_2\mu_1+\mu_2$, system (3.6) has no such solution (β,C), where $0\le\operatorname{Re}\beta<1$, $C=const$. In this case, the normal stresses $\sigma_y^{(1)}(z,t)$ may have the logarithmic singularity or may be bounded in the neighborhood of zero.

In the case $\varkappa_1\mu_2+\mu_1=\varkappa_2\mu_1+\mu_2$, equations (3.6) take the form

$$\cos\pi\beta+e_1+2e_1(1-\beta)^2+D_1C=0\tag{3.7}$$

where $C=\pm\sqrt{\frac{D_2}{D_1}}=\pm\sqrt{\frac{(\varkappa_1-1)(\varkappa_2+1)}{(\varkappa_2-1)(\varkappa_1+1)}}$, and if, additionally, $\mu_1=\mu_2$ (the material of both half-planes is the same), we obtain

$$\cos\pi\beta=\pm2,\quad \beta=-\frac{i}{\pi}\ln(2\pm\sqrt{3}),$$

which means that the functions $f_1(s,t)$ and $f_2(-s,t)$, and accordingly the normal stresses $\sigma_y^{(1)}(z,t)$ can have an integrable singularity or can be bounded in the neighborhood of zero.

The analytical study of equations (3.6) and (3.7) is associated with great difficulties and can be investigated only numerically.

Theorem. *If the system of singular integral equations (2.5) has a solution, then it satisfies the following estimates:*

$$\begin{aligned}
&\begin{cases} f_1(s,t)\\ f_2(-s,t)\end{cases}=O((1-s)^{-\frac12}),\quad \begin{cases} s\to1-,\\ -s\to-1+,\end{cases}\quad 0<s<1,\\
&\begin{cases} f_1(s,t)\\ f_2(-s,t)\end{cases}=O(s^{-\beta}),\quad \begin{cases} s\to0+,\\ -s\to0-,\end{cases}\quad 0<s<1,\ \ 0\le\operatorname{Re}\beta<1,
\end{aligned}$$

where β can be determined from equation (3.7).

4 Exact solution of the problem in a specific case

Let us consider a problem when the crack is located in one half-plane and passes at a right angle on the interface of two materials. Considering

$$\sigma_y^{(1)+}(x)+\sigma_y^{(1)-}(x)=\begin{cases}2N_k(x,t), & x\in[0,1],\\ 2\sigma_y^{(1)}(x), & x\in(1,\infty),\end{cases}$$

the problem is reduced to the following singular integral equation of the first kind:

$$\frac{1}{\pi}\int_0^1 \left\{\frac{1}{x-s} - \frac{A_1}{x+s} - \frac{2e_1x}{(x+s)^2} + \frac{4e_1x^2}{(x+s)^3}\right\} f_1(x,t)dx = N_1(s,t), \quad 0<s<1. \tag{4.1}$$

Using the equality

$$\int_0^1 f_1(x,t)dx = \frac{2\mu_1}{1+\varkappa_1}(I-L)^{-1}\int_0^1 \frac{\partial v_1(x,t)}{\partial x}dx = \frac{2\mu_1}{1+\varkappa_1}(I-L)^{-1}(v_1^+(1,t)-v_1^+(0,t)) = 0,$$

from (4.1) we have

$$\frac{1}{\pi}\int_0^1 \left\{\frac{1}{x-s} + \frac{A_1-2e_1}{x+s} + \frac{6e_1x}{(x+s)^2} - \frac{4e_1x^2}{(x+s)^3}\right\} f(x,t)dx = sN_1(s,t), \quad 0<s<1, \tag{4.2}$$

where $f(x,t) = xf_1(x,t)$.

Due to the fact that the displacement must be limited at the point$x=0$, the desired function $f(x,t)$ is required to satisfy the condition $f(x,t)\to 0$, $x\to 0$.

To solve equation (4.2), we make changes to the variables $x=e^{\varsigma}$, $s=e^{\xi}$, and use a generalized Fourier transform to obtain the Riemann boundary value problem [7]

$$G(y)\Phi^-(y,t) = \Psi^+(y,t) + P(y,t), \quad y = y_0 + i\varepsilon, \quad |y_0|<\infty, \quad \varepsilon>0$$

where

$$\Phi^-(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 f_0(\xi,t)e^{i\xi z}d\xi, \quad \Psi^+(z,t) = \frac{i}{\sqrt{2\pi}}\int_0^{\infty} \psi^+(\xi,t)e^{i\xi z}d\xi,$$

$$P(z,t) = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^0 N_0(\xi,t)e^{i\xi z}d\xi,$$

$$f_0(\xi,t) = f(e^{\xi},t), \quad N_0(\xi,t) = 2e^{\xi}N_1(e^{\xi},t), \quad \psi^+(\xi,t) = \begin{cases} 0, & \xi<0, \\ 2\sigma_y^{(1)}(e^{\xi}), & \xi>0, \end{cases}$$

$$G(z) = \frac{1}{\operatorname{sh}\pi z}[\operatorname{ch}\pi z + (A_1-2e_1) + 2e_1(z+i)(2z+7i)].$$

The problem can be formulated as follows: find the function$\Psi^+(z,t)$, holomorphic in the half-plane $\operatorname{Im} z>0$, and the function $\Phi^-(z,t)$, holomorphic in the half-plane $\operatorname{Im} z<1$ (with the exception of a finite number of roots of the function $G(z)$ in the strip $0<\operatorname{Im} z<1$, at which the function $\Phi^-(z,t)$ has poles of the first order), vanishing at infinity and satisfying condition (4.1) [7, 11].

The boundary condition (4.1) is represented in the form

$$G_0(y)\frac{\Phi^-(y,t)\sqrt{y-i}}{y} = \frac{\Psi^+(y,t)}{\sqrt{y+i}} + \frac{P(y,t)}{\sqrt{y+i}}, \quad y = y_0+i\varepsilon, \quad \varepsilon>0, \tag{4.3}$$

where the function $G_0(y,t) = yG(y)[1+y^2]^{-\frac{1}{2}}$ satisfies the conditions $\operatorname{Re} G_0(y,t)>0$, $G_0(\pm\infty)=1$, $\operatorname{Ind} G_0(y,t)=0$.

The solution of this problem can be represented in the form [10]

$$\Phi^-(z,t) = \frac{zX(z,t)}{\sqrt{z-i}}, \quad \operatorname{Im} z\le 0, \quad \Psi^+(z,t) = X(z,t)\sqrt{z+i}, \quad \operatorname{Im} z>0,$$

$$\Phi^-(z,t) = \frac{\Psi^+(z,t)+P(z,t)}{G(z)}, \quad 0<\operatorname{Im} z<1,$$

where

$$X(z,t)=\frac{X(z)}{2\pi i}\int_{-\infty}^{\infty}\frac{P(y,t)dy}{X^+(y)\sqrt{y+i}(y-z)},\quad X(z)=\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G_0(y)dy}{y-z}\right).$$

The function $\Phi^-(z,t)$ is holomorphic in the half-plane $\operatorname{Im} z<1$, with the exception of a finite number of roots of the function $G(z)$ in the strip $0<\operatorname{Im} z<1$.

The function $\Psi^+(z,t)$ can be represented in the form

$$\Psi^+(z,t)=\frac{X(z)}{2\pi i\sqrt{z+i}}\left\{\int_{-\infty}^{\infty}\frac{\sqrt{y+i}P(y,t)dy}{X^+(y)(y-z)}-\int_{-\infty}^{\infty}\frac{P(y,t)dy}{X^+(y)\sqrt{y+i}}\right\}$$
$$={\Psi_0}^+(z,t)-\frac{X(z)}{2\pi i\sqrt{z+i}}\int_{-\infty}^{\infty}\frac{P(y,t)dy}{X^+(y)\sqrt{y+i}}.$$

The boundary value ${\Psi_0}^+(y,t)$ of the function ${\Psi_0}^+(z,t)$ is the Fourier transform of the bounded function $\psi_0(\xi,t)$, continuous on the semi-axis $\xi\geq 0$, except possibly at the point $\xi=0$, where it may have a logarithmic singularity. Using the inverse Fourier transform [13], from

$$\Psi^+(y,t)=\frac{c(t)}{\sqrt{y+i}}+{\Psi_0}^+(y,t),\tag{4.4}$$

we obtain ${\sigma^{(1)}}_y(x,t)=\frac{c(t)}{\sqrt{ix}\sqrt{x-1}}+\psi_0(\ln x,t)$, $x>1$, and from the function

$$\Phi^-(y,t)=\frac{c_1(t)}{\sqrt{y-i}}+{\Phi_0}^-(y,t),$$

where the function ${\Psi_0}^-(y,t)$ is the Fourier transform of the bounded function $\varphi_0(\xi,t)$, which is continuous on the semi-axis $\xi\leq 0$ (except maybe at the point $\xi=0$, where it may have a logarithmic singularity), the inverse Fourier transform yields

$$f_1(x,t)=\frac{c_1(t)}{\sqrt{1-x}}+\varphi_0(\ln x,t)=O((1-x)^{-\frac{1}{2}}),\quad 0<x<1,\quad x\to 1-.$$

It can be easily shown that for $0\leq\operatorname{Im} z<1$, we have

$$\Phi^-(z,t)=\frac{c_2(t)}{\sqrt{z-i}}+{\Phi_1}^-(z,t),\tag{4.5}$$

where the function ${\Phi_1}^-(z,t)$ is holomorphic in the strip $0<\operatorname{Im} z<1$, except maybe at one point $z_0=i\tau_0+\alpha$, $\tau_0<\beta<1$, where it has a first order pole. Using Cauchy's theorem, we obtain

$$\int_{-N}^{N}{\Phi_1}^-(y,t)e^{-iy\xi}dy=e^{\beta\xi}\int_{-N}^{N}{\Phi_1}^-(y+i\beta,t)e^{-iy\xi}dy+c_3(t)e^{\tau_0\xi}+\varepsilon(N,\xi),$$

where $\varepsilon(N,\xi)\to 0$, $N\to\infty$.

From formula (4.5), the Fourier transform gives

$$e^{\xi}f_1(e^{\xi},t)=\frac{c_2(t)e^{\xi}}{\sqrt{-\xi}}+\frac{e^{\beta\xi}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}{\Phi_0}^-(y+i\beta,t)e^{-iy\xi}dy+\frac{c_3(t)e^{\tau_0\xi}}{\sqrt{2\pi}},\quad \xi<0.$$

Therefore,

$$f_1(x,t)=\frac{c_2(t)}{\sqrt{-\ln x}}+x^{\beta-1}\varphi_1(x,t)+\frac{c_3(t)x^{\tau_0-1}}{\sqrt{2\pi}}=O(x^{\tau_0-1}),\quad 0<x<1,\quad x\to 0+,$$

$$\frac{\partial v_1(x,t)}{\partial x} = \frac{\varkappa_1 + 1}{2\mu_1}(I - L)f_1(x,t) = O(x^{\tau_0 - 1}), \quad x \to 0+ \ .$$

In a particular case $\mu_1 = \mu_2$ ($e_1 = 0$, $A_1 = \frac{\varkappa_1 - \varkappa_2}{2(\varkappa_2 + 1)}$, $|A_1| < 1$), the equation $G(z) = 0$ has a purely imaginary solution (the least distant from the origin) $z_0 = i\tau_0$, $\tau_0 = \frac{1}{\pi}\operatorname{arc}\cos\frac{\varkappa_2 - \varkappa_1}{2(\varkappa_2 + 1)}$, and the following conclusions are valid:

a) if $A_1 > 0$ ($\varkappa_1 > \varkappa_2$), then $\frac{1}{2} < \tau_0 < 1$,

b) if $A_1 < 0$ ($\varkappa_1 < \varkappa_2$), then $0 < \tau_0 < \frac{1}{2}$,

c) if $A_1 = 0$ ($\varkappa_1 = \varkappa_2$, the material of both half-planes is the same), then $\tau_0 = \frac{1}{2}$.

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