Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2024, 1–16

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ON THE LIMIT CYCLES THAT CAN BIFURCATE FROM A UNIFORM ISOCHRONOUS CENTER VIA AVERAGING METHOD

Abstract. The aim of this paper is to determine an upper bound of number of limit cycles that can bifurcate from a uniform isochronous center of a cubic homogeneous planar polynomial differential system when we perturb it inside the class of all quintic polynomial differential systems. We prove that at most 5 limit cycles can bifurcate from the period annulus by using the averaging theory of first order and at most 19 limit cycles by applying the second order averaging method. This study needs many calculations that have been verified by Maple and Mathematica.

2020 Mathematics Subject Classification. 34C07, 34G15, 34C05.

Key words and phrases. Isochronous center, limit cycle, averaging theory, polynomial differential systems.

1 Introduction

The main problem in the qualitative theory of the planar polynomial differential systems is the determination of the limit cycles. Recall that in the phase plan, a *limit cycle* of a planar polynomial differential system is an isolated periodic solution in the set of all its periodic solutions, isolated means that the surrounding trajectories are not closed.

The study of limit cycles has a long history, it was started by the works of Poincaré [15]. In 1900, at the international congress of mathematics in Paris, 23 problems [9] were posed by David Hilbert. One of them concerned the number of limit cycles that a real planar polynomial differential system may have: the second part of the 16^{th} Hilbert problem. It has been one of the main problems in the qualitative theory of planar differential systems in the last century which remains open until now (see, e.g., [7, 8, 12] and the references therein). To attack the second part of 16^{th} Hilbert problem, many researchers investigated the number of limit cycles of various planar polynomial differential systems. Among them is the problem of a number of limit cycles obtained by perturbing polynomial differential systems with centers, through of which limit cycles can bifurcate from the periodic solutions of these centers (see [7, 8, 12]). In order to study limit cycles, some methods have been developed and based on the Poincaré map such as the Poincaré–Pontryagin–Melnikov functions [2, 16], inverse integrating factor [19] and the averaging method [5, 17, 18].

The averaging method is a useful tool in dynamical systems to determine a number of limit cycles that a periodic differential system can have. The idea of averaging method is to approximate the periodic solutions of a non-autonomous periodic differential system by the solutions of its averaged differential system which is autonomous. In particular, it gives a quantitative relation between the hyperbolic equilibrium point of the averaged differential system and the limit cycles of the non-autonomous differential system (see, e.g., [5, 17, 18]) which makes the study more flexible. In other words, under some assumptions, the averaging method transforms the problem of studying the limit cycles of periodic differential system into a problem of zeros of the so-called *averaged function* associated to this system.

Isochronous differential systems constitute a large class of polynomial differential systems with interesting properties which arise in many applications. During the past three decades, wide studies on the bifurcation of limit cycles for planar differential systems with uniform isochronous centers have been published (see, e.g., [4, 10, 12, 13]). Recall that a center is isochronous if all the periodic solutions in a neighborhood of this center have the same period $T \in \mathbb{R}^*$. Moreover, the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, with $P(x, y)$ and $Q(x, y)$ as real polynomials, has a uniform isochronous center if $x\dot{y} - y\dot{x} = k(x^2 + y^2)$, $k \in \mathbb{R} - \{0\}$. In the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, isochronous means that $\dot{\theta} = k$, $k \in \mathbb{R} - \{0\}$ (for more details, see [6]).

In this paper, we consider the following planar polynomial differential system:

$$\begin{aligned} \dot{x} &= -y + x^2y + \varepsilon p(x, y), \\ \dot{y} &= x + xy^2 + \varepsilon q(x, y), \end{aligned} \tag{1.1}$$

where

$$p(x, y) = \sum_{0\le i+j\le n} a_{i,j}x^iy^j \quad \text{and} \quad q(x, y) = \sum_{0\le i+j\le n} b_{i,j}x^iy^j.$$

System $(1.1)_{\varepsilon=0}$ has a center O at the origin. It is clear that the center O is a uniform isochronous center because of $x\dot{y} - y\dot{x} = x^2 + y^2$. Note that system (1.1) has the rational first integral $H(x, y) = \frac{x^2+y^2}{1-x^2}$ with the integrating factor $\mu(x, y) = \frac{2}{(1-x^2)^2}$.

The authors in [13] prove that for $n = 3$ and $\varepsilon \neq 0$, a real parameter sufficiently small, system (1.1) can have at most three limit cycles bifurcating from the center O by using the first order averaging method. In [12], the authors confirm the result of [13] and prove that there are perturbations of $(1.1)_{n=3}$ with only $0, 1, \ldots, 8$ limit cycles by using the second order averaging method.

Motivated by the above paper, in this work, using averaging method, we study the bifurcation of limit cycles of system $(1.1)_{\varepsilon=0}$ under any small quintic homogeneous perturbations. In other words, by using averaging theory of first and second order, our objective is to determine a maximum number of limit cycles that bifurcate from the isochronous center $(1.1)_{\varepsilon=0}$ of the differential system

$$\begin{aligned}\dot{x} &= -y + x^2y + \varepsilon \sum_{0\leq i+j\leq 5} a_{i,j}x^iy^j = P(x,y) + \varepsilon p(x,y),\\ \dot{y} &= x + xy^2 + \varepsilon \sum_{0\leq i+j\leq 5} b_{i,j}x^iy^j = Q(x,y) + \varepsilon q(x,y),\end{aligned} \tag{1.2}$$

where $\varepsilon \neq 0$ is a real positive parameter, sufficiently small, and $p(x,y)$, $q(x,y)$ are two real polynomials.

The following theorem represents the main results of this paper.

Theorem 1.1. *For $\varepsilon \neq 0$, a real parameter sufficiently small,*

(a) *System* (1.2) *can have up to* 5 *limit cycles that bifurcate from the isochronous center* $(1.2)_{\varepsilon=0}$ *by using the averaging method of the first order.*

(b) *By using the averaging method of the second order, the polynomial differential system* (1.2) *has at most* 19 *limit cycles bifurcating from the isochronous center* $(1.2)_{\varepsilon=0}$.

Remark 1.1. Note that the result in statement (a) improves the ones of Proposition 1 of [12] and Theorem 1.1 of [13] by providing 2 more limit cycles, and statement (b) provides 11 more limit cycles than the ones in [12] under the perturbation of system $(1.1)_{n=3}$.

This paper is organised as follows: Theorem 1.1 is proved in Section 3. Its proof is based on the averaging theory of the first and second order (see Section 2).

2 Preliminaries

To prove our main result, we introduce some tools that we have used. The proof is based on the averaging method of the first and second order.

2.1 Averaging theory of the first order

Theorem 2.1. *Consider the following two initial value problems:*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 h(t,x,\varepsilon), \quad x(0) = x_0, \tag{2.1}$$

and

$$\dot{y} = \varepsilon f^0(y), \quad y(0) = x_0, \tag{2.2}$$

where $x, y, x_0 \in D$, D is an open interval of $\mathbb{R}$, $t \in [0,+\infty)$, f and h are T-periodic in the variable t, and $f^0(y)$ is the averaged function of $f(t,x)$ with respect to t, i.e.,

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)\,dt. \tag{2.3}$$

Suppose:

(1) f, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, h *and* $\frac{\partial h}{\partial x}$ *are defined, continuous and bounded by a constant independent of ε in* $[0,+\infty)\times D$ *and* $\varepsilon \in (0,\varepsilon_0]$.

(2) T *is a constant independent of* ε.

(3) $y(t)$ *belongs to D on the timescale* $1/\varepsilon$.

Then the following statements hold:

(1) *On the timescale* $1/\varepsilon$, *we have*

$$|x(t) - y(t)| = o(\varepsilon) \quad as \;\; \varepsilon \to 0.$$

(2) *If p is an equilibrium point of the averaged system* (2.2) *such that*

$$\frac{\partial f^0}{\partial y}\Big|_{y=p} \neq 0, \tag{2.4}$$

then system (2.1) *has a $T-$periodic solution $\Phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(3) *If* (2.4) *is negative, then the corresponding periodic solution $\Phi(t,\varepsilon)$ of* (2.1) *in the space (t,x) is assympotically stable for ε sufficiently small. If* (2.4) *is positive, then it is unstable.*

2.2 Averaging theory of the second order

Theorem 2.2. *Consider the following two initial value problems:*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 h(t,x,\varepsilon), \quad x(0) = x_0, \tag{2.5}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2 (f^{10}(y) + g^0(y)), \quad y(0) = x_0, \tag{2.6}$$

D is an open subset of $\mathbb{R}$, f, g and h are periodic of period T in t, and

$$f^1(t,x) = \frac{\partial f}{\partial x}\, y^1(t,x) - \frac{\partial y^1}{\partial x}\, f^0(t,x),$$

where

$$y^1(t,x) = \int_0^t \big(f(s,x) - f^0(x)\big)\, ds + z(x),$$

with $z(x) \in C^1$ such that the averaging of $y^1(\theta,r)$ is zero. Besides, f^0, f^{10}, g^0 denote the averaging functions of f, f^1 and g, respectively, defined as in (2.3)*. Suppose that:*

(1) *$\frac{\partial f}{\partial x}$ is Lipschitz in x and all these functions are continuous on their domain of definition.*

(2) *$|h(t,x,\varepsilon|$ is bounded by a constant uniformly in $[0,L/\varepsilon) \times D \times (0,\varepsilon_0]$.*

(3) *T is a constant independent of ε, and $y(t)$ belongs to D on the timescale $1/\varepsilon$.*

Then the following statements hold:

(1) *On the timescale $1/\varepsilon$, we have*

$$x(t) = y(t) + \varepsilon y^1(t,y(t)) + O(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0.$$

If, in addition, $f^0(y) \equiv 0$, then:

(2) *If p is an equilibrium point of the averaged system* (2.6) *such that*

$$\frac{\partial}{\partial y}(f^{10}(y) + g^0(y))\Big|_{y=p} \neq 0, \tag{2.7}$$

then there exists a T-periodic solution $\Phi(t,\varepsilon)$ of (2.5) *which is close to p such that $\Phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(3) *If* (2.7) *is negative, then the corresponding periodic solution $\Phi(t,\varepsilon)$ of* (2.5) *in the space (t,x) is asymptotically stable for ε sufficiently small. If ε is positive, then it is unstable.*

For details about the Theorems 2.1 and 2.2 see the references [17, 18].

Now, we recall the Descartes Theorem about the determination of a number of zeros of a real polynomial (for a proof, see, e.g., [3]).

Theorem 2.3 (Descartes Theorem)**.** *Consider the polynomial $p(y) = a_{i_1} y^{i_1} + a_{i_2} y^{i_2} + \cdots + a_{i_n} y^{i_n}$ with $0 \leq i_1 < i_2 < \cdots < i_n$ and the real constant $a_{i_k} \neq 0 \ \forall\, k \in \{1,2,\ldots,n\}$. When $a_{i_k} a_{i_{k+1}} < 0$, we say that a_{i_k} and $a_{i_{k+1}}$ have a variation of sign. If the number of the variation is $m \in \mathbb{N}$, then $p(y)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficient of $p(y)$ in such a way that $p(y)$ has exactly $n-1$ positive real roots.*

3 Proof of Theorem 1.1

3.1 Proof of statement (a) of Theorem 1.1

In order to apply the averaging method for studying limit cycles of system (1.2), we should write system (1.2) in the standard form (2.1). The following result provides how to transform system (1.2) into the standard form.

Lemma 3.1 ([5])**.** *Let H be the first integral of $(1.2)_{\varepsilon=0}$. Assume that:*

- *System $(1.2)_{\varepsilon=0}$ has a continuous family of ovals*

$$\{\Gamma_h\} \subset \big\{(x,y):\ H(x,y)=h,\ h_1<h<h_2\big\}.$$

- *$xQ(x,y)-yP(x,y)\neq 0$ for all (x,y) in the period annulus formed by the ovals $\{\Gamma_h\}$. Let $\rho:(\sqrt{h_1},\sqrt{h_2})\times[0,2\pi)\to[0,\infty)$ be a continuous function such that*

$$H\big(\rho(R,\phi)\cos\phi,\rho(R,\phi)\sin\phi\big)=R^2,$$

for all $R\in(\sqrt{h_1},\sqrt{h_2})$ and all $\phi\in[0,2\pi)$.

Then the differential equation which describes the dependence between the square root of energy, $R=\sqrt{h}$, and the angle ϕ for system (1.2) is

$$\frac{dR}{d\phi}=\varepsilon\,\frac{\mu(x^2+y^2)(Qp-Pq)}{2R(Qx-Py)}-\varepsilon^2\,\frac{\mu(x^2+y^2)(Qp-Pq)(qx-py)}{2R(Qx-Py)^2}$$

with

$$x=\rho(R,\phi)\cos\phi\ \ \textit{and}\ \ y=\rho(R,\phi)\sin\phi.$$

Using Lemma 3.1, we take

$$x=\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}}\ \ \text{and}\ \ y=\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}.$$

Then system (1.2) can be written as

$$\frac{dr}{d\theta}=\varepsilon F_1(\theta,r)+O(\varepsilon^2), \tag{3.1}$$

where

$$F_1(\theta,r)=\sqrt{1+r^2\cos^2\theta}\,\Big((1+r^2)\cos\theta\,\overline{p}(\theta,r)+\sin\theta\,\overline{q}(\theta,r)\Big)$$

with

$$\overline{p}(\theta,r)=p\Big(\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}},\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big)$$

and

$$\overline{q}(\theta,r)=q\Big(\frac{r\cos\theta}{\sqrt{1+r^2\cos^2\theta}},\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big).$$

The assumptions of Theorem 2.1 are verified for equation (3.1). Then, by direct computations, the expression of $f^\circ(r)$ is given by

$$
\begin{aligned}
f^\circ(r) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r)\,d\theta \\
&= \frac{1}{2r\sqrt{1+r^2}}\Bigg(\Big((a_{3,2}+a_{1,0}+a_{3,0}+a_{5,0}+a_{1,2}-3\,a_{1,4})r^4 \\
&\quad + \big(5\,a_{3,2}-3\,b_{2,3}+b_{2,1}+b_{4,1}+a_{1,0}-a_{3,0}-3\,a_{5,0}+b_{0,1}-3\,b_{0,3}+5\,b_{0,5}+3\,a_{1,2}-7\,a_{1,4}\big)r^2 \\
&\qquad + 4\,a_{3,2}+2\,b_{2,1}-4\,b_{2,3}+4\,b_{4,1}-2\,a_{3,0}-4\,a_{5,0}-2\,b_{0,3}+4\,b_{0,5}+2\,a_{1,2}-4\,a_{1,4}\Big)\sqrt{r^2+1} \\
&\quad + (b_{0,5}+a_{1,4})r^6 + \big(-3\,a_{3,2}+b_{2,3}+2\,b_{0,3}-2\,b_{0,5}-2\,a_{1,2}+6\,a_{1,4}\big)r^4 \\
&\quad + \big(-7\,a_{3,2}-2\,b_{2,1}+5\,b_{2,3}-3\,b_{4,1}+2\,a_{3,0}+5\,a_{5,0}+4\,b_{0,3}-7\,b_{0,5}-4\,a_{1,2}+9\,a_{1,4}\big)r^2 \\
&\qquad -4\,a_{3,2}-2\,b_{2,1}+4\,b_{2,3}-4\,b_{4,1}+2\,a_{3,0}+4\,a_{5,0}+2\,b_{0,3}-4\,b_{0,5}-2\,a_{1,2}+4\,a_{1,4}\Bigg). \qquad (3.2)
\end{aligned}
$$

Taking $s=\sqrt{1+r^2}$, we have

$$
\begin{aligned}
f^\circ(s) &= \frac{1}{2s\sqrt{s^2-1}}\Big((a_{1,4}+b_{0,5})s^6 + \big(a_{1,0}+a_{1,2}-3\,a_{1,4}+a_{3,0}+a_{3,2}+a_{5,0}\big)s^5 \\
&\quad + \big(-2\,a_{1,2}+3\,a_{1,4}-3\,a_{3,2}+2\,b_{0,3}-5\,b_{0,5}+b_{2,3}\big)s^4 \\
&\quad + \big(-3\,a_{3,0}+3\,a_{3,2}-a_{1,0}+a_{1,2}-a_{1,4}-5\,a_{5,0}+b_{0,1}-3\,b_{0,3}+5\,b_{0,5}+b_{2,1}-3\,b_{2,3}+b_{4,1}\big)s^3 \\
&\quad + \big(2\,a_{3,0}-a_{3,2}+5\,a_{5,0}-2\,b_{2,1}+3\,b_{2,3}-3\,b_{4,1}\big)s^2 \\
&\quad + \big(-b_{0,1}+b_{0,3}-b_{0,5}+b_{2,1}-b_{2,3}+3\,b_{4,1}\big)s - a_{5,0}-b_{4,1}\Big) \\
&= \frac{1}{2s\sqrt{s^2-1}}\,(A_6s^6+A_5s^5+A_4s^4+A_3s^3+A_2s^2+A_1s+A_0), \qquad (3.3)
\end{aligned}
$$

where

$$
\begin{aligned}
A_0 &= -a_{5,0}-b_{4,1}, \\
A_1 &= -b_{0,1}+b_{0,3}-b_{0,5}+b_{2,1}-b_{2,3}+3\,b_{4,1}, \\
A_2 &= 2\,a_{3,0}-a_{3,2}+5\,a_{5,0}-2\,b_{2,1}+3\,b_{2,3}-3\,b_{4,1}, \\
A_3 &= -3\,a_{3,0}+3\,a_{3,2}-a_{1,0}+a_{1,2}-a_{1,4}-5\,a_{5,0}+b_{0,1}-3\,b_{0,3}+5\,b_{0,5}+b_{2,1}-3\,b_{2,3}+b_{4,1}, \\
A_4 &= -2\,a_{1,2}+3\,a_{1,4}-3\,a_{3,2}+2\,b_{0,3}-5\,b_{0,5}+b_{2,3}, \\
A_5 &= a_{1,0}+a_{1,2}-3\,a_{1,4}+a_{3,0}+a_{3,2}+a_{5,0}, \\
A_6 &= a_{1,4}+b_{0,5}.
\end{aligned}
$$

Using Theorem 2.3, the function $f^0(s)$ can have at most 6 real zeros. Since $A_0+A_1+A_2+A_3+A_4+A_5+A_6=0$, $f^0(s)$ can have at most 5 zeros $s\in(1,\infty)$. This allows us to conclude that system (1.2) has at most 5 limit cycles bifurcating from the isochronous center $(1.2)_{\varepsilon=0}$ using the averaging method of first order and there are perturbations with only $0,1,2,\ldots,5$ limit cycles.

3.2 Proof of statement (b) of Theorem 1.1

To apply the averaging method of second order, we use the same changes of variable as in the proof of statement (a) of Theorem 1.1. Then system (1.2) can be written as follows:

$$\frac{dr}{d\theta} = \varepsilon F_1(\theta,r)+\varepsilon^2F_2(\theta,r)$$

with

$$F_1(\theta,r)=\sqrt{1+r^2\cos^2\theta}\,\Big((1+r^2)\cos\theta\,\overline{p}(\theta,r)+\sin\theta\,\overline{q}(\theta,r)\Big)$$

and

$$F_2(\theta,r)=\frac{1+r^2\cos^2\theta}{r}\Big(\sin(\theta)\overline{p}(\theta,r)-\cos(\theta)\overline{q}(\theta,r)\Big)\Big((r^2+1)\cos(\theta)\overline{p}(\theta,r)+\sin(\theta)\overline{q}(\theta,r)\Big).$$

To compute the second order averaged function, the first order averaged function $f^{\circ}(r)$ must be vanished.

Lemma 3.2. *For*

$$a_{1,4}=-b_{0,5},\ \ a_{5,0}=-b_{4,1},\ \ b_{0,1}=-a_{1,0},$$
$$b_{2,1}=-3\,a_{3,0}-4\,a_{1,0}-a_{1,2}-3\,b_{0,3},\ \ a_{3,2}=-a_{1,0}-a_{3,0}+b_{4,1}-a_{1,2}-3\,b_{0,5},$$
$$b_{2,3}=-2\,b_{0,3}-b_{0,5}-3\,a_{1,0}-3\,a_{3,0}+3\,b_{4,1}-a_{1,2},$$

the function $f^0(r)$ defined in (3.2) *is identically zero.*

Proof. From (3.3), it is easy to verify that $f^0(r)=0$ if and only if $a_{1,4}=-b_{0,5}$, $a_{5,0}=-b_{4,1}$, $b_{0,1}=-a_{1,0}$, $b_{2,1}=-3\,a_{3,0}-4\,a_{1,0}-a_{1,2}-3\,b_{0,3}$, $a_{3,2}=-a_{1,0}-a_{3,0}+b_{4,1}-a_{1,2}-3\,b_{0,5}$, $b_{2,3}=-2\,b_{0,3}-b_{0,5}-3\,a_{1,0}-3\,a_{3,0}+3\,b_{4,1}-a_{1,2}$. □

Now, the following lemma presents some useful integrals to determine the explicit expression of $y^1(\theta,r)$.

Lemma 3.3. *For $r>0$,*

$$I_1=\int\frac{\cos^3\theta}{\sqrt{1+r^2\cos^2\theta}}\,d\theta=\frac{r^2-1}{2r^3}\arctan\Big(\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big)+\frac{1}{2r^2}\,\sin\theta\sqrt{1+r^2\cos^2\theta}\,,$$
$$I_2=\int\frac{\cos^5\theta}{(1+r^2\cos^2\theta)^{3/2}}\,d\theta=\frac{r^2-3}{2r^5}\,\arctan\Big(\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big)+\frac{\sin\theta\sqrt{1+r^2\cos^2\theta}}{2r^4}+\frac{1}{r^4(r^2+1)}\,\frac{\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\,,$$
$$I_3=\int\frac{\sin\theta\cos^2\theta}{(1+r^2\cos^2\theta)^{3/2}}\,d\theta=-\frac{1}{r^3}\,\ln\Big(r\cos\theta+\sqrt{1+r^2\cos^2\theta}\Big)+\frac{\cos\theta}{r^2\sqrt{1+r^2\cos^2\theta}}\,.$$

Proof. To compute the integral I_1, we need the following integrals (see [12]):

$$K_1=\int\frac{\cos\theta}{\sqrt{1+r^2\cos^2\theta}}=\frac{1}{r}\,\arctan\Big(\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big),$$

and

$$K_2=\int\cos\theta\sqrt{1+r^2\cos^2\theta}=\frac{1+r^2}{r}\,\arctan\Big(\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big)+\frac{1}{2}\,\sin\theta\sqrt{1+r^2\cos^2\theta}\,.$$

We have

$$I_1=\int\frac{\cos^3\theta}{\sqrt{1+r^2\cos^2\theta}}\,d\theta=\frac{1}{r^2}(K_2-K_1)=\frac{r^2-1}{2r^3}\,\arctan\Big(\frac{r\sin\theta}{\sqrt{1+r^2\cos^2\theta}}\Big)+\frac{1}{2r^2}\,\sin\theta\sqrt{1+r^2\cos^2\theta}\,.$$

Taking the derivative of the both sides of I_1 with respect to r, we get

$$\frac{dI_1}{dr}=-rI_2\Longrightarrow I_2=-\frac{1}{r}\,I_1.$$

The other integrals can be computed in the same way. □

Because of $f^0(r) \equiv 0$, we have

$$y^1(\theta, r) = \int_0^\theta f(\phi, r)\, d\phi + z(r).$$

The explicit expression of $y^1(\theta, r)$ is

$$\begin{aligned} y^1(\theta, r) &= \widetilde{A}_1 \ln\Big(r\cos\theta + \sqrt{1 + r^2\cos^2\theta}\Big) + \widetilde{A}_2 \arctan\Big(\frac{r\sin\theta}{\sqrt{1 + r^2\cos^2\theta}}\Big) + \widetilde{A}_3 \sin\theta\sqrt{1 + r^2\cos^2\theta} \\ &+ \widetilde{A}_4 \cos\theta\sqrt{1 + r^2\cos^2\theta} + \widetilde{A}_5\Big(\ln(1 + r^2\cos^2\theta) + 2\ln(2) - 2\ln\Big(1 + \sqrt{r^2+1}\Big)\Big) \\ &+ \widetilde{A}_6 \frac{\sin\theta}{\sqrt{1 + r^2\cos^2\theta}} + \widetilde{A}_7 \frac{\cos\theta}{\sqrt{1 + r^2\cos^2\theta}} + \widetilde{A}_8\Big(\frac{1}{1 + r^2\cos^2\theta} - \frac{1}{\sqrt{r^2+1}}\Big) \\ &+ \widetilde{A}_9 \frac{\sin\theta\cos\theta}{1 + r^2\cos^2\theta} + \widetilde{A}_{10} \frac{\sin\theta\cos^3\theta}{1 + r^2\cos^2\theta} + \widetilde{A}_{11}\Big(\cos^2\theta - \frac{1}{2}\Big) + \widetilde{A}_{12}\sin\theta\cos\theta. \end{aligned}$$

The coefficients $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_{14}$ are given in Appendix A. (The computations of $y^1(\theta, r)$ can be revised by using Maple or Mathematica.)

If the conditions of Lemma 3.2 are satisfied, then we can compute $F_{20}(r) = f^{10}(r) + g^0(r)$, the second order averaged function, where $f^{10}(r) = \int_0^{2\pi} \frac{dF_1(\theta,r)}{dr}\, y^1(\theta, r)\, d\theta$. The following lemma lists some integrals which appear in $f^{10}(r)$.

Lemma 3.4. *The following integrals hold:*

$$\begin{aligned} L_1 &= \int_0^{2\pi} \frac{\cos^3\theta}{(1 + r^2\cos^2\theta)^{3/2}} \arctan\Big(\frac{\sin\theta}{\sqrt{1 + r^2\cos^2\theta}}\Big) \\ &= -\frac{\pi}{r^3}\Big(\ln(1 + r^2) - 2\Big(r^2 + 1 - \sqrt{r^2+1}\Big)\Big), \\ L_2 &= \int_0^{2\pi} \frac{\cos^3\theta}{(1 + r^2\cos^2\theta)^{3/2}} \ln\Big(r\cos\theta + \sqrt{1 + r^2\cos^2\theta}\Big)\, d\theta \\ &= \frac{\pi}{r^3}\Big(\ln(1 + r^2) + \frac{2(1 - \sqrt{(1 + r^2)})}{1 + r^2}\Big), \\ L_3 &= \int_0^{2\pi} \frac{\cos^5\theta}{(1 + r^2\cos^2\theta)^{3/2}} \ln(r\cos\theta + \sqrt{1 + r^2\cos^2\theta})\, d\theta \\ &= \frac{\pi}{3r^5}\Big(3\ln(1 + r^2) + \frac{-11r^4 - 19r^2 - 8}{(r^2+1)^{5/2}} + \frac{12r^2 + 8}{(r^2+1)^2}\Big), \\ L_4 &= \int_0^{2\pi} \frac{\cos^4\theta}{1 + r^2\cos^2\theta} \ln(1 + r^2\cos^2\theta)\, d\theta \\ &= \frac{\pi}{r^4}\Big(\Big(r^2 - 2 - \frac{2}{\sqrt{r^2+1}}\Big)\ln\Big(1 + \sqrt{r^2+1}\Big) + \frac{2\ln(r^2+1)}{\sqrt{r^2+1}} \\ &\quad + \ln(2)\Big(\frac{-2r^4 + 2r^2 + 8}{\sqrt{r^2+1}\,(1 + \sqrt{r^2+1})} + \frac{-2r^2 + 8}{1 + \sqrt{r^2+1}}\Big) + \frac{r^2(\sqrt{r^2+1} - 1)}{1 + \sqrt{r^2+1}}\Big). \end{aligned}$$

Proof. We just compute the integral L_2 and the other oness can be computed in the same way.

Firstly, the integral

$$J_1 = \int_0^{2\pi} \frac{\cos\theta}{\sqrt{1 + r^2\cos^2\theta}} \ln\Big(r\cos\theta + \sqrt{1 + r^2\cos^2\theta}\Big)\, d\theta = \frac{\pi}{r}\ln(1 + r^2)$$

holds (see [12]). Taking the derivative in the both sides of J_1 with respect to r, we get

$$\frac{dJ_1}{dr} = -rL_2 + \frac{\cos^2\theta}{1+r^2\cos^2\theta} = -\frac{\pi\ln(r^2+1)}{r^2} + \frac{2\pi}{r^2+1}\,,$$

whence

$$L_2 = \frac{\pi}{r^3}\left(\ln(1+r^2) + \frac{2(1-\sqrt{(1+r^2)})}{1+r^2}\right). \qquad \square$$

Now, using the ideas in the proof of Lemma 3.4, we compute $f^{10}(r)$. As a consequence, the second order averaged function $F_{20}(r)$ is given by

$$\begin{aligned}
F_{20}(r) &= \frac{1}{2\sqrt{s^2-1}}\,(B_1 + B_2 s^2 + B_3 s^4)\ln(s) \\
&\quad + \frac{1}{8s^5(s+1)\sqrt{s^2-1}}\,\big(B_4 s^2 + B_5 s^3 + B_6 s^4 + B_7 s^5 + B_8 s^6 + B_9 s^7 \\
&\qquad + B_{10}s^8 + B_{11}s^9 + B_{12}s^{10} + B_{13}s^{11} + B_{14}s^{12} + B_{15}s^{13}\big) \\
&= \frac{1}{8s^5\sqrt{s^2-1}\,(s+1)}\Big(4\left((s^5+s^6)B_1 + (s^7+s^8)B_2 + (s^9+s^{10})B_3\right)\ln(s) \\
&\qquad + B_4 s^2 + B_5 s^3 + B_6 s^4 + B_7 s^5 + B_8 s^6 + B_9 s^7 + B_{10}s^8 + B_{11}s^9 \\
&\qquad + B_{12}s^{10} + B_{13}s^{11} + B_{14}s^{12} + B_{15}s^{13}\Big), \\
&= \frac{1}{8s^5\sqrt{s^2-1}(s+1)}F(s),
\end{aligned}$$

where

$$\begin{aligned}
F(s) = 4\Big((s^5+s^6)B_1 + (s^7+s^8)B_2 + (s^9+s^{10})B_3\Big)\ln(s) + B_4 s^2 + B_5 s^3 + B_6 s^4 \\
+ B_7 s^5 + B_8 s^6 + B_9 s^7 + B_{10}s^8 + B_{11}s^9 + B_{12}s^{10} + B_{13}s^{11} + B_{14}s^{12} + B_{15}s^{13},
\end{aligned}$$

with the coefficients $B_1, B_2, \ldots, B_{15}$ given in Appendix B.

Taking the 11^{th} order derivative of $F(s)$ with respect to s, we obtain

$$\begin{aligned}
F^{(11)}(s) = \frac{5760}{s^6}\Big(&540540\,B_{15}s^8 + 83160\,B_{14}s^7 + 6930\,B_3 s^6 \\
&+ 2520\,B_3 s^5 - 252\,B_3 s^4 + 56\,B_2 s^3 - 21\,B_2 s^2 + 12\,B_1 s - 10\,B_1\Big).
\end{aligned}$$

Using Theorem 2.3, we conclude that $F^{(11)}(s)$ can have at most 8 zeros $s \in (1,+\infty)$. By applying Rolle's rule, the function $F(s)$ can have at most 19 zeros, taking into account their multiplicities in $(1,+\infty)$. Hence the second average function $F_{20}(r)$ has at most 19 zeros $r \in (0,+\infty)$. This allows us to conclude that system (1.2) has at most 19 limit cycles that can bifurcate from the center $(1.2)_{\varepsilon=0}$.

Appendix A

In this appendix, we present the coefficients $A_1, A_2, \ldots, A_{13}$ which appear in $y^1(\theta,r)$.

$$\begin{aligned}
\widetilde{A}_1 &= -\frac{(r^2{+}1)(2(r^2{+}1)a_{1,3}{+}2b_{2,2}{-}4b_{0,4}{+}2b_{0,2}-3a_{3,1}{+}a_{1,3}{-}a_{1,1}){+}b_{0,0}{-}b_{2,0}{-}3b_{4,0}{+}b_{2,2}{+}b_{0,4}{-}b_{0,2}}{2r}\,, \\
\widetilde{A}_2 &= \frac{(r^2{+}1)(b_{3,1}{-}3b_{1,3}{+}b_{1,1}{-}4a_{4,0}{+}(r^2{+}1)(a_{4,0}{+}a_{2,2}{+}a_{2,0}{+}a_{0,2}{+}a_{0,0}{-}3a_{0,4}){+}2a_{2,2}{-}2a_{2,0}){+}2b_{3,1}}{2r}\,, \\
\widetilde{A}_3 &= \frac{(r^2+1)(a_{2,0}+a_{0,4}-a_{0,2}+a_{0,0}-a_{2,2}+a_{4,0}) - 1/2\,b_{1,1} + 1/2\,b_{1,3} - 1/2\,b_{3,1}}{2}\,, \\
\widetilde{A}_4 &= \frac{(r^2+1)(-a_{3,1}+a_{1,3}-a_{1,1}) - 1/2\,b_{0,0} + 1/2\,b_{0,2} - 1/2\,b_{0,4} - 1/2\,b_{2,0} + 1/2\,b_{2,2} - 1/2\,b_{4,0}}{2}\,,
\end{aligned}$$

$$\widetilde{A}_5 = -\frac{(r^2+1)((r^2+1)(a_{2,3}-2\,a_{0,5}+a_{0,3})+b_{3,2}-2\,b_{1,4}+b_{1,2}-2\,a_{4,1}+a_{2,3}-a_{2,1})-b_{3,0}-2\,b_{5,0}+b_{3,2}}{2r},$$
$$\widetilde{A}_6 = \frac{(r^2+1)(2\,(r^2+1)a_{0,4}+2\,b_{1,3}-2\,a_{2,2})-b_{3,1}+a_{4,0}}{2},$$
$$\widetilde{A}_7 = (r^2+1)\big((r^2+1)(a_{1,3}-b_{0,4})+b_{2,2}-a_{3,1}\big)-b_{4,0},$$
$$\widetilde{A}_8 = \frac{(r^2+1)((r^2+1)((r^2+1)a_{0,5}+b_{1,4}-a_{2,3})-b_{3,2}+a_{4,1})+b_{5,0}}{2r},$$
$$\widetilde{A}_9 = -\frac{r((2\,r^2+4)b_{0,3}+(r^2+4)a_{3,0}+(r^2+4)a_{1,0}+r^2(2\,r^2b_{0,5}-a_{1,2}))}{2},$$
$$\widetilde{A}_{10} = \frac{r^3((r^2-2)(a_{3,0}+a_{1,0})+r^2(a_{1,2}+2\,b_{0,5}-2\,b_{4,1})-2\,b_{0,3})}{2},$$
$$\widetilde{A}_{11} = -\frac{r((r^2+1)(a_{4,1}+a_{2,1}+a_{0,5}-a_{0,3}+a_{0,1}-a_{2,3})+b_{1,0}-b_{1,2}+b_{1,4}+b_{3,0}-b_{3,2}+b_{5,0})}{2},$$
$$\widetilde{A}_{12} = \frac{r((r^2+1)(a_{3,0}+a_{1,0}-a_{1,2})+5\,a_{1,0}+3\,a_{3,0}+a_{1,2}+4\,b_{0,3})}{2}.$$

Appendix B

This appendix lists the coefficients $B_1, B_2, \dots, B_{15}$ which appear in $F_{20}(r)$.

$$\begin{aligned}
B_1 = &\big(-b_{0,0}-6\,a_{3,1}+3\,b_{2,2}+5\,b_{0,2}-9\,b_{0,4}+b_{2,0}+3\,b_{4,0}-2\,a_{1,1}+2\,a_{1,3}\big)b_{3,1}\\
&-\big(2\,a_{2,2}+b_{1,1}-3\,b_{1,3}-2\,a_{2,0}-4\,a_{4,0}\big)\big(b_{0,0}+b_{2,2}-b_{0,2}+b_{0,4}-b_{2,0}-3\,b_{4,0}\big),
\end{aligned}$$

$$\begin{aligned}
B_2 = &\big(6\,b_{4,0}-2\,b_{2,2}-2\,b_{0,0}+2\,b_{0,2}-2\,b_{0,4}+2\,b_{2,0}\big)a_{2,2}\\
&+\big(-2\,a_{0,0}-2\,a_{0,2}+6\,a_{0,4}-2\,a_{2,0}-2\,a_{4,0}\big)b_{2,2}\\
&+\big(6\,b_{4,0}-2\,b_{0,0}+2\,b_{0,2}-2\,b_{0,4}+2\,b_{2,0}\big)a_{0,2}\\
&+\big(-18\,b_{4,0}+6\,b_{0,0}-6\,b_{0,2}+6\,b_{0,4}-6\,b_{2,0}\big)a_{0,4}\\
&+\big(6\,b_{4,0}-2\,b_{0,0}+2\,b_{0,2}-2\,b_{0,4}+2\,b_{2,0}\big)a_{2,0}\\
&+\big(6\,b_{4,0}-2\,b_{0,0}+2\,b_{0,2}-2\,b_{0,4}+2\,b_{2,0}\big)a_{4,0}\\
&+6\,a_{0,0}b_{4,0}-2\,a_{0,0}b_{0,0}+2\,a_{0,0}b_{0,2}-2\,a_{0,0}b_{0,4}+2\,a_{0,0}b_{2,0}+8\,a_{1,3}b_{3,1},
\end{aligned}$$

$$\begin{aligned}
B_3 = &\big(-a_{0,0}+3\,a_{2,2}+2\,b_{1,1}-6\,b_{1,3}+2\,b_{3,1}-a_{0,2}+3\,a_{0,4}-5\,a_{2,0}-9\,a_{4,0}\big)a_{1,3}\\
&+\big(a_{0,0}+a_{2,2}+a_{0,2}-3\,a_{0,4}+a_{2,0}+a_{4,0}\big)\big(3\,a_{3,1}-2\,b_{2,2}-2\,b_{0,2}+4\,b_{0,4}+a_{1,1}\big),
\end{aligned}$$

$$B_4 = \big(4\,a_{1,0}+4\,a_{3,0}+3\,b_{0,3}+a_{1,2}-4\,b_{4,1}\big)b_{5,0}-4\,b_{3,1}b_{4,0}+2\,b_{3,2}b_{4,1},$$

$$\begin{aligned}
B_5 = &\big(-4\,b_{0,0}-4\,b_{2,2}+4\,b_{0,2}-4\,b_{0,4}+4\,b_{2,0}+8\,b_{4,0}\big)b_{3,1}\\
&+\big(4\,a_{1,0}+4\,a_{3,0}+3\,b_{0,3}+a_{1,2}-4\,b_{4,1}\big)b_{5,0}+2\,b_{3,2}b_{4,1},
\end{aligned}$$

$$\begin{aligned}
B_6 = &\big(16\,b_{3,0}+56\,b_{5,0}+8\,a_{2,1}-6\,a_{2,3}+12\,a_{4,1}-8\,b_{1,2}+12\,b_{1,4}-34\,b_{3,2}\big)b_{4,1}\\
&+\big(-4\,b_{0,0}-4\,b_{2,0}+8\,b_{4,0}-8\,a_{3,1}+16\,b_{2,2}+4\,b_{0,2}-4\,b_{0,4}\big)b_{3,1}\\
&+\big(3\,a_{1,2}+6\,a_{1,0}+14\,a_{3,0}+3\,b_{0,3}+6\,b_{0,5}\big)b_{3,2}\\
&+\big(8\,a_{2,2}+8\,b_{1,1}-12\,b_{1,3}-16\,a_{2,0}-32\,a_{4,0}\big)b_{4,0}\\
&+\big(-10\,a_{1,2}-56\,a_{1,0}-56\,a_{3,0}-30\,b_{0,3}-12\,b_{0,5}\big)b_{5,0}\\
&+\big(a_{1,2}+4\,a_{1,0}+4\,a_{3,0}+3\,b_{0,3}\big)a_{4,1}\\
&+\big(-4\,a_{1,2}-16\,a_{1,0}-16\,a_{3,0}-12\,b_{0,3}\big)b_{3,0}-4(a_{3,1}-2\,b_{2,2})a_{4,0},
\end{aligned}$$

$$\begin{aligned}
B_7 = &\big(-12\,a_{0,5}-44\,b_{3,0}+28\,b_{1,2}-12\,b_{1,0}-84\,b_{5,0}-28\,a_{2,1}+30\,a_{2,3}-48\,b_{1,4}\\
&\qquad-12\,a_{0,1}+66\,b_{3,2}+12\,a_{0,3}-48\,a_{4,1}\big)b_{4,1}\\
&+\big(44\,b_{3,0}-8\,a_{4,1}+4\,a_{2,3}-30\,b_{3,2}-4\,b_{1,4}-4\,a_{2,1}+12\,b_{1,0}-4\,a_{0,3}
\end{aligned}$$

$$
\begin{aligned}
&\quad - 4\, b_{1,2} + 84\, b_{5,0} + 4\, a_{0,5} + 4\, a_{0,1}\big) a_{1,0}\\
&+ \big(- 4\, a_{2,1} - 4\, a_{0,3} + 30\, b_{5,0} - 9\, b_{3,2} + 4\, a_{0,5} + 4\, a_{2,3} - 4\, b_{1,4} + 24\, b_{3,0}\\
&\quad - 4\, b_{1,2} - 9\, a_{4,1} + 4\, a_{0,1} + 12\, b_{1,0}\big) b_{0,3}\\
&+ \big(- 4\, a_{0,5} + 4\, a_{2,1} - 4\, a_{2,3} + 12\, a_{4,1} - 4\, b_{1,2} + 12\, b_{1,4} - 30\, b_{3,2}\\
&\quad + 48\, b_{5,0} - 4\, a_{0,1} + 12\, b_{3,0} - 4\, b_{1,0} + 4\, a_{0,3}\big) b_{0,5}\\
&+ \big(2\, a_{0,0} + 2\, a_{4,0} - 14\, a_{2,2} - 6\, b_{1,1} + 22\, b_{1,3} + 2\, b_{3,1} - 2\, a_{0,2} + 2\, a_{0,4} + 6\, a_{2,0}\big) b_{2,2}\\
&+ \big(6\, b_{0,2} - 18\, b_{0,4} - 2\, b_{2,0} - 6\, b_{4,0} + 6\, b_{0,0} - 12\, a_{1,1} + 12\, a_{1,3} - 20\, a_{3,1}\big) b_{3,1}\\
&+ \big(- 14\, b_{0,2} + 30\, b_{0,4} - 10\, b_{2,0} - 2\, b_{4,0} - 2\, b_{0,0} + 8\, a_{1,1} - 8\, a_{1,3} + 20\, a_{3,1}\big) a_{4,0}\\
&+ \big(- 6\, a_{0,0} + 10\, a_{2,2} + 2\, b_{1,1} - 10\, b_{1,3} + 6\, a_{0,2} - 6\, a_{0,4} - 2\, a_{2,0}\big) b_{0,2}\\
&+ \big(6\, a_{0,0} - 10\, a_{2,2} - 6\, b_{1,1} + 14\, b_{1,3} - 6\, a_{0,2} + 6\, a_{0,4} + 2\, a_{2,0}\big) b_{0,4}\\
&+ \big(- 2\, a_{0,0} + 14\, a_{2,2} + 2\, b_{1,1} - 18\, b_{1,3} + 2\, a_{0,2} - 2\, a_{0,4} - 6\, a_{2,0}\big) b_{2,0}\\
&+ \big(6\, a_{0,0} + 14\, a_{2,2} + 2\, b_{1,1} - 30\, b_{1,3} - 6\, a_{0,2} + 6\, a_{0,4} + 2\, a_{2,0}\big) b_{4,0}\\
&+ \big(6\, a_{0,0} - 10\, a_{2,2} + 2\, b_{1,1} + 6\, b_{1,3} - 6\, a_{0,2} + 6\, a_{0,4} + 2\, a_{2,0}\big) b_{0,0}\\
&+ \big(4\, b_{1,0} + 8\, b_{3,0} + 10\, b_{5,0} + a_{4,1} - 4\, b_{1,2} + 4\, b_{1,4} - 9\, b_{3,2}\big) a_{1,2}\\
&+ \big(12\, b_{1,0} + 44\, b_{3,0} + 84\, b_{5,0} + 4\, a_{4,1} - 12\, b_{1,2} + 12\, b_{1,4} - 46\, b_{3,2}\big) a_{3,0},
\end{aligned}
$$

$$
\begin{aligned}
B_8 = \big(&- 4\, a_{0,5} - 20\, a_{2,1} + 26\, a_{2,3} - 48\, a_{4,1} + 20\, b_{1,2} - 48\, b_{1,4} + 54\, b_{3,2} - 60\, b_{5,0}\\
&\quad - 4\, a_{0,1} - 36\, b_{3,0} - 4\, b_{1,0} + 4\, a_{0,3}\big) b_{0,5}\\
&+ \big(48\, a_{0,5} + 20\, b_{3,0} - 28\, b_{1,2} + 20\, b_{1,0} + 28\, a_{2,1} - 54\, a_{2,3} + 60\, b_{1,4}\\
&\quad + 20\, a_{0,1} - 30\, b_{3,2} - 36\, a_{0,3} + 60\, a_{4,1}\big) b_{4,1}\\
&+ \big(- 20\, b_{3,0} - 20\, a_{4,1} - 14\, a_{2,3} + 40\, b_{3,2} - 4\, b_{1,4} + 4\, a_{2,1} - 20\, b_{1,0}\\
&\quad - 4\, a_{0,3} + 20\, b_{1,2} + 4\, a_{0,5} + 4\, a_{0,1}\big) a_{1,0}\\
&+ \big(2\, a_{0,0} + 2\, a_{4,0} + 2\, a_{2,2} + 2\, b_{1,1} - 14\, b_{1,3} - 14\, b_{3,1} - 2\, a_{0,2} + 2\, a_{0,4} + 6\, a_{2,0}\big) b_{2,2}\\
&+ \big(- 2\, b_{0,2} - 22\, b_{0,4} + 6\, b_{2,0} - 2\, b_{4,0} + 6\, b_{0,0} + 4\, a_{1,1} + 12\, a_{1,3} + 12\, a_{3,1}\big) b_{3,1}\\
&+ \big(8\, a_{2,1} - 4\, a_{0,3} + 15\, b_{5,0} + 6\, b_{3,2} + 4\, a_{0,5} - 17\, a_{2,3} + 11\, b_{1,4} + 8\, b_{1,2}\\
&\quad + 6\, a_{4,1} + 4\, a_{0,1} - 12\, b_{1,0}\big) b_{0,3}\\
&+ \big(2\, b_{0,2} - 18\, b_{0,4} - 10\, b_{2,0} + 30\, b_{4,0} - 2\, b_{0,0} - 16\, a_{1,1} + 28\, a_{1,3} - 28\, a_{3,1}\big) a_{4,0}\\
&+ \big(- 6\, a_{0,0} + 10\, a_{2,2} + 2\, b_{1,1} - 10\, b_{1,3} + 6\, a_{0,2} - 6\, a_{0,4} - 2\, a_{2,0}\big) b_{0,2}\\
&+ \big(6\, a_{0,0} - 10\, a_{2,2} - 6\, b_{1,1} + 14\, b_{1,3} - 6\, a_{0,2} + 6\, a_{0,4} + 2\, a_{2,0}\big) b_{0,4}\\
&+ \big(- 2\, a_{0,0} - 2\, a_{2,2} + 2\, b_{1,1} + 6\, b_{1,3} + 2\, a_{0,2} - 2\, a_{0,4} - 6\, a_{2,0}\big) b_{2,0}\\
&+ \big(6\, a_{0,0} + 14\, a_{2,2} - 6\, b_{1,1} + 18\, b_{1,3} + 26\, a_{0,2} - 42\, a_{0,4} + 18\, a_{2,0}\big) b_{4,0}\\
&+ \big(6\, a_{0,0} - 10\, a_{2,2} + 2\, b_{1,1} + 6\, b_{1,3} - 6\, a_{0,2} + 6\, a_{0,4} + 2\, a_{2,0}\big) b_{0,0}\\
&+ \big(- 4\, b_{1,0} + 5\, b_{5,0} - 4\, a_{2,1} + 3\, a_{2,3} - 10\, a_{4,1} + 8\, b_{1,2} - 11\, b_{1,4} + 6\, b_{3,2}\big) a_{1,2}\\
&+ \big(- 20\, b_{1,0} - 20\, b_{3,0} - 8\, a_{2,1} + 6\, a_{2,3} - 40\, a_{4,1} + 28\, b_{1,2} - 36\, b_{1,4} + 40\, b_{3,2}\big) a_{3,0}\\
&+ 4\, a_{3,1} (a_{2,2} - 2\, a_{2,0}),
\end{aligned}
$$

$$
\begin{aligned}
B_9 = \big(&6\, a_{2,3} + 4\, b_{3,2} - 5\, a_{0,1} - 8\, b_{1,0} - 40\, b_{3,0} - 84\, b_{5,0} + 16\, a_{2,1} + 48\, a_{4,1}\\
&\quad - 16\, b_{1,2} + 24\, b_{1,4} + 24\, a_{0,3} - 40\, a_{0,5}\big) a_{1,0}\\
&+ \big(- 11\, b_{5,0} - 9\, a_{2,3} + 10\, a_{4,1} + 5\, b_{1,4} - 4\, b_{1,0} + 4\, a_{0,1} + 8\, a_{2,1}\\
&\quad - 4\, a_{0,3} + 4\, a_{0,5} + 6\, b_{3,2} - 8\, b_{3,0}\big) a_{1,2}\\
&+ \big(60\, b_{1,4} - 58\, a_{2,3} - 32\, b_{1,2} + 24\, a_{0,1} + 60\, a_{4,1} - 30\, b_{3,2} + 56\, a_{0,5}\\
&\quad + 32\, a_{2,1} - 40\, a_{0,3} + 24\, b_{3,0} + 24\, b_{1,0}\big) b_{0,5}
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(8\,a_{0,1} + 8\,a_{2,1} + 84\,b_{5,0} - 60\,a_{0,5} + 30\,a_{2,3} + 24\,a_{0,3} - 42\,b_{3,2} \\
&\qquad + 40\,b_{3,0} - 8\,b_{1,2} + 8\,b_{1,0}\big)b_{4,1} \\
&+ \big(- 33\,b_{5,0} + 23\,a_{2,3} + 6\,b_{3,2} - 5\,b_{1,4} - 44\,a_{0,5} - 12\,b_{1,0} + 28\,a_{0,3} \\
&\qquad + 6\,a_{4,1} - 12\,a_{0,1} - 24\,b_{3,0}\big)b_{0,3} \\
&+ \big(- 16\,b_{0,2}+20\,b_{0,4}-12\,b_{2,0}-12\,b_{4,0}-2\,a_{1,1}+2\,a_{1,3}-14\,a_{3,1}+12\,b_{0,0}+16\,b_{2,2}\big)a_{2,2} \\
&+ \big(- 16\,a_{2,0} - 12\,a_{4,0} - 12\,a_{0,0} + 6\,b_{1,1} - 18\,b_{1,3} + 2\,b_{3,1} + 4\,a_{0,2} + 20\,a_{0,4}\big)b_{2,2} \\
&+ \big(- 8\,b_{1,0}-40\,b_{3,0}-84\,b_{5,0}+16\,a_{2,1}-18\,a_{2,3}+40\,a_{4,1}-8\,b_{1,2}+24\,b_{1,4}+20\,b_{3,2}\big)a_{3,0} \\
&+ \big(12\,b_{0,2} - 48\,b_{0,4} + 16\,b_{2,0} + 8\,b_{4,0} - 2\,a_{1,1} - 18\,a_{1,3} - 6\,a_{3,1} + 8\,b_{0,0}\big)a_{4,0} \\
&+ \big(- 4\,a_{2,0} - 8\,a_{0,0} + 22\,b_{1,1} - 46\,b_{1,3} + 50\,b_{3,1} + 16\,a_{0,2} - 8\,a_{0,4}\big)b_{0,4} \\
&+ \big(- 2\,a_{2,0} + 2\,a_{0,0} - 4\,b_{1,1} + 4\,b_{1,3} - 2\,a_{0,2} + 2\,a_{0,4}\big)a_{1,3} \\
&+ \big(- 10\,b_{0,2} - 2\,b_{2,0} - 2\,b_{4,0} + 12\,a_{1,1} + 20\,a_{3,1} - 2\,b_{0,0}\big)b_{3,1} \\
&+ \big(- 12\,b_{0,2} + 8\,b_{4,0} + 2\,a_{1,1} - 6\,a_{3,1} + 8\,b_{0,0}\big)a_{0,2} \\
&+ \big(4\,b_{0,2} - 24\,b_{2,0} - 48\,b_{4,0} - 2\,a_{1,1} + 6\,a_{3,1}\big)a_{0,4} \\
&+ \big(12\,b_{2,0} + 4\,b_{4,0} + 2\,a_{1,1} + 10\,a_{3,1} + 4\,b_{0,0}\big)a_{2,0} \\
&+ \big(4\,a_{0,0} - 2\,b_{1,1} + 18\,b_{1,3}\big)b_{0,2} + \big(8\,a_{0,0} - 2\,b_{1,1} + 18\,b_{1,3}\big)b_{2,0} \\
&+ (-2\,b_{1,1} + 30\,b_{1,3})b_{4,0} - 2(b_{1,1} + 3\,b_{1,3})\,b_{0,0} + 2(-a_{1,1} + 3a_{3,1})\,a_{0,0},
\end{aligned}
$$

$$
\begin{aligned}
B_{10} = &\big(20\,a_{2,3} - 16\,b_{1,2} - 42\,b_{3,2} - 5\,a_{0,1} + 24\,b_{1,0} + 40\,b_{3,0} + 56\,b_{5,0} - 16\,a_{2,1} \\
&\qquad - 4\,a_{4,1} - 8\,b_{1,4} - 8\,a_{0,3} + 40\,a_{0,5}\big)a_{1,0} \\
&+ \big(12\,b_{5,0} - 2\,a_{2,3} - 9\,b_{3,2} - 10\,b_{1,4} + 51\,a_{0,5} + 12\,b_{1,0} - 16\,a_{0,3} - 8\,a_{2,1} \\
&\qquad - 8\,b_{1,2} - 9\,a_{4,1} - 4\,a_{0,1} + 12\,b_{3,0}\big)b_{0,3} \\
&+ \big(4\,b_{5,0} + 6\,a_{2,3} + 5\,a_{4,1} + 10\,b_{1,4} + 4\,b_{1,0} - 4\,a_{0,1} - 8\,b_{1,2} + 8\,a_{0,3} \\
&\qquad - 11\,a_{0,5} - 9\,b_{3,2} + 4\,b_{3,0}\big)a_{1,2} \\
&+ \big(- 24\,a_{0,1} - 32\,a_{2,1} - 56\,b_{5,0} + 30\,a_{2,3} + 24\,a_{0,3} - 60\,b_{1,4} - 60\,a_{4,1} \\
&\qquad + 58\,b_{3,2} - 40\,b_{3,0} + 32\,b_{1,2} - 24\,b_{1,0}\big)b_{4,1} \\
&+ \big(42\,a_{2,3} + 8\,b_{1,2} - 8\,a_{0,1} - 30\,b_{3,2} + 60\,b_{5,0} - 84\,a_{0,5} \\
&\qquad - 8\,a_{2,1} + 40\,a_{0,3} + 24\,b_{3,0} - 8\,b_{1,0}\big)b_{0,5} \\
&+ \big(- 16\,b_{0,2} + 12\,b_{0,4} + 4\,b_{2,0} - 20\,b_{4,0} + 6\,a_{1,1} - 2\,a_{1,3} \\
&\qquad + 18\,a_{3,1} + 12\,b_{0,0} - 16\,b_{2,2}\big)a_{2,2} \\
&+ \big(- 16\,a_{2,0} - 20\,a_{4,0} - 12\,a_{0,0} - 2\,b_{1,1} + 14\,b_{1,3} - 2\,b_{3,1} - 12\,a_{0,2} + 12\,a_{0,4}\big)b_{2,2} \\
&+ \big(- 4\,b_{0,2} + 48\,b_{0,4} + 16\,b_{2,0} + 8\,b_{4,0} + 22\,a_{1,1} - 50\,a_{1,3} + 46\,a_{3,1} + 8\,b_{0,0}\big)a_{4,0} \\
&+ \big(- 10\,a_{2,0} + 2\,a_{0,0} + 12\,b_{1,1} - 20\,b_{1,3} - 2\,a_{0,2} + 2\,a_{0,4}\big)a_{1,3} \\
&+ \big(12\,a_{2,0} - 8\,a_{0,0} - 2\,b_{1,1} + 6\,b_{1,3} + 18\,b_{3,1} + 16\,a_{0,2} - 8\,a_{0,4}\big)b_{0,4} \\
&+ \big(- 2\,b_{0,2} - 2\,b_{2,0} - 2\,b_{4,0} - 4\,a_{1,1} - 4\,a_{3,1} - 2\,b_{0,0}\big)b_{3,1} \\
&+ \big(4\,b_{0,2} + 8\,b_{2,0} + 48\,b_{4,0} - 2\,a_{1,1} - 30\,a_{3,1}\big)a_{0,4} \\
&+ \big(- 12\,b_{0,2} - 24\,b_{4,0} + 2\,a_{1,1} + 18\,a_{3,1} + 8\,b_{0,0}\big)a_{0,2} \\
&+ \big(24\,b_{1,0} + 40\,b_{3,0} + 56\,b_{5,0} + 12\,a_{2,3} + 20\,a_{4,1} - 24\,b_{1,2} + 24\,b_{1,4} - 50\,b_{3,2}\big)a_{3,0} \\
&+ \big(12\,b_{2,0} + 4\,b_{4,0} + 2\,a_{1,1} + 18\,a_{3,1} + 4\,b_{0,0}\big)a_{2,0} + \big(4\,a_{0,0} - 2\,b_{1,1} + 10\,b_{1,3}\big)b_{0,2} \\
&+ \big(8\,a_{0,0} - 2\,b_{1,1} - 6\,b_{1,3}\big)b_{2,0} + (-2\,b_{1,1} - 6\,b_{1,3})b_{4,0} \\
&- 2\,(-a_{1,1} + 3a_{3,1})a_{0,0} - 2\,b_{0,0}(b_{1,1} + 3\,b_{1,3}),
\end{aligned}
$$

$$
B_{11} = \big(- 12\,a_{2,1} - 16\,a_{2,3} - 40\,a_{4,1} + 20\,b_{1,2} - 20\,b_{1,4} + 26\,b_{3,2} + a_{0,1}
$$

$$
\begin{aligned}
& - 4\,b_{1,0} - 4\,b_{3,0} - 4\,b_{5,0} - 20\,a_{0,3} + 20\,a_{0,5}\big)a_{1,0} \\
&+ \big(4\,a_{0,1} + 20\,a_{2,1} + 4\,b_{5,0} + 60\,a_{0,5} - 54\,a_{2,3} - 36\,a_{0,3} \\
&\quad + 48\,b_{1,4} + 48\,a_{4,1} - 26\,b_{3,2} + 4\,b_{3,0} - 20\,b_{1,2} + 4\,b_{1,0}\big)b_{4,1} \\
&+ \big(- 60\,b_{1,4} + 30\,a_{2,3} + 28\,b_{1,2} - 20\,a_{0,1} - 60\,a_{4,1} + 54\,b_{3,2} \\
&\quad - 48\,b_{5,0} - 28\,a_{2,1} + 20\,a_{0,3} - 36\,b_{3,0} - 20\,b_{1,0}\big)b_{0,5} \\
&+ \big(- 12\,b_{3,1} - 2\,a_{2,0} + 2\,a_{0,4} - 6\,a_{0,0} + 22\,a_{4,0} - 12\,b_{1,3} + 6\,a_{0,2} + 14\,a_{2,2} + 4\,b_{1,1}\big)a_{1,3} \\
&+ \big(2\,a_{1,1} + 14\,a_{3,1} - 2\,b_{4,0} - 2\,b_{2,0} + 6\,b_{0,2} - 2\,b_{0,4} - 2\,b_{2,2} - 2\,b_{0,0}\big)a_{2,2} \\
&+ \big(- 2\,b_{2,0} - 2\,b_{2,2} - 2\,b_{4,0} + 6\,b_{0,2} - 10\,b_{0,4} - 2\,a_{1,1} + 6\,a_{3,1} - 2\,b_{0,0}\big)a_{0,2} \\
&+ \big(- 6\,b_{0,0} + 26\,b_{2,0} + 42\,b_{4,0} - 6\,a_{1,1} - 18\,a_{3,1} - 30\,b_{0,4} + 18\,b_{0,2} - 14\,b_{2,2}\big)a_{0,4} \\
&+ \big(2\,b_{0,2} - 2\,a_{1,1} - 6\,b_{0,0} - 6\,b_{4,0} - 10\,a_{3,1} - 6\,b_{2,0} + 10\,b_{2,2} + 2\,b_{0,4}\big)a_{2,0} \\
&+ \big(- 4\,b_{1,0} - 4\,b_{3,0} - 4\,b_{5,0} - 16\,a_{2,1} + 12\,a_{2,3} - 44\,a_{4,1} + 20\,b_{1,2} - 36\,b_{1,4} + 26\,b_{3,2}\big)a_{3,0} \\
&+ \big(2\,b_{0,2} - 6\,b_{2,0} + 18\,b_{0,4} - 14\,a_{3,1} + 10\,b_{2,2} - 6\,a_{1,1} - 6\,b_{4,0} - 6\,b_{0,0}\big)a_{4,0} \\
&+ \big(4\,a_{2,1} - 22\,a_{2,3} + 3\,a_{4,1} + 4\,b_{1,2} + 10\,b_{1,4} + 3\,b_{3,2} + 8\,a_{0,1} - 20\,a_{0,3} + 15\,a_{0,5}\big)b_{0,3} \\
&+ \big(2\,b_{0,2} - 6\,b_{2,0} - 6\,b_{4,0} + 2\,a_{1,1} - 6\,a_{3,1} - 6\,b_{0,0} + 10\,b_{2,2}\big)a_{0,0} \\
&+ \big(2\,a_{0,0} - 16\,b_{1,1} + 28\,b_{1,3} - 28\,b_{3,1}\big)b_{0,4} \\
&+ \big(- 8\,a_{2,1} + 6\,a_{2,3} - 11\,a_{4,1} + 4\,b_{1,2} - 10\,b_{1,4} + 3\,b_{3,2} - 4\,a_{0,1} + 5\,a_{0,5}\big)a_{1,2} \\
&- (8\,b_{0,2} + 4\,b_{2,2})b_{1,3},
\end{aligned}
$$

$$
\begin{aligned}
B_{12} = \big(& - 6\,a_{2,3} + 12\,b_{1,4} + 12\,a_{2,1} + 20\,a_{4,1} + a_{0,1} - 4\,b_{1,0} - 4\,b_{3,0} \\
&\quad - 4\,b_{5,0} - 4\,b_{1,2} - 4\,b_{3,2} + 12\,a_{0,3} - 44\,a_{0,5}\big)a_{1,0} \\
&+ \big(4\,a_{0,1} - 4\,a_{2,1} + 4\,b_{5,0} - 48\,a_{0,5} + 30\,a_{2,3} + 12\,a_{0,3} - 12\,b_{1,4} \\
&\quad - 12\,a_{4,1} + 4\,b_{3,2} + 4\,b_{3,0} + 4\,b_{1,2} + 4\,b_{1,0}\big)b_{4,1} \\
&+ \big(48\,b_{1,4} - 66\,a_{2,3} - 28\,b_{1,2} + 12\,a_{0,1} + 48\,a_{4,1} - 30\,b_{3,2} \\
&\quad + 12\,b_{5,0} + 84\,a_{0,5} + 28\,a_{2,1} - 44\,a_{0,3} + 12\,b_{3,0} + 12\,b_{1,0}\big)b_{0,5} \\
&+ \big(- 12\,b_{3,1} + 6\,a_{2,0} + 6\,a_{0,4} - 6\,a_{0,0} + 18\,a_{4,0} + 20\,b_{1,3} - 2\,a_{0,2} - 2\,a_{2,2} - 12\,b_{1,1}\big)a_{1,3} \\
&+ \big(- 6\,a_{1,1} - 22\,a_{3,1} - 2\,b_{4,0} - 2\,b_{2,0} + 6\,b_{0,2} - 2\,b_{0,4} + 14\,b_{2,2} - 2\,b_{0,0}\big)a_{2,2} \\
&+ \big(- 2\,b_{2,0} + 14\,b_{2,2} - 2\,b_{4,0} + 6\,b_{0,2} - 10\,b_{0,4} - 2\,a_{1,1} - 18\,a_{3,1} - 2\,b_{0,0}\big)a_{0,2} \\
&+ \big(- 6\,b_{0,0} - 6\,b_{2,0} - 6\,b_{4,0} + 2\,a_{1,1} + 30\,a_{3,1} + 2\,b_{0,4} + 2\,b_{0,2} - 14\,b_{2,2}\big)a_{0,4} \\
&+ \big(2\,b_{0,2} - 2\,a_{1,1} - 6\,b_{0,0} - 6\,b_{4,0} - 10\,a_{3,1} - 6\,b_{2,0} + 10\,b_{2,2} - 14\,b_{0,4}\big)a_{2,0} \\
&+ \big(- 4\,b_{1,0} - 4\,b_{3,0} - 4\,b_{5,0} + 8\,a_{2,1} - 18\,a_{2,3} + 16\,a_{4,1} - 4\,b_{1,2} + 12\,b_{1,4} - 4\,b_{3,2}\big)a_{3,0} \\
&+ \big(2\,b_{0,2} - 6\,b_{2,0} - 30\,b_{0,4} - 14\,a_{3,1} + 10\,b_{2,2} - 6\,a_{1,1} - 6\,b_{4,0} - 6\,b_{0,0}\big)a_{4,0} \\
&+ \big(2\,a_{0,0} + 8\,b_{1,1} - 20\,b_{1,3} + 8\,b_{3,1}\big)b_{0,4} \\
&+ \big(2\,b_{0,2} - 6\,b_{2,0} - 6\,b_{4,0} + 2\,a_{1,1} - 6\,a_{3,1} - 6\,b_{0,0} + 10\,b_{2,2}\big)a_{0,0} \\
&+ \big(4\,a_{2,1} - 9\,a_{2,3} + 4\,a_{4,1} + b_{1,4} + 4\,a_{0,1} - 8\,a_{0,3} + 10\,a_{0,5}\big)a_{1,2} \\
&+ \big(19\,a_{2,3} - b_{1,4} + 20\,a_{0,3} - 54\,a_{0,5}\big)b_{0,3},
\end{aligned}
$$

$$
\begin{aligned}
B_{13} = \big(& - 8\,a_{2,1} + 34\,a_{2,3} - 12\,a_{4,1} + 8\,b_{1,2} - 12\,b_{1,4} + 6\,b_{3,2} + 16\,a_{0,3} - 56\,a_{0,5}\big)b_{0,5} \\
&+ \big(4\,a_{0,0} - 16\,a_{2,2} - 4\,a_{0,2} + 8\,b_{1,3} - 8\,a_{0,4} + 4\,a_{2,0} + 4\,a_{4,0}\big)a_{1,3} \\
&+ \big(- 5\,b_{0,3} + 3\,a_{1,2} - 6\,b_{4,1} + 6\,a_{1,0} + 6\,a_{3,0}\big)a_{2,3} \\
&+ \big(- 16\,b_{0,2} + 32\,b_{0,4} + 8\,a_{1,1} + 12\,a_{3,1} - 8\,b_{2,2}\big)a_{0,4} \\
&+ \big(26\,b_{0,3} - 10\,a_{1,2} + 12\,b_{4,1} + 16\,a_{1,0}\big)a_{0,5} + (b_{1,4} + 4\,a_{0,3})a_{1,2} \\
&+ (-b_{1,4} - 4\,a_{0,3})b_{0,3} - (8\,a_{2,2} - 4\,b_{1,3})b_{0,4},
\end{aligned}
$$

$$B_{14} = \big(4\,a_{0,0}+4\,a_{2,2}+4\,a_{0,2}-8\,a_{0,4}+4\,a_{2,0}+4\,a_{4,0}\big)a_{1,3}+\big(-b_{0,3}+4\,b_{0,5}+a_{1,2}\big)a_{0,5}-2\,a_{2,3}b_{0,5},$$

$$B_{15} = \big(-b_{0,3}+4\,b_{0,5}+a_{1,2}\big)a_{0,5}+4\,a_{0,4}a_{1,3}-2\,a_{2,3}b_{0,5}.$$

Acknowledgements

We would like to express our gratitude to Dr F. Khelili, 20 août 1955 University of Skikda for his critial contribution to the success of this research.

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(Received 25.042023; accepted 01.08.2023)

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