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EXISTENCE RESULTS FOR NONLINEAR WEIGHTED ANISOTROPIC ELLIPTIC SYSTEMS WITH VARIABLE EXPONENTS

Abstract. This paper establishes the existence of distributional solutions for a class of anisotropic elliptic $\vec{p}(\cdot)$-systems. These systems are weighted by a positive function that belongs to the $\vec{p}(\cdot)$-Sobolev space and involves $L^1(\Omega;\mathbb{R}^m)$-data associated with the $L^1(\Omega)$-coefficient of the zero order term.

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1 Introduction

This study focuses on establishing the existence of at least one distributional solution $u=(u_1,\dots,u_m)^\top$ for a particular class of weighted elliptic $\vec{p}(\,\cdot\,)$-systems expressed in the form

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i\big(\upsilon(x)\sigma_i(x,\partial_i u)\big) + A(x)g(x,u) = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded Lipschitz domain, $\partial_i u = \frac{\partial u}{\partial x_i}$, $i = 1,\dots,N$, $f \in L^1(\Omega;\mathbb{R}^m)$, $A(\,\cdot\,) \in L^1(\Omega)$, and $\upsilon(\,\cdot\,)$ is in $\overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega)$ such that there exists $\alpha,\beta > 0$:

$$|f(x)| \leq \alpha A(x), \tag{1.2}$$

$$\upsilon(x) \geq \beta. \tag{1.3}$$

$\sigma_i : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, $i = 1,\dots,N$, are the Carathéodory functions such that, for almost every $x \in \Omega$ and every $s, s' \in \mathbb{R}^m$ $(s,s') \neq (0,0)$, we have

$$\sigma_i(x,s)\cdot s \geq c_1|s|^{p_i(x)}, \tag{1.4}$$

$$|\sigma_i(x,s)| \leq c_2\big(|s|^{p_i(x)} + |h|\big)^{1-\frac{1}{p_i(x)}}, \quad h \in L^1(\Omega) \tag{1.5}$$

$$\big(\sigma_i(x,s) - \sigma_i(x,s')\big)\cdot(s - s') \geq \begin{cases} c_3|s-s'|^{p_i(x)}, & \text{if } p_i(x) \geq 2, \\ c_4\,\dfrac{|s-s'|^2}{(|s|+|s'|)^{2-p_i(x)}}, & \text{if } 1 < p_i(x) < 2, \end{cases} \tag{1.6}$$

where c_l, c_2, c_3, c_4 are the positive constants.

The function $g : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function and meets the following conditions almost everywhere for $x \in \Omega$:

$$g(x,s)\cdot(s - s') \geq 0, \quad \forall\, s,s' \in \mathbb{R}^m, \quad |s| = |s'|, \tag{1.7}$$

$$\sup_{|s|\leq t} |g(x,s)| \in L^1(\Omega;\mathbb{R}^m), \quad \forall\, s \in \mathbb{R}^m, \quad \text{and} \quad \forall\, t > 0, \tag{1.8}$$

$$g(x,s)\cdot s \geq \sum_{i=1}^{N} |\rho|^{p_i(x)+1}, \quad \forall\, s \in \mathbb{R}^m. \tag{1.9}$$

As a typical example, consider the following:

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i\big(\upsilon(x)|\partial_i u|^{p_i(x)-2}\partial_i u\big) + A(x)u\sum_{i=1}^{N}|u|^{p_i(x)-1} = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{aligned}$$

where f, $A(\,\cdot\,)$, $\upsilon(\,\cdot\,)$, and $p_i(\,\cdot\,)$, $i = 1,\dots,N$, are restricted as in Theorem 3.1.

In this work, we focus on the $\vec{p}(x)$-anisotropic differential operator, which has a wide range of applications in applied sciences. For example, such operators are frequently employed in the study of electro-rheological fluids and image processing, as highlighted in references [10, 17, 27]. Various existence results for systems involving these operators under different conditions and data have been established, as detailed in [5, 19, 24–26]. For the anisotropic scalar case, the related results can be found in [1–4, 13]. Furthermore, the existence of solutions for variable exponent anisotropic nonlinear weighted elliptic equations has been demonstrated in [20–23], with the corresponding isotropic scalar case discussed in [6].

In the present paper, we establish existence results for distributional solutions to a class of anisotropic nonlinear elliptic systems with variable exponents described by (1.1). These systems are weighted by a positive function $\upsilon(\,\cdot\,) \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega)$, and the given datum satisfies $f \in L^1(\Omega;\mathbb{R}^m)$. A key assumption is the interaction described in (1.2) between the datum and the $L^1(\Omega)$ coefficient $A(\,\cdot\,)$ of the zero-order term inducing a regularizing effect on (1.1). Our approach to proving the existence of solutions depends critically on the requirement that the weight function belongs to the $\vec{p}(\,\cdot\,)$-Sobolev space.

We developed our proof based on a sequence of approximate solutions (u_n), which requires demonstrating their existence through the main theorem on pseudo-monotone operators and the findings presented in [25]. Subsequently, we employ a priori estimates to establish the boundedness of (u_n) and the almost everywhere convergence of their partial derivatives $\partial_i u_n$ for $i = 1, \dots, N$, which can be interpreted as a strong L^1-convergence. With this convergence established, we take the limit in the strong L^1 sense for $\upsilon_n(x)\sigma_i(x, \partial_i u_n)$ for $i = 1, \dots, N$ and in $A_n(x)g(x, u_n)$. Ultimately, we conclude that the approximate solutions u_n converge to the solution of (1.1).

Our work is organized as follows. Section 2 focuses on the mathematical preliminaries, covering isotropic and anisotropic variable exponent Lebesgue–Sobolev spaces, along with several embedding theorems. The primary theorem, along with its proof, is presented in Section 3.

2 Preliminaries

This section aims to introduce fundamental definitions and properties related to isotropic and anisotropic variable exponent Lebesgue–Sobolev spaces (refer to [11, 12, 15]).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset. We denote

$$\mathcal{C}_+(\overline{\Omega}) = \Big\{\text{continuous function } \; p(\,\cdot\,) : \overline{\Omega} \mapsto \mathbb{R}, \;\; / \;\; 1 < p^- \; (= p^- = \min_{x \in \overline{\Omega}} p(x))\Big\}.$$

Let $p(\,\cdot\,) \in \mathcal{C}_+(\overline{\Omega})$. Variable exponent Lebesgue space with $L^{p(\,\cdot\,)}(\Omega)$ is defined by

$$L^{p(\,\cdot\,)}(\Omega) := \Big\{\text{measurable functions } u : \Omega \mapsto \mathbb{R};\ \int\limits_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$

where

$$u \longmapsto \varrho_{p(\,\cdot\,)}(u) := \int\limits_\Omega |u(x)|^{p(x)}\,dx \;\; \text{is called the convex modular.}$$

It forms a reflexive Banach space when equipped with the Luxemburg norm

$$u \longmapsto \|u\|_{p(\,\cdot\,)} := \inf\Big\{s > 0 \mid \;\; \varrho_{p(\,\cdot\,)}\Big(\frac{u}{s}\Big) \leq 1\Big\}.$$

The Hölder type inequality

$$\Big|\int\limits_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(\,\cdot\,)}\|v\|_{p'(\,\cdot\,)} \leq 2\|u\|_{p(\,\cdot\,)}\|v\|_{p'(\,\cdot\,)}$$

holds true.

The subsequent results are presented in [11, 12].

If $u \in L^{p(\,\cdot\,)}(\Omega)$, then we obtain

$$\begin{gathered}\min \leq \big(\varrho_{p(\,\cdot\,)}(u)^{\frac{1}{p^+}}, \varrho_{p(\,\cdot\,)}(u)^{\frac{1}{p^-}}\big) \leq \|u\|_{p(\,\cdot\,)} \leq \max\big(\varrho_{p(\,\cdot\,)}(u)^{\frac{1}{p^+}}, \varrho_{p(\,\cdot\,)}(u)^{\frac{1}{p^-}}\big), \\ \min\big(\|u\|_{p(\,\cdot\,)}^{p^-}, \|u\|_{p(\,\cdot\,)}^{p^+}\big) \leq \varrho_{p(\,\cdot\,)}(u) \leq \max\big(\|u\|_{p(\,\cdot\,)}^{p^-}, \|u\|_{p(\,\cdot\,)}^{p^+}\big).\end{gathered} \tag{2.1}$$

We now proceed to introduce the $\vec{p}(\,\cdot\,)$-Sobolev spaces $W^{1,\vec{p}(\,\cdot\,)}(\Omega)$.

Let $\vec{p}(x) = (p_1(x), \dots, p_N(x)) \in (C(\overline{\Omega}, [1, +\infty)))^N$, and for every x in $\overline{\Omega}$ we set

$$p_+(x) = \max_{1 \le i \le N} p_i(x), \quad p_-(x) = \min_{1 \le i \le N} p_i(x),$$
$$p_-^- = \min_{x \in \overline{\Omega}} p_-(x), \quad p_+^+ = \max_{x \in \overline{\Omega}} p_+(x),$$
$$\overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}}, \quad \overline{p}^\star(x) = \begin{cases} \dfrac{N\overline{p}(x)}{N - \overline{p}(x)} & \text{for } \overline{p}(x) < N, \\ +\infty & \text{for } \overline{p}(x) \ge N. \end{cases}$$

The Banach space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\vec{p}(\cdot)}(\Omega) = \Big\{ u \in L^{p_+(\cdot)}(\Omega),\ \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N \Big\}$$

under the norm

$$u \longmapsto \|u\|_{\vec{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}. \tag{2.2}$$

The Banach space $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows:

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega) = W^{1,\vec{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega),$$

when equipped with the norm (2.2).

The following results can be found in [14, 15].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\vec{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$.

($*$) If $r \in \mathcal{C}_+(\overline{\Omega})$ and $\forall\, x \in \overline{\Omega}$, $r(x) < \max(p_+(x), \overline{p}^\star(x))$, then the embedding

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$$

is compact.

($*$) If we have

$$\forall\, x \in \overline{\Omega}, \quad p_+(x) < \overline{p}^\star(x), \tag{2.3}$$

then the following inequality holds:

$$\|u\|_{p_+(\cdot)} \le C \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}, \quad \forall\, u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega),$$

where $C > 0$ is independent of u. Thus

$$u \longmapsto \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)} \ \text{ is an equivalent norm to (2.2) on } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega). \tag{2.4}$$

In our paper, we denote by $L^{p(\cdot)}(\Omega, \mathbb{R}^m)$ and $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega, \mathbb{R}^m)$ ($m \ge 1$) the $\mathbb{R}^m$-valued version of $L^{p(\cdot)}(\Omega)$ and $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, respectively.

The space $X = \mathring{W}^{1,\vec{p}(\cdot)}(\Omega, \mathbb{R}^m)$ becomes a Banach space, when equipped with the norm

$$\|\cdot\|_X = \||\cdot|\|_{\vec{p}(\cdot)}.$$

Also $Y = L^{p(\cdot)}(\Omega, \mathbb{R}^m)$, when equipped with the norm

$$\|\cdot\|_Y = \||\cdot|\|_{p(\cdot)}.$$

For any $t > 0$, the scalar truncation function T_t on $[0, \infty)$ is defined as

$$T_t(s) := \begin{cases} s & \text{if } s \le t, \\ t & \text{if } s > t. \end{cases}$$

For any $t > 0$, define the spherial truncation function $T_t : \mathbb{R}^m \to \mathbb{R}^m$ by

$$T_t(s) := \begin{cases} s & \text{if } |s| \le t, \\ \dfrac{s}{|s|}\, t & \text{if } |s| > t. \end{cases} \tag{2.5}$$

3 Statement of results and proof

Definition. The vector-valued function $u = (u_1, \dots, u_m)^\top : \Omega \longrightarrow \mathbb{R}^m$ is a solution of system (1.1) in the sense of distributions if and only if $u \in W_0^{1,1}(\Omega; \mathbb{R}^m)$, and for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$,

$$\int_\Omega \sum_{i=1}^N \upsilon(x)\sigma_i(x, \partial_i u) \cdot \partial_i \varphi \, dx + \int_\Omega \sum_{i=1}^N A(x) g(x, u) \cdot \varphi \, dx = \int_\Omega f(x) \cdot \varphi \, dx.$$

Our main Theorem is the following.

Theorem 3.1. *Let $p_i(\cdot)$, $i = 1, \dots, N$ be in $\mathcal{C}_+(\overline{\Omega})$ such that $\overline{p}(\cdot) < N$ in $\overline{\Omega}$, and let (2.3) hold. Assume f is in $L^1(\Omega; \mathbb{R}^m)$, $A(\cdot)$ is in $L^1(\Omega)$, and $\upsilon(\cdot)$ is in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ such that (1.2), (1.3) and (2.3) hold.*

Let σ_i, $i = 1, \dots, N$, and g be the Carathéodory functions, where σ_i satisfy (1.4)–(1.6), and g satisfies (1.7)–(1.9). Then problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega; \mathbb{R}^m)$ in the sense of distributions.

3.1 Existence of approximate solutions

We consider the function $\theta(\cdot)$ defined as follows:

$$\theta(x) = \frac{nx}{x+n}, \quad x \ge 0,$$

$$f_n(x) = \theta(f(x)), \quad A_n(x) = \frac{1}{\alpha}\theta(A(x)), \quad \upsilon_n(x) = \theta(\upsilon(x)), \quad n \in \mathbb{N}^*. \tag{3.1}$$

Noticing the increase of θ, (1.2) and (1.3), we can obtain

$$|f_n(x)| \le \theta(\alpha A(x)) = \alpha A_n(x)$$

and

$$\theta(\beta) \le \upsilon_n(x) \le \theta(n).$$

Then we conclude that for all $x \in \overline{\Omega}$,

$$|f_n(x)| \le \alpha A_n(x) \tag{3.2}$$

and

$$\frac{\beta}{1+\beta} \le \upsilon_n(x) \le n. \tag{3.3}$$

Lemma 3.1. *Let $p_i(\cdot)$, $i = 1, \dots, N$, be in $\mathcal{C}_+(\overline{\Omega})$ such that $\overline{p}(\cdot) < N$ in $\overline{\Omega}$, and let (2.3) hold. Assume f is in $L^1(\Omega; \mathbb{R}^m)$, $A(\cdot)$ is in $L^1(\Omega)$, and $\upsilon(\cdot)$ is in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ such that (1.2), (1.3) and (2.3) hold.*

Let σ_i, $i = 1, \dots, N$, and g be the Carathéodory functions, where σ_i satisfy (1.4)–(1.6), and g satisfies (1.7)–(1.9).

Then there exists at least one weak solution $u_n \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)$ *to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^{N}\partial_i\big(\upsilon_n(x)\sigma_i(x,\partial_i u_n)\big)+A_n(x)g(x,u_n))&=f_n \ \text{ in } \Omega,\\ u_n&=0 \ \text{ on } \partial\Omega, \end{aligned} \tag{3.4}$$

in the following sense:

For every $\varphi \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$,

$$\sum_{i=1}^{N}\int_\Omega \upsilon_n(x)\sigma_i(x,\partial_i u_n)\cdot\partial_i\varphi\,dx+\int_\Omega A_n(x)g(x,u_n)\cdot\varphi\,dx=\int_\Omega f_n\cdot\varphi\,dx. \tag{3.5}$$

Proof. Consider the system

$$\begin{aligned} -\sum_{i=1}^{N}\partial_i\big(\upsilon_n(x)\sigma_i(x,\partial_i u_{n_k})\big)+A_n(x)T_k(g(x,u_{n_k}))&=f_n \ \text{ in } \Omega,\\ u_{n_k}&=0 \ \text{ on } \partial\Omega. \end{aligned} \tag{3.6}$$

In a similar manner to the results obtained in [25], applying the main Theorem on the pseudo-monotone operators ((Theorem 27.A in [28], see also [8, 9, 18])), we conclude that there exists a solution $u_{n_k} \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)$ to system (3.6), which satisfies

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega \upsilon_n(x)\sigma_i(x,\partial_i u_{n_k})\cdot\partial_i\varphi\,dx\\ &\qquad+\int_\Omega A_n(x)T_k\big(g(x,u_{n_k})\big)\cdot\varphi\,dx=\int_\Omega f_n\cdot\varphi\,dx,\ \ \forall\,\varphi\in\overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m). \end{aligned} \tag{3.7}$$

Now, choosing $\varphi = u_{n_k}$ as a test function in (3.7) and using (3.3), (1.4), after dropping the nonegative term (since $A_n(x)T_k(g(x,u_{n_k}))\cdot u_{n_k} \geq 0$, due to (1.9) and the fact that $A_n(x)\geq 0$), we get

$$\frac{c_1\beta}{(1+\beta)}\sum_{i=1}^{N}\int_\Omega|\partial_i u_{n_k}|^{p_i(x)}\,dx\leq n\int_\Omega|u_{n_k}|\,dx.$$

Using Young's inequality, for all $\varepsilon>0$, we get

$$\begin{aligned} \sum_{i=1}^{N}\int_\Omega|\partial_i u_{n_k}|^{p_i(x)}\,dx&\leq\frac{n(1+\beta)}{c_1\beta}\int_\Omega|u_{n_k}|\,dx\\ &\leq\frac{n(1+\beta)}{c_1\beta}\Big(\varepsilon\int_\Omega|u_{n_k}|^{p^-_-}\,dx+C(\varepsilon)\Big)\leq\frac{n(1+\beta)}{c_1\beta}\Big(\varepsilon c\int_\Omega|\partial_i u_{n_k}|^{p^-_-}\,dx+C(\varepsilon)\Big)\\ &\leq\frac{n(1+\beta)}{c_1\beta}\Big(\varepsilon c\Big(1+\int_\Omega|\partial_i u_{n_k}|^{p_i(x)}\,dx\Big)+C(\varepsilon)\Big)\\ &\leq\frac{n(1+\beta)}{c_1\beta}\Big(\varepsilon c\Big(1+\sum_{i=1}^{N}\int_\Omega|\partial_i u_{n_k}|^{p_i(x)}\,dx\Big)+C(\varepsilon)\Big). \end{aligned}$$

Choosing $\varepsilon=\frac{c_1\beta}{2nc(1+\beta)}$, we obtain

$$\sum_{i=1}^{N}\int_\Omega|\partial_i u_{n_k}|^{p_i(x)}\,dx\leq c(n). \tag{3.8}$$

On the other hand, from (2.1) for all $i = 1, \dots, N$ we have

$$1 + \int_\Omega |\partial_i u_{n_k}|^{p_i(x)}\, dx \ge |||\partial_i u_{n_k}|||_{p_i(x)}^{p_i^-} \tag{3.9}$$

and

$$1 + |||\partial_i u_{n_k}|||_{p_i(x)}^{p_i^-} \ge |||\partial_i u_{n_k}|||_{p_i(x)}^{p_-^-}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\sum_{i=1}^N \int_\Omega |\partial_i u_{n_k}|^{p_i(x)}\, dx \ge \sum_{i=1}^N |||\partial_i u_{n_k}|||_{p_i(x)}^{p_-^-} - 2N. \tag{3.11}$$

By (3.11) and (2.4) (due (2.3)), we deduce that

$$\sum_{i=1}^N \int_\Omega |\partial_i u_{n_k}|^{p_i(x)}\, dx \ge \Big(\frac{1}{N}\, |||u_{n_k}|||_{\vec{p}(\,\cdot\,)}\Big)^{p_-^-} - 2N. \tag{3.12}$$

From (3.8) and (3.12), we conclude

$$|||u_{n_k}|||_{\vec{p}(\,\cdot\,)} \le C(n). \tag{3.13}$$

Through (3.13) we can conclude that there is a subsequence (still denoted by u_{n_k}) $u_n \in \mathring{W}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)$ such that

$$u_{n_k} \rightharpoonup u_n \ \text{ weakly in } \mathring{W}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m) \text{ and a.e in } \Omega.$$

In a similar manner to the results obtained in [25], thanks to (1.4)–(1.6) and (1.7)–(1.9), we can obtain

$$\partial_i u_{n_k} \to \partial_i u_n \ \text{ strongly in } L^{p_i(\,\cdot\,)}(\Omega;\mathbb{R}^m) \text{ and a.e. in } \Omega.$$

So,

$$\upsilon_n(x)\sigma_i(x,\partial_i u_{n_k}) \rightharpoonup \upsilon_n(x)\sigma_i(x,\partial_i u_n) \ \text{ in } L^{p_i'(\,\cdot\,)}(\Omega;\mathbb{R}^m).$$

Taking $T_t(u_{n_k})$ as a test function in (3.7), by (1.8), (2.5), and the fact that

$$|T_t(s)| \le M + t1_{\{|s|>M\}}, \quad \forall\, s \in \mathbb{R}^m \ \text{ and } \ 0 < M < t,$$

for all $0 < M < t$ we get

$$\int_{\{|u_{n_k}|>t\}} A_n(x)|T_k(g(x,u_{n_k}))|\, dx \le \frac{M}{t}\, \|f\|_{L^1(\Omega;\mathbb{R}^m)} + \int_{\{|u_{n_k}|>M\}} |f_n|. \tag{3.14}$$

Let $E \subset \Omega$ be any measurable set, we write

$$\int_E A_n(x)|T_k(g(x,u_{n_k}))|\, dx = \int_{E\cap\{|u_{n_k}|\le t\}} A_n(x)|g(x,u_{n_k})|\, dx + \int_{E\cap\{|u_{n_k}|>t\}} A_n(x)|T_k(g(x,u_{n_k}))|\, dx.$$

Then, by (1.8) and the decomposition (3.14), we deduce that the sequence $\{A_n(x)T_k(g(x,u_{n_k}))\}$ is equi-integrable in $L^1(\Omega;\mathbb{R}^m)$, and since $T_k(g(x,u_{n_k})) \to g(x,u_n)$ a.e. in Ω, Vitali's theorem implies that

$$A_n(x)T_k(g(x,u_{n_k})) \to A_n(x)g(x,u_n) \ \text{ in } L^1(\Omega;\mathbb{R}^m).$$

Therefore, we can obtain (3.5) by passing to the limit in (3.7). □

3.1.1 A priori estimates

Lemma 3.2. *Let f, A, υ, g and p_i, σ_i, $i = 1, \dots, N$, be restricted as in Theorem 3.1. Then*

$$|g(x, u_n)| \leq \alpha, \tag{3.15}$$

$$|u_n| \leq \Big(\frac{\alpha}{N} + 1\Big)^{\frac{1}{p_-^-}}, \tag{3.16}$$

where u_n is the weak solution to problem (3.4).

Proof. After choosing $\varphi = u_n$ in the weak formulation (3.5), and dropping the nonnegative term (since $\upsilon_n(x)\sigma_i(x, \partial_i u_n) \cdot \partial_i u_n \geq 0$, $i = 1, \dots, N$, due (1.4), and (3.3)), we obtain

$$\int\limits_\Omega A_n(x) g(x, u_n) \cdot u_n \, dx \leq \int\limits_\Omega |f_n||u_n| \, dx.$$

Using (3.2) and the fact that

$$g(x, u_n) \cdot u_n \geq |g(x, u_n)|\,|u_n|$$

$\Big($it is produced through the following: by (1.7), we get

$$\frac{u_n}{|u_n|} \cdot g(x, u_n) - |g(x, u_n)| = \frac{1}{|u_n|} g(x, u_n) \cdot \Big(u_n - |u_n| \frac{g(x, u_n)}{|g(x, u_n)|}\Big) \geq 0 \Big),$$

we obtain

$$\int\limits_\Omega A_n(x)|g(x, u_n)||u_n| \, dx \leq \alpha \int\limits_\Omega A_n(x)|u_n| \, dx.$$

whence

$$\int\limits_\Omega A_n(x)\big(|g(x, u_n)| - \alpha\big)|u_n| \, dx \leq 0. \tag{3.17}$$

Then (3.17) implies (3.15).

Also, by the fact that $1 + |u_n|^{p_i(x)} \geq |u_n|^{p_-^-}$, $i = 1, \dots, N$, due to (1.9) and (3.15), we get (3.16). □

Remark 3.1. (3.15) and (3.16) imply that

$$\begin{aligned} (g(x, u_n)) \ \text{ is bounded in } L^\infty(\Omega, \mathbb{R}^m), \\ (u_n) \ \text{ is bounded in } L^\infty(\Omega, \mathbb{R}^m). \end{aligned} \tag{3.18}$$

Lemma 3.3.

$$\big(A_n(x) g(x, u_n)\big) \ \textit{ is bounded in } L^1(\Omega, \mathbb{R}^m). \tag{3.19}$$

Proof. Through (3.15) and (3.1), we get

$$\int\limits_\Omega |A_n(x) g(x, u_n)| \, dx \leq \alpha \int\limits_\Omega |A_n(x)| \, dx \leq \|A\|_{L^1(\Omega)}. \tag{3.20}$$

So, (3.20) implies (3.19). □

Lemma 3.4.

$$\upsilon_n \ \textit{ is bounded in } \mathring{W}^{1,\vec{p}(\,\cdot\,)}(\Omega) \tag{3.21}$$

and

$$\upsilon_n \to \upsilon \ \textit{ strongly in } \mathring{W}^{1,\vec{p}(\,\cdot\,)}(\Omega). \tag{3.22}$$

Proof. Since, for all $x \in \overline{\Omega}$,

$$\partial_i \upsilon_n(x) = \frac{\partial_i \upsilon(x)}{(1+\frac{\upsilon(x)}{n})^2}\,, \quad i=1,\dots,N,$$

we have

$$|\partial_i \upsilon_n(x)| \le |\partial_i \upsilon(x)|$$

and, therefore,

$$\upsilon_n(\,\cdot\,) \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega).$$

From this and the fact that $0 \le \upsilon_n(x) \le \upsilon(x)$, we get (3.21) and (3.22). □

Lemma 3.5. *Let f, A, υ, g and p_i, σ_i, $i=1,\dots,N$, be restricted as in Theorem 3.1. Then*

$$u_n \ \text{ is bounded in } \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m), \tag{3.23}$$

where u_n is the weak solution to problem (3.4).

Proof. After choosing $\varphi = u_n$ in the weak formulation (3.5), and dropping the nonnegative term (since $A_n(x)g(x,u_n)\cdot u_n \ge 0$, due to (1.9) and the fact that $A_n(x) \ge 0$), and using (3.16), (1.4), (3.1) and (3.3), we can get

$$\frac{c_1\beta}{(1+\beta)} \sum_{i=1}^{N} \int\limits_{\Omega} |\partial_i u_n|^{p_i(x)}\,dx \le \Big(\frac{\alpha}{N}+1\Big)^{\frac{1}{p_-^-}} \|f\|_{L^1(\Omega,\mathbb{R}^m)}.$$

Then we have

$$\sum_{i=1}^{N} \int\limits_{\Omega} |\partial_i u_n|^{p_i(x)}\,dx \le c. \tag{3.24}$$

By a proof similar to the that of (3.12), we can get

$$\sum_{i=1}^{N} \int\limits_{\Omega} |\partial_i u_n|^{p_i(x)}\,dx \ge \Big(\frac{1}{N}\,\||u_n|\|_{\vec{p}(\,\cdot\,)}\Big)^{p_-^-} - 2N. \tag{3.25}$$

Combining (3.24) and (3.25), we obtain

$$\||u_n|\|_{\vec{p}(\,\cdot\,)} \le C, \tag{3.26}$$

where $C>0$ is independent of n.

Then (3.26) implies (3.23). □

Remark 3.2. It follows from (3.23) that there exist a function $u \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \ \text{ weakly in } \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m) \text{ and a.e in } \Omega. \tag{3.27}$$

Lemma 3.6. *For all $i=1,\dots,N$,*

$$\partial_i u_n \to \partial_i u \ \ a.e. \ in \ \overline{\Omega}, \tag{3.28}$$

where u is the weak limit of the sequence (u_n) in $\overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)$.

Proof. By (3.3), we obtain

$$\frac{1}{\upsilon_n(x)}\,\varphi \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega,\mathbb{R}^m) \ \text{ for all } \ \varphi \in \overset{\circ}{W}{}^{1,\vec{p}(\,\cdot\,)}(\Omega,\mathbb{R}^m) \cap L^{\infty}(\Omega,\mathbb{R}^m).$$

Therefore, we can choose it as a test function in (3.5) and get

$$\sum_{i=1}^{N} \int_{\Omega} \sigma_i(x, \partial_i u_n) \cdot \partial_i \varphi \, dx = \int_{\Omega} \Phi_n(x) \varphi \, dx,$$

where Φ_n is defined by

$$\Phi_n(x) = \frac{1}{\upsilon_n(x)} \Big(f_n(x) - A_n(x) g(x, u_n) + \sum_{i=1}^{N} \sigma_i(x, \partial_i u_n) \partial_i \upsilon_n(x) \Big).$$

By Young's inequality, (1.5), and since $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, for all $\varepsilon > 0$ we get

$$\int_{\Omega} \big|\sigma_i(x, \partial_i u_n)\big| \, dx = c' + \int_{\Omega} |\partial_i u_n|^{p_i(x)-1} \, dx$$
$$\leq c' + C(\varepsilon) + \varepsilon \int_{\Omega} |\partial_i u_n|^{p_i(x)} \, dx \leq c' + C(\varepsilon) + \varepsilon c = C'(\varepsilon).$$

Then, for any fixed choice for $\varepsilon > 0$, we conclude that for all $i = 1, \dots, N$,

$$(\sigma_i(x, \partial_i u_n)) \text{ is bounded in } L^1(\Omega, \mathbb{R}^m). \tag{3.29}$$

From (3.1) (implying that $f_n \in L^1(\Omega, \mathbb{R}^m)$), (3.29) and (3.19), we conclude that

$$\Big(f_n(x) - A_n(x) g(x, u_n) + \sum_{i=1}^{N} \sigma_i(x, \partial_i u_n) \partial_i \upsilon_n(x) \Big) \text{ is bounded in } L^1(\Omega, \mathbb{R}^m). \tag{3.30}$$

Through (3.30) and the boundedness of the sequence $(\frac{1}{\upsilon_n(x)})$ (due to (3.3)), we obtain

$$(\Phi_n) \text{ is bounded in } L^1(\Omega, \mathbb{R}^m).$$

So, applying the results obtained in [7] to the sequence (u_n), we can simply obtain (3.28). □

3.2 Proof of Theorem 3.1

For all $i = 1, \dots, N$, we put

$$\sigma_i(x, \partial_i u_n) = \big(\sigma_i^{(1)}(x, \partial_i u_n), \dots, \sigma_i^{(m)}(x, \partial_i u_n)\big)$$

and

$$\sigma_i(x, \partial_i u) = \big(\sigma_i^{(1)}(x, \partial_i u), \dots, \sigma_i^{(m)}(x, \partial_i u)\big).$$

By (3.28), for all $i = 1, \dots, N$ we have

$$\sigma_i(x, \partial_i u_n) \rightharpoonup \sigma_i(x, \partial_i u) \text{ weakly in } L^{p_i'(\cdot)}(\Omega; \mathbb{R}^m).$$

Then we conclude that, for all $i = 1, \dots, N$ and all $j = 1, \dots, m$,

$$\sigma_i^{(j)}(x, \partial_i u_n) \rightharpoonup \sigma_i^{(j)}(x, \partial_i u) \text{ weakly in } L^{p_i'(\cdot)}(\Omega), \tag{3.31}$$

where $p_i'(\cdot)$ denotes the Hölder conjugate of $p_i(\cdot)$ in $\overline{\Omega}$.

By (3.22), we conclude that for all $i = 1, \dots, N$,

$$\upsilon_n(\cdot) \to \upsilon(\cdot) \text{ strongly in } L^{p_i(\cdot)}(\Omega). \tag{3.32}$$

Then, from (3.31) and (3.32), for all $i = 1, \dots, N$ and all $j = 1, \dots, m$, we obtain

$$\upsilon_n(x) \sigma_i^{(j)}(x, \partial_i u_n) \to \upsilon(x) \sigma_i^{(j)}(x, \partial_i u) \text{ strongly in } L^1(\Omega). \tag{3.33}$$

So, (3.33) implies that

$$v_n(x)\sigma_i(x,\partial_i u_n) \to v(x)\sigma_i(x,\partial_i u) \text{ strongly in } L^1(\Omega;\mathbb{R}^m). \tag{3.34}$$

Now, we put

$$g(x,u_n) = \big(g_1(x,u_n),\dots,g_m(x,u_n)\big)$$

and

$$g(x,u) = \big(g_1(x,u),\dots,g_m(x,u)\big).$$

Through (3.18) and the fact that $|g_j(x,u_n)| \le |g(x,u_n)|$, $j = 1,\dots,m$, we conclude that

$$(g_j(x,u_n)) \text{ is bounded in } L^\infty(\Omega). \tag{3.35}$$

Then, as $A_n \in L^1(\Omega)$, from (3.35) and (3.27), for all $j = 1,\dots,m$, we obtain

$$A_n(x)g_j(x,u_n) \to A(x)g_j(x,u) \text{ strongly in } L^1(\Omega). \tag{3.36}$$

So, (3.36) implies that

$$A_n(x)g(x,u_n) \to A(x)g(x,u) \text{ strongly in } L^1(\Omega;\mathbb{R}^m). \tag{3.37}$$

Then, through (3.34) and (3.37), we can pass to the limit in (3.5). Thus Theorem 3.1 is proved.

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