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EXISTENCE AND UNIQUENESS OF A SOLUTION OF A SELF-REFERRED DIFFERENTIAL EQUATION WITH WEIGHT

Abstract. In this paper, the existence and uniquenesss of a solution of an initial-value problem of a self-referred differential equation with weight is investigated. In addition, the Lipschitzian continuity of this unique solution is also considered.

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1 Introduction

Let X be a space of functions, $A : X \to \mathbb{R}$ and $B : X \to \mathbb{R}$ be funtionals. Consider the equation

$$Au(x,t) = u(Bu(x,t),t), \tag{1.1}$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R} \times [0,+\infty)$, is an unknown function satisfying some initial conditions at $t = 0$ for every $x \in \mathbb{R}$. Equation (1.1) is called a self-referred equation. Many authors have investigated (1.1) for different A and B (see [1–5] and the references therein).

In [3], the authors considered the existence and uniqueness of a local solution of the following initial-value problem of a self-referred differential equation associated with an integral operator:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\displaystyle\int_0^t u(x,s)\,ds, t\right), & t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ and u_0 is a given function. Then, replacing the integral operator in (1.2) by some integral operators with weight, the authors also got the existence and uniqueness of a local solution of the initial-value problems

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\dfrac{1}{t}\displaystyle\int_0^t u(x,s)\,ds, t\right), & t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\displaystyle\int_0^t \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)\,d\xi\,ds, t\right), & t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ and u_0 is a given function.

In [6], the authors studied the following system of two partial-differential equations with self-reference and weighted hereditary:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(f(u(x,t)) + v\left(\dfrac{1}{t}\displaystyle\int_0^t u(x,s)\,ds + \varphi(u(x,t)), t\right), t\right), \\ \dfrac{\partial}{\partial t} v(x,t) = v\left(g(v(x,t)) + u\left(\dfrac{1}{t}\displaystyle\int_0^t v(x,s)\,ds + \psi(u(x,t)), t\right), t\right), \end{cases} \tag{1.5}$$

associated with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x).$$

With some appropriate conditions on the given functions f, g, φ, ψ, u_0 and v_0, the uniqueness of a local solution and the existence of a global solution of this problem were proved.

By considering (1.1) in one-dimensional space setting and the Chandrase–Khar kernel, the general state-dependent integral equation via Chandrase–Khar kernel

$$x(t) = b(t) + \lambda x\left(\int_0^t \frac{t}{t+s} g(s,x(s))\,ds\right), \quad t \in [0,1], \tag{1.6}$$

was investigated in [7] under some conditions on the given functions b and g. The existence of a solution of (1.6) and the continuous dependence of this solution were obtained.

As a generalization of (1.3), (1.4), (1.5) and (1.6), this paper deals with the following initial-value problem of a self-referred differential equation with weight:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(au(x,t) + \displaystyle\int_0^t \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)\, d\xi\, ds, t\right), \\ u(x,0) = u_0(x), \end{cases} \tag{1.7}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^\star$, $\mathbb{R}^\star = \mathbb{R}^+ \cup \{0\}$, $a \in \mathbb{R}$; u_0 and $\delta > 0$ are the given real functions and u is an unknown function. Using a fixed-point method, the existence and uniqueness of a solution of (1.7) are studied. In addition, the Lipschitz property of this solution is also obtained.

2 Main results

In this section, we study the existence and uniqueness of a solution of the following initial-value problem of a self-referred differential equation with weight

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(au(x,t) + \displaystyle\int_0^t \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)\, d\xi\, ds, t\right), & t \geq 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $x \in \mathbb{R}$, $t \geq 0$, $a \in \mathbb{R}$, $u_0(0)$ and $\delta(s) > 0$ are two given functions, u is an unknow function.

First, denote

$$\mathbb{C}(\mathbb{R} \times [0,+\infty), \mathbb{R}) = \big\{u : \ \mathbb{R} \times [0,+\infty) \to \mathbb{R}, \ u \text{ is continuous and bounded}\big\}.$$

For $u \in \mathbb{C}(\mathbb{R} \times [0,+\infty), \mathbb{R})$, consider the following operator:

$$Tu(x,t) = u_0(x) + \int_0^t u\left(au(x,t) + \int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u(\xi,s)\, d\xi\, ds, \tau\right) d\tau,$$

where u_0 and δ are the given real continuous functions.

Remark. Given a continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ and a non-negative function $\delta : \mathbb{R} \to [0,+\infty)$, it follows that

$$\begin{aligned} &\left| \int_{x-\delta(s)}^{x+\delta(s)} f(\xi)\, d\xi - \int_{y-\delta(s)}^{y+\delta(s)} f(\xi)\, d\xi \right| \\ &\qquad = \left| \int_{x-\delta(s)}^{0} f(\xi)\, d\xi + \int_{0}^{x+\delta(s)} f(\xi)\, d\xi + \int_{0}^{y-\delta(s)} f(\xi)\, d\xi - \int_{0}^{y+\delta(s)} f(\xi)\, d\xi \right| \\ &\qquad = \left| \int_{x-\delta(s)}^{y-\delta(s)} f(\xi)\, d\xi + \int_{y+\delta(s)}^{x+\delta(s)} f(\xi)\, d\xi \right| \leq 2\|f\|_\infty |x-y|. \end{aligned} \tag{2.1}$$

Suppose that

(M_1) $a \in \mathbb{R}$;

(M_2) u_0 is a bounded function in $\mathbb{R}$, $\|u_0\|_{L^\infty(\mathbb{R},\mathbb{R})} < +\infty$;

(M_3) u_0 is Lipschitz, which means that there exists $L_0 \geq 0$ such that

$$|u_0(x) - u_0(y)| \leq L_0|x - y|$$

for all $x, y \in \mathbb{R}$;

(M_4) $\delta : \mathbb{R} \to [0, +\infty)$ satisfies the condition

$$\int_0^t \frac{1}{\delta(s)}\, ds < +\infty$$

for all $t > 0$.

We consider the sequence of functions $(u_n)_n$ defined by recurrence formula:

$$u_1(x,t) = u_0(x) + \int_0^t u_0\bigg(au_0(x) + \int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_0(\xi)d\xi\, ds\bigg)\, d\tau,$$

$$u_{n+1}(x,t) = u_0(x) + \int_0^t u_n\bigg(au_n(x,\tau) + \int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_n(\xi, s)d\xi\, ds, \tau\bigg)\, d\tau,$$

where $t \geq 0$, $x \in \mathbb{R}$, $n \geq 1$.

We see that

$$|u_1(x,t)| \leq \|u_0\|_\infty(1 + t),$$
$$|u_2(x,t)| \leq \|u_0\|_\infty\Big(1 + t + \frac{t^2}{2}\Big).$$

By induction on n, we get

$$|u_n(x,t)| \leq \|u_0\|_\infty \sum_{i=1}^{n} \frac{t^i}{i!}\,.$$

Therefore, we can choose $T_1 > 0$ such that

$$\|u_n\|_{L^\infty(\mathbb{R}\times[0,T_1])} \leq \|u_0\|_\infty \sum_{i=1}^{n} \frac{T_1^i}{i!} \leq e^{T_1}\|u_0\|_\infty. \tag{2.2}$$

On the other hand, we have

$$0 \leq |u_1(x,t) - u_0(x)| \leq \|u_0\|_\infty t = A_1(t), \ \ \forall\, x \in \mathbb{R}, \ \ t > 0.$$

Furthermore,

$$|u_1(x,t) - u_1(y,t)| \leq \bigg[L_0 + \int_0^t L_0\bigg(|a|L_0 + \|u_0\|_\infty \int_0^\tau \frac{1}{\delta(s)}\, ds\bigg)\, d\tau\bigg]\, |x - y| := L_1(t)|x - y|,$$

where

$$L_1(t) = L_0 + \int_0^t L_0\bigg(|a|L_0 + \|u_0\|_\infty \int_0^\tau \frac{1}{\delta(s)}\, ds\bigg)\, d\tau.$$

Since

$$u_2(x,t) = u_0(x) + \int_0^t u_1\bigg(au_1(x,\tau) + \int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} u_1(\xi, s)\, d\xi\, ds, \tau\bigg)\, d\tau,$$

we have

$$
\begin{aligned}
|u_2(x,t)-u_1(x,t)| \le \int\limits_0^t \Bigg[\Bigg| & u_1\Bigg(au_1(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)} \int\limits_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)\, d\xi\, ds, \tau \Bigg) \\
& - u_0\Bigg(au_1(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)} \int\limits_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)\, d\xi\, ds \Bigg)\Bigg| \\
& + \Bigg| u_0\Bigg(au_1(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)} \int\limits_{x-\delta(s)}^{x+\delta(s)} u_1(\xi,s)\, d\xi\, ds \Bigg) \\
& - u_0\Bigg(au_0(x) + \int\limits_0^\tau \frac{1}{2\delta(s)} \int\limits_{x-\delta(s)}^{x+\delta(s)} u_0(\xi)\, d\xi\, ds \Bigg)\Bigg|\Bigg]\, d\tau \\
\le & \int\limits_0^t \Bigg[A_1(\tau) + L_0\Bigg(|a| A_1(\tau) + \int\limits_0^\tau A_1(s)\, ds \Bigg)\Bigg] d\tau \\
= & \int\limits_0^t \Bigg[\big(1 + L_0|a|\big) A_1(\tau) + L_0 \int\limits_0^\tau A_1(s)\, ds \Bigg] d\tau \\
:= & A_2(t).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& |u_2(x,t)-u_2(y,t)| \\
& \quad \le L_0|x-y| + \int\limits_0^t L_1(\tau)\Bigg[|a|L_1(\tau) + \|u_1\|_{L^\infty(\mathbb{R}\times[0,T_1)])} \int\limits_0^\tau \frac{1}{\delta(s)}\, ds \Bigg] |x-y|\, d\tau \\
& \quad = |x-y| \Bigg[L_0 + \int\limits_0^t L_1(\tau) \Bigg(|a| L_1(\tau) + e^{T_1}\|u_0\|_\infty \int\limits_0^\tau \frac{1}{\delta(s)}\, ds \Bigg) d\tau \Bigg] \\
& \quad = L_2(t)|x-y|,
\end{aligned}
$$

where

$$
L_2(t) = L_0 + \int\limits_0^t L_1(\tau) \Bigg(|a| L_1(\tau) + e^{T_1}\|u_0\|_\infty \int\limits_0^\tau \frac{1}{\delta(s)}\, ds \Bigg) d\tau.
$$

By induction on n, we get

$$
\begin{aligned}
& |u_{n+1}(x,t)-u_{n+1}(y,t)| \\
& \quad \le L_0|x-y| + \int\limits_0^t L_n(\tau)\Bigg[|a|L_n(\tau) + \|u_1\|_{L^\infty(\mathbb{R}\times[0,T_1])} \int\limits_0^\tau \frac{1}{\delta(s)}\, ds \Bigg] |x-y|\, d\tau \\
& \quad = |x-y| \Bigg[L_0 + \int\limits_0^t L_n(\tau) \Bigg(|a| L_n(\tau) + e^{T_1}\|u_0\|_\infty \int\limits_0^\tau \frac{1}{\delta(s)}\, ds \Bigg) d\tau \Bigg] \\
& \quad = L_{n+1}(t)|x-y|,
\end{aligned}
$$

where

$$
L_{n+1}(t) = L_0 + \int\limits_0^t L_n(\tau) \Bigg(|a| L_n(\tau) + e^{T_1}\|u_0\|_\infty \int\limits_0^\tau \frac{1}{\delta(s)}\, ds \Bigg) d\tau.
$$

It is easy also to prove that

$$|u_{n+1}(x,t) - u_n(x,t)| \leq A_{n+1}(t), \ \ \forall\, x \in \mathbb{R}, \ \ 0 \leq t \leq T_1, \ \ \forall\, n \in \mathbb{N},$$

and

$$|u_{n+1}(x,t) - u_{n+1}(y,t)| \leq L_{n+1}|x-y|, \ \ \forall\, x, y \in \mathbb{R}, \ \ 0 \leq t \leq T_1, \ \ \forall\, n \in \mathbb{N},$$

where

$$A_{n+1}(t) = \int_0^t \left[\big(1 + |a| L_n(\tau)\big) A_n(\tau) + L_n(\tau) \int_0^\tau A_n(s)\, ds \right] d\tau,$$

$$L_{n+1}(t) = L_0 + \int_0^t L_n(t) \left(|a| L_n(\tau) + e^{T_1} \|u_0\|_\infty \int_0^\tau \frac{1}{\delta(s)}\, ds \right) d\tau.$$

Let $M_0 > 0$, putting $c^\star = \int_0^{T_1} \frac{1}{\delta(s)}\, ds$, $T_2 \leq T_1$, such that for every $0 \leq t \leq T_2$,

$$L_0 \Big(|a| L_0 + e^{T_1} \|u_0\|_\infty c^\star \Big) t \leq M_0,$$
$$(M_0 + L_0) \Big[\, |a|(M_0 + L_0) + e^{T_1} \|u_0\|_\infty c^\star \Big] t \leq M_0,$$
$$\frac{t^2}{2} (M_0 + L_0) + \big((M_0 + L_0)|a| + 1 \big) t \leq h < 1,$$

we can see that

$$0 \leq L_1((t) - L_0 \leq \int_0^t L_0 \left(|a| L_0 + e^{T_1} \|u_0\|_\infty \int_0^{T_1} \frac{1}{\delta(s)}\, ds \right) d\tau$$
$$\leq \int_0^t L_0 \Big(|a| L_0 + e^{T_1} \|u_0\|_\infty c^\star \Big)\, d\tau \leq L_0 \Big(|a| L_0 + e^{T_1} \|u_0\|_\infty c^\star \Big) t \leq M_0$$

and

$$0 \leq L_2(t) - L_0$$
$$\leq \int_0^t L_1(\tau) \left(|a| L_1(\tau) + e^{T_1} \|u_0\|_\infty \int_0^{T_1} \frac{1}{\delta(s)}\, ds \right) d\tau \leq \int_0^t L_1(\tau) \Big(|a| L_1(\tau) + e^{T_1} \|u_0\|_\infty c^\star \Big)\, d\tau$$
$$= \int_0^t \big[L_1(\tau) - L_0 + L_0 \big] \Big(|a| \big[L_1(\tau) - L_0 + L_0 \big] + e^{T_1} \|u_0\|_\infty c^\star \Big)\, d\tau$$
$$\leq \int_0^t (M_0 + L_0) \Big[\, |a|(M_0 + L_0) + e^{T_1} \|u_0\|_\infty c^\star \Big]\, d\tau$$
$$\leq (M_0 + L_0) \Big[\, |a|(M_0 + L_0) + e^{T_1} \|u_0\|_\infty c^\star \Big] t \leq M_0.$$

By induction on n, we get

$$0 \leq t \leq T_2, \quad 0 \leq L_n(t) - L_0 \leq M_0, \ \ \forall\, n \in \mathbb{N}.$$

So, we obtain

$$0 \leq t \leq T_2, \quad 0 \leq L_n(t) \leq L_0 + M_0, \ \ \forall\, n \in \mathbb{N}.$$

As a result, it follows that

$$\begin{aligned}A_{n+1}(t) &= \int\limits_0^t \left[\big(1+|a|L_n(\tau)\big)A_n(\tau) + L_n(\tau)\int\limits_0^\tau A_n(s)\,ds\right]d\tau \\ &\le \int\limits_0^t \left[\big(1+|a|(M_0+L_0)\big)\|A_n\|_{L^\infty([0,T_2])} + \tau(M_0+L_0)\|A_n\|_{L^\infty([0,T_2])}\right]d\tau \\ &= \|A_n\|_{L^\infty([0,T_2])}\left[\big(1+|a|(M_0+L_0)\big)t + (M_0+L_0)\frac{t^2}{2}\right] \le h\|A_n\|_{L^\infty([0,T_2])}.\end{aligned}$$

In conclusion, the series

$$\sum_{n=0}^\infty \|A_n\|_{L^\infty([0,T_2])}$$

is convergent and we deduce that the sequence (u_n) is uniformly convergent to a function $u^\star \in X$. It is easy to prove that $u^\star$ is the solution of Problem (1.7), $\forall\, x\in\mathbb{R},\ t\in[0,T_2]$. Moreover, the solution $u^\star$ is unique. In fact, we assume that $\overline{u}(x,t)$ is also the solution of Problem (1.7). Then we have

$$\begin{aligned}\big|\overline{u}(x,t)-u^\star(x,t)\big| &= \Bigg|u_0(x) + \int\limits_0^t \overline{u}\bigg(a\overline{u}(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} \overline{u}(\xi,\tau)\,d\xi\,ds,\tau\bigg)d\tau \\ &\qquad - \bigg[u_0(x) + \int\limits_0^t u^\star\bigg(au^\star(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} u^\star(\xi,\tau)\,d\xi\,ds,\tau\bigg)d\tau\bigg]\Bigg| \\ &\le \Bigg|\int\limits_0^t \overline{u}\bigg(a\overline{u}(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} \overline{u}(\xi,\tau)\,d\xi\,ds,\tau\bigg)d\tau \\ &\qquad - \int\limits_0^t u^\star\bigg(au^\star(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} u^\star(\xi,\tau)\,d\xi\,ds,\tau\bigg)d\tau\Bigg| \\ &\le \int\limits_0^t \Bigg|\overline{u}\bigg(a\overline{u}(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} \overline{u}(\xi,\tau)\,d\xi\,ds,\tau\bigg) \\ &\qquad\qquad - u^\star\bigg(a\overline{u}(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} \overline{u}(\xi,\tau)\,d\xi\,ds,\tau\bigg)\Bigg|d\tau \\ &\qquad + \int\limits_0^t \Bigg|u^\star\bigg(a\overline{u}(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} \overline{u}(\xi,\tau)\,d\xi\,ds,\tau\bigg) \\ &\qquad\qquad - u^\star\bigg(au^\star(x,\tau) + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} u^\star(\xi,\tau)\,d\xi\,ds,\tau\bigg)\Bigg|d\tau \\ &\le \int\limits_0^t \|\overline{u}-u^\star\|_\infty\,d\tau \\ &\qquad + \int\limits_0^t (M_0+L_0)\bigg[|a|\,\|\overline{u}-u^\star\|_\infty + \int\limits_0^\tau \frac{1}{2\delta(s)}\int\limits_{x-\delta(s)}^{x+\delta(s)} \|\overline{u}-u^\star\|_\infty\,d\xi\,ds\bigg]d\tau\end{aligned}$$

$$
\begin{aligned}
&= \|\overline{u} - u^\star\|_\infty t + (M_0 + L_0)\|\overline{u} - u^\star\|_\infty \int_0^t \left[|a| + \int_0^\tau \frac{1}{2\delta(s)} \int_{x-\delta(s)}^{x+\delta(s)} d\xi \, ds \right] d\tau \\
&= \|\overline{u} - u^\star\|_\infty t + (M_0 + L_0)\|\overline{u} - u^\star\|_\infty \int_0^t \left[|a| + \int_0^\tau ds \right] d\tau \\
&= \|\overline{u} - u^\star\|_\infty t + (M_0 + L_0)\|\overline{u} - u^\star\|_\infty \int_0^t \left[|a| + \tau \right] d\tau \\
&= \|\overline{u} - u^\star\|_\infty \Big[(M_0 + L_0)\Big(|a|t + \frac{t^2}{2} \Big) + t \Big] \\
&= \|\overline{u} - u^\star\|_\infty \Big[\frac{t^2}{2} (M_0 + L_0) + \big((M_0 + L_0)|a| + 1 \big) t \Big] \\
&\le \|\overline{u} - u^\star\|_\infty \cdot h.
\end{aligned}
$$

Since $0 < h < 1$, we deduce $\|\overline{u} - u^\star\|_\infty = 0$. We conclude that $u^\star(x,t)$ is the unique solution of (1.7).

Moreover, from (2.1) it follows that

$$|u^\star(x,t) - u^\star(y,t)| \le (M_0 + L_0)|x - y|, \quad \forall\, x, y \in \mathbb{R}, \quad t \in [0, T_2],$$

and from (2.2) it follows

$$|u^\star(x,t_2) - u^\star(x,t_1)| \le e^{T_1} \|u_0\|_\infty |t_2 - t_1|,$$

hence $u^\star$ is Lipschitz in the first variable (uniformly with respect to the second one) and is Lipschitz in the second variable (uniformly with respect to the first one).

To summarize, the following theorem is proved.

Theorem. *Suppose* (M_1)–(M_4) *hold. There exists* $T > 0$ *such that Problem* (1.7) *has a unique solution* $u \in \mathbb{C}(\mathbb{R} \times [0,T], \mathbb{R}) \cap L^\infty(\mathbb{R} \times [0,T], \mathbb{R})$*. In addition,* u *is Lipschitz in the first variable (uniformly with respect to the second one), and Lipschitz in the second variable (uniformly with respect to the first one).*

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