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NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS AND PARAMETER DEPENDENCE

Abstract. This work is concerned with studying the existence of positive solutions for nonlinear fractional differential equations problems with integral boundary conditions and parameter dependence:

$$D_{0^+}^{\beta} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 \psi(r) u(r)\, dr,$$

where $2 < \beta \leq 3$ and $\lambda > 0$, $D_{0^+}^{\beta}$ is the Riemann–Liouville fractional derivative, f is a continuous function and ψ is a continuous function on $[0, 1]$. Using the fixed point theorem on the cone, we show when this type of problem has at least one solution.

Some examples are included to illustrate the main results.

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1 Introduction

Fractional differential equations have become a highly intriguing research field in recent years. These equations find applications in numerous areas, including engineering and sciences, such as rheology, viscoelasticity, and electromagnetism (see [6–10, 10, 11, 16]).

Recently, several research papers have explored the existence and multiplicity of positive solutions for nonlinear fractional differential equations using nonlinear analysis techniques, particularly fixed-point theorems (see [1–4, 12–15, 17]).

In the paper, the authors presented several results utilizing the properties of the Green function and the fixed-point theorem on a cone. Moreover, their goal was to determine a λ-interval such that the following problem exhibits both the existence and multiplicity of positive solutions:

$$D_{0+}^{\beta}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 \psi(r)u(r)\,dr, \tag{1.2}$$

where $2 < \beta \leq 3$ and $\lambda > 0$, D_{0+}^{β} is the Riemann–Liouville fractional derivative of order β and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function and ψ is a continuous function on $[0,1]$.

In [17], Yige Zhao et al. examined the existence of positive solutions for the following problem:

$$D_{0+}^{\beta}u(t) + \lambda f(u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u(1) = 0,$$

where $2 < \beta \leq 3$ and $\lambda > 0$, D_{0+}^{β} is the Riemann–Liouville fractional derivative and $f \in [0,1] \times [0,+\infty) \to [0,+\infty)$. In their paper, they presented some results using the properties of the Green function and the fixed point theorem on the cone, furthermore, their purpose was to derive a $\lambda-$interval such that the problem has both the existence and multiplicity of positive solutions.

In their papers, Hafida Abbas et al. [1] and Hamza Tabti et al. [13] utilized the integral boundary conditions and parameter dependence

$$u(0) = 0, \quad u(1) = \lambda \int_0^1 h(r)u(r)\,dr.$$

The paper is divided into four sections. In Section 2, the Green function is presented by the study of the problem

$$D_{0+}^{\beta}u(t) + y(t) = 0, \quad 0 < t < 1, \tag{1.3}$$

with condition (1.2), where $2 < \beta \leq 3$ and $\lambda > 0$. We will give some results and properties that will be use later. In Section 3, we present the main results of this paper by utilizing the properties of the Green function obtained in Section 2 and the fixed-point theorem. This aims to establish the sufficient conditions for problem (1.1), (1.2) to have at least one solution. We conclude the study with some examples to illustrate the obtained results.

2 Preliminaries

In the following, we will give some definitions and lemmas of fractional integrals and derivatives, concentrating on the Riemann–Liouville fractional derivative, which can be found in the recent literature (see [7, 10, 11]).

Definition 2.1 ([7])**.** The Riemann–Liouville fractional integral operator of order $\beta > 0$ for a function $f : (0,+\infty) \to \mathbb{R}$ is defined as

$$I_{0+}^{\beta}f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,ds,$$

provided that the right-hand side is defined pointwise on $(0,+\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([7])**.** The Riemann–Liouville fractional derivative operator of order $\beta > 0$ of a continuous function $f:(0,+\infty)\to\mathbb{R}$ is given by

$$D^{\beta}_{0+}f(t)=\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^{n}\int\limits_{0}^{t}(t-s)^{n-\beta-1}f(s)\,ds,$$

where $n=[\beta]+1$, $[\beta]$ denotes the integral part of the number β, provided the right-hand side is defined pointwise on $(0,+\infty)$.

Lemma 2.1 ([10])**.** *Let $\beta>0$ and $f\in\mathcal{L}^1(0,1)$, then*

$$D^{\beta}_{0+}I^{\beta}_{0+}f(t)=f(t)$$

holds almost everywhere on $(0,1)$.

Lemma 2.2 ([10])**.** *Let $\beta>0$. If we assume that $f\in\mathcal{C}(0,1)\cap\mathcal{L}(0,1)$, then the solutions of the fractional differential equation*

$$D^{\beta}_{0+}f(t)=0$$

are given by the following expression:

$$f(t)=c_1t^{\beta-1}+c_2t^{\beta-2}+\dots+c_nt^{\beta-n},$$

where $c_i\in\mathbb{R}$; $i=1,2,\dots,n$ with $n-1<\beta\le n$.

From Lemma 2.2, we conclude the following result.

Lemma 2.3 ([10])**.** *Assume that $f\in\mathcal{C}(0,1)\cap\mathcal{L}(0,1)$ with a fractional derivative of order $\beta>0$ that belongs to $\mathcal{C}(0,1)\cap\mathcal{L}(0,1)$. Then*

$$I^{\beta}_{0+}D^{\beta}_{0+}f(t)=f(t)+c_1t^{\beta-1}+c_2t^{\beta-2}+\dots+c_nt^{\beta-n},$$

where $c_i\in\mathbb{R}$; $i=1,2,\dots,n$ with $n-1<\beta\le n$.

Below, we give the Green function for problem (1.3) with conditions (1.2) and some of its properties that we use to prove our main results.

Lemma 2.4. *Let $2<\beta\le 3$. Suppose that $1-\lambda\int\limits_0^1\psi(r)r^{\beta-1}\,dr\neq 0$ and $y\in\mathcal{C}[0,1]$, then the boundary value problem (1.3), (1.2) has the unique solution $u\in\mathcal{C}[0,1]$ defined by the expression*

$$u(t)=\int\limits_0^1 G(t,s)y(s)\,ds,$$

where $G(t,s)$ is the Green function given by

$$G(t,s)=G_1(t,s)+G_2(t,s)$$

with

$$G_1(t,s)=\begin{cases}\dfrac{t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0\le s\le t\le 1,\\[2ex] \dfrac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0\le t\le s\le 1,\end{cases}$$

and

$$G_2(t,s)=\frac{\lambda t^{\beta-1}}{1-\lambda\int\limits_0^1\psi(r)r^{\beta-1}\,dr}\int\limits_0^1\psi(r)G_1(r,s)\,dr. \tag{2.1}$$

Proof. By Lemma 2.3, we find that u is a solution of the linear equation (1.3) if and only if it satisfies

$$u(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + c_1 t^{\beta-1} + c_2 t^{\beta-2} + c_3 t^{\beta-3}.$$

Since $u(0) = u'(0) = 0$, we have $c_2 = c_3 = 0$, and conclude that

$$u(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + c_1 t^{\beta-1}.$$

By the condition $u(1) = \lambda \int_0^1 \psi(r) u(r)\, dr$, we obtain

$$c_1 = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)\, ds + \lambda c_1 \int_0^1 \psi(r) r^{\beta-1}\, dr - \frac{\lambda}{\Gamma(\beta)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr.$$

Now, since $1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr \neq 0$, we have

$$c_1 = \frac{1}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \left(\int_0^1 (1-s)^{\beta-1} y(s)\, ds - \lambda \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr \right).$$

With the same calculation as used in [1], we obtain the following form:

$$\begin{aligned}
u(t) = &-\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 (1-s)^{\beta-1} y(s)\, ds \\
&- \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr \\
= &-\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr + \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 (1-s)^{\beta-1} y(s)\, ds \\
&- \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr \\
= &-\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)\, ds \\
&+ \frac{\lambda t^{\beta-1} \int_0^1 \psi(r) r^{\beta-1}\, dr}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 (1-s)^{\beta-1} y(s)\, ds \\
&- \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1 - \lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^t (1-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_t^1 (1-s)^{\beta-1} y(s)\, ds \\
&\qquad + \frac{\lambda t^{\beta-1}}{\Gamma(\beta)\Big(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr\Big)} \int_0^1 \psi(r) r^{\beta-1}\, dr \cdot \int_0^1 (1-s)^{\beta-1} y(s)\, ds \\
&\qquad - \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr \\
&= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^t (1-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_t^1 (1-s)^{\beta-1} y(s)\, ds \\
&\qquad + \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^1 r^{\beta-1}(1-s)^{\beta-1} y(s)\, ds\, dr \\
&\qquad - \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr \\
&= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^t (1-s)^{\beta-1} y(s)\, ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_t^1 (1-s)^{\beta-1} y(s)\, ds \\
&\qquad + \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r r^{\beta-1}(1-s)^{\beta-1} y(s)\, ds\, dr \\
&\qquad - \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_0^r (r-s)^{\beta-1} y(s)\, ds\, dr \\
&\qquad + \frac{\lambda t^{\beta-1}}{\Gamma(\beta)(1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr)} \int_0^1 \psi(r) \int_r^1 r^{\beta-1}(1-s)^{\beta-1} y(s)\, ds\, dr \\
&= \int_0^1 G_1(t,s) y(s)\, ds + \frac{\lambda t^{\beta-1}}{1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr} \int_0^1 \psi(r) \int_0^1 G_1(r,s) y(s)\, ds\, dr \\
&= \int_0^1 G_1(t,s) y(s)\, ds + \int_0^1 \frac{\lambda t^{\beta-1}}{1-\lambda \int_0^1 \psi(r) r^{\beta-1}\, dr} \int_0^1 \psi(r) G_1(r,s)\, dr y(s)\, ds \\
&= \int_0^1 G_1(t,s) y(s)\, ds + \int_0^1 G_2(t,s) y(s)\, ds \\
&= \int_0^1 G(t,s) y(s)\, ds. \qquad \square
\end{aligned}
$$

Lemma 2.5 ([17]). *The function G_1 presented in Lemma 2.4 has the following properties:*

(i) $G_1(t,s) > 0 \ \forall\, t,s \in (0,1)$.

(ii) $G_1(t,s)$ *is continuous* $\forall\, t,s \in [0,1]$.

(iii) $G_1(t,s) = G_1(1-s, 1-t) \ \forall\, t,s \in [0,1]$.

(iv) $t^{\beta-1}(1-t)s(1-s)^{\beta-1} \le \Gamma(\beta)G_1(t,s) \le (\beta-1)s(1-s)^{\beta-1}$ *for all* $t,s \in [0,1]$.

In the following lemma, we give an important inequality that will be used in this paper.

Lemma 2.6. *Denote* $K = \int_0^1 \psi(r) r^{\beta-1}\, dr$, $M = \int_0^1 \psi(r)\, dr$, $N = \int_0^1 \psi(r) r^{\beta-1}(1-r)\, dr$, *and assume that* $\psi \ge 0$ *on* $[0,1]$ *and* $1 - \lambda K > 0$ *with* $\lambda > 0$. *Then the Green function* $G(t,s)$ *defined in Lemma 2.4 satisfies the following inequalities:*

$$\frac{\lambda t^{\beta-1} N s(1-s)^{\beta-1}}{\Gamma(\beta)(1-\lambda K)} \le G(t,s) \le \frac{(\beta-1)s(1-s)^{\beta-1}}{\Gamma(\beta)} \left[1 + \frac{\lambda M}{1-\lambda K}\right] \quad \forall\, t,s \in [0,1]. \tag{2.2}$$

Proof. From the definition of G and using Lemma 2.5 part (iv), we obtain

$$\begin{aligned} G(t,s) = G_1(t,s) + G_2(t,s) &\le \frac{(\beta-1)s(1-s)^{\beta-1}}{\Gamma(\beta)} + \frac{\lambda t^{\beta-1}}{1-\lambda K} \int_0^1 \psi(r) G_1(r,s)\, dr \\ &\le \frac{(\beta-1)s(1-s)^{\beta-1}}{\Gamma(\beta)} + \frac{\lambda t^{\beta-1}}{1-\lambda K} \int_0^1 \psi(r) \frac{(\beta-1)s(1-s)^{\beta-1}}{\Gamma(\beta)}\, dr \\ &\le \frac{(\beta-1)s(1-s)^{\beta-1}}{\Gamma(\beta)} \left[1 + \frac{\lambda M}{1-\lambda K}\right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} G(t,s) = G_1(t,s) + G_2(t,s) \ge G_2(t,s) &= \frac{\lambda t^{\beta-1}}{1-\lambda K} \int_0^1 \psi(r) G_1(r,s)\, dr \\ &\ge \frac{\lambda t^{\beta-1}}{1-\lambda K} \int_0^1 \psi(r) \frac{r^{\beta-1}(1-r)s(1-s)^{\beta-1}}{\Gamma(\beta)}\, dr \ge \frac{\lambda t^{\beta-1} N s(1-s)^{\beta-1}}{\Gamma(\beta)(1-\lambda K)}. \end{aligned}$$

The proof is complete. □

The aim of this work is to establish the existence of positive solutions for the boundary value problem (1.1), 1.2 by the following fixed point theorem.

Firstly, let us define the concept of a cone.

Definition 2.3 ([5]). Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone if it satisfies the following two conditions:

(1) $u \in \mathcal{P}$, $\varrho \ge 0$ implies $\varrho u \in \mathcal{P}$;

(2) $u \in \mathcal{P}$, $-u \in \mathcal{P}$ implies $u = 0$.

Theorem 2.1 ([5]). *Let E be a Banach space and let $\mathcal{P} \subset E$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $\mathcal{T} : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that one of following assertions is fulfilled:*

(i) $\|\mathcal{T}u\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|\mathcal{T}u\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$;

(ii) $\|\mathcal{T}u\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|\mathcal{T}u\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then the operator $\mathcal{T}$ has at least one fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we establish the existence of positive solutions for problem (1.1), (1.2).

Let $E = C[0,1]$ be the Banach space endowed with the norm $\|u\| = \max 0 \le t \le 1|u(t)|$. Let us consider the following supposition:

(f) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function.

Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} = \left\{ u \in E : \ u(t) \ge 0, \ \text{for all} \ t \in [0,1], \ u(t) \ge \frac{t^{\beta-1}\Lambda}{\beta - 1}\|u\|, \ \text{for all} \ t \in \left[\frac{1}{4}, 1\right] \right\},$$

where

$$\Lambda = \frac{\lambda N}{1 + \lambda(M - K)},$$

K M and N are defined in Lemma 2.6. Set

$$f^0 = \lim_{u\to 0^+} \left\{ \max_{0\le t\le 1} \frac{f(t,u)}{u} \right\} \ \text{and} \ f_\infty = \lim_{u\to\infty} \left\{ \min_{1/4\le t\le 1} \frac{f(t,u)}{u} \right\}.$$

We define the operator $\mathcal{T} : \mathcal{P} \to E$ by

$$\mathcal{T}u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds,$$

with G given in Lemma 2.4.

It is simple to see that the fixed points of the operator $\mathcal{T}$ are the solutions of problem (1.1), (1.2).

Now, we are in a position to show the main result of our work.

Theorem 3.1. *Suppose that condition (f) holds coupled with the following hypothesis:*

(i) *(superlinear case)* $f^0 = 0, \ f_\infty = \infty.$

Then, for all $2 \le \beta < 3$, $\lambda > 0$ and $1 - \lambda K > 0$, problem (1.1), (1.2) has at least one positive solution $u \in \mathcal{P}$.

Proof. First, we prove that the operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Since f and $G(t,s)$ are continuous and positive functions, it follows that if $u \in \mathcal{P}$, than $\mathcal{T}u \in E$ and $\mathcal{T}u(t) \ge 0$ for all $t \in [0,1]$.

Let us demonstrate that $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$. Take $u \in \mathcal{P}$. By Lemma 2.6, we have

$$\begin{aligned}
\mathcal{T}u(t) &= \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\ge \lambda N t^{\beta-1} \int_0^1 \frac{s(1-s)^{\beta-1}}{\Gamma(\beta)(1-\lambda K)} f(s,u(s))\, ds \ge \frac{t^{\beta-1}\Lambda}{\beta-1} \int_0^1 \max_{0\le t\le 1}\{G(t,s)\} f(s,u(s))\, ds \\
&= \frac{t^{\beta-1}\Lambda}{\beta-1} \max_{0\le t\le 1} \left\{ \int_0^1 G(t,s) f(s,u(s))\, ds \right\} = \frac{t^{\beta-1}\Lambda}{\beta-1} \|\mathcal{T}u\|.
\end{aligned}$$

Thus $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$. In view of the continuity of the functions G and f, the operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is continuous.

Let $\mathcal{F} \subset \mathcal{P}$ be a bounded set which means that there exists $L > 0$ such that $\mathcal{F} = \{u \in \mathcal{P} : \ \|u\| \le L\}$. Let

$$R = \max_{0\le t\le 1,\, 0\le u\le L} |f(t,u)|.$$

From inequality (2.2) and for all $u \in \mathcal{F}$, we have

$$|\mathcal{T}u(t)| = \Big| \int_0^1 G(t,s) f(s,u(s))\, ds \Big| \le \frac{R(\beta-1)}{\Gamma(\beta)} \Big(1 + \frac{\lambda M}{1-\lambda K}\Big) \int_0^1 s(1-s)^{\beta-1}\, ds = \frac{R(\beta-1)}{\Gamma(\beta+2)} \Big(1 + \frac{\lambda M}{1-\lambda K}\Big).$$

Thus $\mathcal{T}(\mathcal{F})$ is bounded. On the other hand, let $\epsilon > 0$ and set

$$\delta = \frac{1}{2} \Big(\frac{\epsilon}{\vartheta R}\Big)^{1/(\beta-1)},$$

where the value of ϑ will be given later.

Now, we prove that $\mathcal{T}$ is equicontinuous, i.e., there exists a constant δ such that whenever $t_1, t_2 \in [0,1]$ and $0 < t_2 - t_1 < \delta$, we have $|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)| < \epsilon$.

We have

$$\begin{aligned} \big|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)\big| &= \Big| \int_0^1 \big[G(t_2,s) - G(t_1,s)\big] f(s,u(s))\, ds \Big| \\ &\le \int_0^1 \big|G(t_2,s) - G(t_1,s)\big| f(s,u(s))\, ds \le R \int_0^1 \big|G(t_2,s) - G(t_1,s)\big|\, ds \\ &\le R\Big(\int_0^1 \big|G_1(t_2,s) - G_1(t_1,s)\big|\, ds + \int_0^1 \big|G_2(t_2,s) - G_2(t_1,s)\big|\, ds \Big). \end{aligned}$$

Using the same method applied in [13], from the definition of $G_1(t,s)$, we obtain

$$\begin{aligned} \int_0^1 \big|G_1(t_2,s) - G_1(t_1,s)\big|\, ds &= \int_0^{t_1} \big|G_1(t_2,s) - G_1(t_1,s)\big|\, ds + \int_{t_1}^{t_2} \big|G_1(t_2,s) - G_1(t_1,s)\big|\, ds \\ &\quad + \int_{t_2}^1 \big|G_1(t_2,s) - G_1(t_1,s)\big|\, ds \le \frac{1}{\Gamma(\beta+1)} (t_2^{\beta-1} - t_1^{\beta-1}). \end{aligned}$$

Denote $\psi^* = \max_{0 \le t \le 1} \psi(t)$, so from the expression of $G_2(t,s)$ represented by equation (2.1) and Lemma 2.5, we get

$$\begin{aligned} \int_0^1 \big|G_2(t_2,s) - G_2(t_1,s)\big|\, ds &= \frac{\lambda(t_2^{\beta-1} - t_1^{\beta-1})}{1-\lambda K} \int_0^1 \int_0^1 \psi(r) G_1(r,s)\, dr\, ds \\ &\le \frac{\lambda(\beta-1)(t_2^{\beta-1} - t_1^{\beta-1})}{1-\lambda K} \frac{\psi^*}{\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1}\, ds \le \frac{\beta-1}{\Gamma(\beta+2)} \cdot \frac{\lambda \psi^*}{1-\lambda K} (t_2^{\beta-1} - t_1^{\beta-1}). \end{aligned}$$

Hence, we conclude that

$$\big|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)\big| < R\Big(\frac{1}{\Gamma(\beta+1)} + \frac{\beta-1}{\Gamma(\beta+2)} \cdot \frac{\lambda\psi^*}{1-\lambda A}\Big)(t_2^{\beta-1} - t_1^{\beta-1}) = \vartheta R (t_2^{\beta-1} - t_1^{\beta-1}),$$

where

$$\vartheta = \frac{1}{\Gamma(\beta+1)} + \frac{\beta-1}{\Gamma(\beta+2)} \cdot \frac{\lambda\psi^*}{1-\lambda A}.$$

To estimate $t_2^{\beta-1} - t_1^{\beta-1}$, we apply a method used in [2].

Case 01. $\delta \le t_1 < t_2 < 1$,

$$\left|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)\right| < \vartheta R(t_2^{\beta-1} - t_1^{\beta-1}) \le \vartheta R\, \frac{\beta-1}{\delta^{2-\beta}}\,(t_2 - t_1) \le \vartheta R\delta^{\beta-1} < \epsilon.$$

Case 02. $0 \le t_1 < \delta$, $t_2 < 2\delta$,

$$\left|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)\right| < \vartheta R(t_2^{\beta-1} - t_1^{\beta-1}) \le \vartheta R t_2^{\beta-1} \le \vartheta R(2\delta)^{\beta-1} \le \epsilon.$$

This means that $\mathcal{T}(\mathcal{F})$ is equicontinuous in E.

So, by the Arzelà–Ascoli theorem, the operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Consider now the condition:

(i) (superlinear case) $f^0 = 0,\ f_\infty = \infty$.

Choose the constant $\alpha_2 > 0$ defined as

$$\alpha_2 = \frac{(1-\lambda K)\Gamma(\beta+2)}{(\beta-1)[1+\lambda(M-K)]}\,.$$

Since $f^0 = 0$, we have that there exists a constant $\sigma_1 > 0$ such that $f(t,u) \le \alpha_2 u$ for all $0 \le u \le \sigma_1$.

Taking $u \in \mathcal{P}$ such that $\|u\| = \sigma_1$, we have

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1}\left\{\int_0^1 G(t,s) f(s,u(s))\,ds\right\}\\ &\le \alpha_2\|u\|\,\frac{(\beta-1)[1+\lambda(M-K)]}{1-\lambda K}\int_0^1 \frac{s(1-s)^{\beta-1}}{\Gamma(\beta)}\,ds \le \alpha_2\|u\|\,\frac{(\beta-1)[1+\lambda(M-K)]}{(1-\lambda K)\Gamma(\beta+2)} = \|u\|.\end{aligned}$$

We define $\alpha_1 > 0$ as follows:

$$\alpha_1 = \frac{16(1-\lambda K)\Gamma(\beta+2)}{\lambda N}\,. \tag{3.1}$$

The fact $f_\infty = \infty$ implies that there exists a constant $\sigma_2 > \sigma_1 > 0$ with $16(\beta-1)\sigma_2 > \Lambda\sigma_1$ such that $f(t,u) \ge \alpha_1 u$ for all $u \ge \sigma_2$.

Let now $u \in \mathcal{P}$ be such that $\|u\| = \sigma_2 \frac{16(\beta-1)}{\Lambda}$. Based on the definition of the cone $\mathcal{P}$, we have $u(t) \ge \sigma_2$ for all $t \in [1/4, 1]$.

Then, condition (i) along with equation (3.1) lead to the following properties:

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1}\left\{\int_0^1 G(t,s) f(s,u(s))\,ds\right\} \ge \max_{1/4\le t\le 1}\left\{\int_0^1 G(t,s) f(s,u(s))\,ds\right\}\\ &\ge \alpha_1 \max_{1/4\le t\le 1}\left\{\frac{\lambda N t^{\beta-1}}{1-\lambda K}\right\}\int_0^1 \frac{s(1-s)^{\beta-1}}{\Gamma(\beta)}\,u(s)\,ds \ge \alpha_1\|u\|\,\frac{\lambda N}{16(1-\lambda K)\Gamma(\beta+2)} = \|u\|.\end{aligned}$$

Thus, bymultlinethe second part of the Guo–Krasnoselskii fixed point theorem, we conclude that problem (1.1), (1.2) has at least one positive solution. □

4 Example

Now, we give an example to illustrate our results.

Example. Consider this fractional differential equation

$$\begin{cases} D_{0^+}^{5/2} u(t) + u^3(t) + u^2(t)\ln(t+1) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \int\limits_0^1 s\sqrt{s}\, u(s)\, ds. \end{cases} \tag{4.1}$$

In system (4.1), we see that $\beta = \frac{5}{2}$, $\lambda = 1$ and $\psi(t) = t\sqrt{t}$. Then

$$K = \int\limits_0^1 h(r) r^{3/2}\, dr = \int\limits_0^1 r^3\, dr = \frac{1}{4}$$

satisfies the condition

$$1 - \lambda K = 1 - \frac{1}{4} > 0.$$

Clearly, for every $u > 0$, it is easy to see that

$$\min_{1/4 \le t \le 1} \frac{f(t,u)}{u} = u^2 + u \ln \frac{5}{4}$$

and

$$\max_{0 \le t \le 1} \frac{f(t,u)}{u} = u^2 + u \ln 2.$$

Obviously, $f^0 = 0$, $f_\infty = \infty$. From the second part of Theorem 3.1, we can conclude that problem (4.1) has at least one positive solution.

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