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SOME ESTIMATES FOR HARDY-STEKLOV TYPE OPERATORS

Abstract. The aim of this work is to establish some new integral inequalities for $0 < p < 1$ under weaker condition than monotonicity via Hardy–Steklov-type operators.

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1 Introduction

It is well-known that for Lebesgue spaces L_p with $0 < p < 1$, the Hardy inequality is not satisfied for arbitrary non-negative measurable functions, but is satisfied for monotone functions (see [2]). In 2007, the Hardy type inequality was obtained under a still weaker condition than monotonicity (see [3]). Namely, the following statements were proved.

Lemma 1.1. *Let $0 < p < 1$, $c_1 > 0$ and f be a non-negative measurable function on $(0, \infty)$ such that for all $x > 0$,*

$$f(x) \leq \frac{c_1}{x} \left(\int_0^x f^p(y) y^{p-1} \, dy \right)^{\frac{1}{p}}. \tag{1.1}$$

Then

$$\left(\int_0^x f(y) \, dy \right)^p \leq c_2 \int_0^x f^p(y) y^{p-1} \, dy, \tag{1.2}$$

where

$$c_2 = c_1^{p(1-p)}.$$

The classical Hardy operators are defined as follows:

$$(H_1 f)(x) = \frac{1}{x} \int_0^x f(y) \, dy, \quad (H_2 f)(x) = \frac{1}{x} \int_x^\infty f(y) \, dy.$$

Theorem 1.1 ([3])**.** *Let $0 < p < 1$, $\alpha < 1 - \frac{1}{p}$ and $c_1 > 0$. If f is non-negative measurable function on $(0, \infty)$ and satisfies (1.1) for all $x > 0$, then*

$$\left\| x^\alpha (H_1 f)(x) \right\|_{L_p(0,\infty)} \leq c_3 \| x^\alpha f(x) \|_{L_p(0,\infty)}, \tag{1.3}$$

where

$$c_3 = c_1^{1-p} \left(1 - \alpha - \frac{1}{p} \right)^{-\frac{1}{p}} p^{-\frac{1}{p}}.$$

The constant c_3 is sharp (the best possible).

Remark 1.1. If f is a non-increasing function on $(0, \infty)$, then (1.1) is satisfied with $c_1 = p^{\frac{1}{p}}$. For such functions inequality (1.3) takes the form

$$\| x^\alpha (H_1 f)(x) \|_{L_p(0,\infty)} \leq \left(p^p \left(1 - \alpha - \frac{1}{p} \right) \right)^{-\frac{1}{p}} \| x^\alpha f(x) \|_{L_p(0,\infty)}. \tag{1.4}$$

The factor $(p^p(1 - \alpha - \frac{1}{p}))^{-\frac{1}{p}}$ is sharp. Inequality (1.4) was proved earlier (for more details, see [2]).

The well-known Hardy–Steklov operator is defined as

$$(Tf)(x) = \frac{1}{x} \int_{a(x)}^{b(x)} f(y) \, dy$$

with the boundary functions $a(x)$, $b(x)$ satisfying the following conditions:

(1) $a(x)$, $b(x)$ are differentiable and strictly increasing functions on $[0, \infty]$,

(2) $0 < a(x) < b(x) < \infty$ for $0 < x < \infty$, $a(0) = b(0) = 0$ and $a(\infty) = b(\infty) = \infty$,

where f is a non-negative Lebesgue measurable function on $(0,\infty)$.

The objective of this work is to extend the results of [3] to Hardy–Steklov type operators T_1, T_2 and T_3 defined as follows:

$$(T_1 f)(x) = \frac{1}{x}\int_0^{b(x)} f(y)\,dy$$

with boundary function $b(x)$ satisfying the following conditions:

(1) $b(x)$ is differentiable and strictly increasing function on $(0,\infty]$,

(2) $0 < b(x) < \infty$ for $0 < x < \infty$ and $b(\infty) = \infty$;

$$(T_2 f)(x) = \frac{1}{x}\int_{a(x)}^{\infty} f(y)\,dy$$

with boundary function $a(x)$ satisfying the following conditions:

(1) $a(x)$ is differentiable and strictly increasing function on $[0,\infty)$,

(2) $0 < a(x) < \infty$ for $0 < x < \infty$ and $a(0) = 0$;

$$(T_3 f)(x) = \frac{1}{x}\int_{a(x)}^{b(x)} f(y)\,dy,$$

where

(1) $a(x)$, $b(x)$ are differentiable and strictly increasing functions on $(0,\infty)$,

(2) $0 < a(x) < b(x) < \infty$ for $0 < x < \infty$.

2 Main results

Throughout the paper, we assume that the function f is a non-negative Lebesgue measurable function on $(0,\infty)$.

Theorem 2.1. *Let $0 < p < 1$, $\alpha < 1 - \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If f is a non-negative measurable function on $(0,\infty)$ and satisfies (1.1) for all $x > 0$, then*

$$\left\| x^{\alpha}(T_1 f)(x)\right\|_{L_p(0,\infty)} \le c_4 \left\| x^{\frac{1}{p'}}(b^{-1}(x))^{\alpha-\frac{1}{p'}} f(x)\right\|_{L_p(0,\infty)},$$

where

$$c_4 = c_1^{1-p}\big((1-\alpha)p-1\big)^{-\frac{1}{p}}.$$

Proof. Choose $t = b(x)$, hence $x = b^{-1}(t)$, where $b^{-1}(t)$ is the reciprocal function of $b(t)$. Applying (1.2) and Fubini's Theorem, we get

$$\begin{aligned}
\left\| x^{\alpha}(T_1 f)(x)\right\|_{L_p(0,\infty)} &= \left(\int_0^{\infty} (b^{-1}(t))^{(\alpha-1)p}\left(\int_0^t f(y)\,dy\right)^p (b^{-1}(t))'\,dt\right)^{\frac{1}{p}} \\
&\le (c_2)^{\frac{1}{p}}\left(\int_0^{\infty} (b^{-1}(t))^{(\alpha-1)p}\left(\int_0^t f^p(y)y^{p-1}\,dy\right)(b^{-1}(t))'\,dt\right)^{\frac{1}{p}} \\
&= (c_2)^{\frac{1}{p}}\left(\int_0^{\infty} f^p(y)y^{p-1}\left(\int_y^{\infty} (b^{-1}(t))'(b^{-1}(t))^{(\alpha-1)p}\,dt\right)dy\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\alpha < 1 - \frac{1}{p}$ and $b^{-1}(\infty) = \infty$, we have

$$\int\limits_y^\infty (b^{-1}(t))'(b^{-1}(t))^{(\alpha-1)p}\,dt = \frac{1}{(1-\alpha)p-1}\,[b^{-1}(y)]^{(\alpha-1)p+1},$$

consequently,

$$\begin{aligned}\big\|x^\alpha(T_1f)(x)\big\|_{L_p(0,\infty)} &\le \left(\frac{c_1^{p(1-p)}}{(1-\alpha)p-1}\right)^{\frac{1}{p}}\left[\int\limits_0^\infty f^p(y)y^{p-1}(b^{-1}(y))^{(\alpha-1)p+1}\,dy\right]^{\frac{1}{p}}\\ &= c_1^{1-p}\big((1-\alpha)p-1\big)^{-\frac{1}{p}}\left[\int\limits_0^\infty \Big(f(y)y^{1-\frac{1}{p}}(b^{-1}(y))^{\alpha-1+\frac{1}{p}}\Big)^p\,dy\right]^{\frac{1}{p}}.\end{aligned}$$

We get the desired inequality. □

Remark 2.1. If f is a non-increasing function on $(0,\infty)$, we obtain the following inequality:

$$\big\|x^\alpha(T_1f)(x)\big\|_{L_p(0,\infty)} \le \left(\frac{p^{1-p}}{(1-\alpha)p-1}\right)^{\frac{1}{p}}\Big\|x^{\frac{1}{p'}}(b^{-1})^{\alpha-\frac{1}{p'}}(x)f(x)\Big\|_{L_p(0,\infty)}.$$

Choosing $b(x) = \beta x$ in Theorem 2.1, where $\beta > 0$, we have the following

Corollary 1. *Let f satisfy the assumptions of Theorem 2.1 and*

$$(S_1f)(x) = \frac{1}{x}\int\limits_0^{\beta x} f(y)\,dy \quad for \quad x > 0,$$

then

$$\big\|x^\alpha(S_1f)(x)\big\|_{L_p(0,\infty)} \le \left(\frac{1}{\beta}\right)^{\alpha-\frac{1}{p'}} c_4\|x^\alpha f(x)\|_{L_p(0,\infty)}.$$

Remark 2.2. Taking $\beta = 1$ in the above corollary, we get Theorem 1.1.

For the next results we need the following

Lemma 2.1. *Let $0 < p < 1$. Suppose that a non-negative function f satisfies the condition: there is a positive constant c_5 such that for all $x > 0$,*

$$f(x) \le \frac{c_5}{x}\left(\int\limits_x^\infty f^p(y)y^{p-1}\,dy\right)^{\frac{1}{p}}, \tag{2.1}$$

then

$$\left(\int\limits_x^\infty f(y)\,dy\right)^p \le c_6\int\limits_x^\infty f^p(y)y^{p-1}\,dy, \tag{2.2}$$

where

$$c_6 = c_5^{p(1-p)}.$$

Proof. Note that

$$f(x) = \big(f^p(x)x^p\big)^{\frac{1}{p}-1}f^p(x)x^{p-1}.$$

Using (2.1), we have

$$x^pf^p(x) \le c_5^p\left(\int\limits_x^\infty f^p(y)y^{p-1}\,dy\right),$$

therefore,

$$\left(x^p f^p(x)\right)^{\frac{1}{p}-1} \le c_5^{1-p}\left(\int_x^\infty f^p(y)y^{p-1}\,dy\right)^{\frac{1}{p}-1}.$$

Multiplying by $f^p(x)x^{p-1}$ and putting $0 < t \le x$, we get

$$f(x) \le c_5^{1-p}\left(\int_t^\infty f^p(y)y^{p-1}\,dy\right)^{\frac{1}{p}-1} f^p(x)x^{p-1},$$

consequently,

$$\begin{aligned}\int_t^\infty f(x)\,dx &\le c_5^{1-p}\left(\int_t^\infty f^p(y)y^{p-1}\,dy\right)^{\frac{1}{p}-1}\int_t^\infty f^p(x)x^{p-1}\,dx\\ &= c_5^{1-p}\left(\int_t^\infty f^p(x)x^{p-1}\,dx\right)^{\frac{1}{p}-1}\int_t^\infty f^p(x)x^{p-1}\,dx\\ &= c_5^{1-p}\left(\int_t^\infty f^p(x)x^{p-1}\,dx\right)^{\frac{1}{p}}.\end{aligned}$$ □

Theorem 2.2. *Let $0 < p < 1$, $\alpha > 1 - \frac{1}{p}$ and $c_1 > 0$. If f is a non-negative measurable function on $(0,\infty)$ and satisfies (2.1) for all $x > 0$, then*

$$\left\|x^\alpha (T_2 f)(x)\right\|_{L_p(0,\infty)} \le c_7 \left\|x^{\frac{1}{p'}}\left(a^{-1}(x)\right)^{\alpha-\frac{1}{p'}} f(x)\right\|_{L_p(0,\infty)},$$

where

$$c_7 = c_5^{1-p}\left((\alpha-1)p+1\right)^{-\frac{1}{p}}.$$

Proof. Put $t = a(x)$, then $x = a^{-1}(t)$, where $a^{-1}(t)$ is the reciprocal function of $a(t)$. Applying inequality (2.2) and Fubini's Theorem, we get

$$\begin{aligned}\left\|x^\alpha (T_2 f)(x)\right\|_{L_p(0,\infty)} &= \left(\int_0^\infty (a^{-1}(t))^{(\alpha-1)p}\left(\int_t^\infty f(y)\,dy\right)^p (a^{-1}(t))'\,dt\right)^{\frac{1}{p}}\\ &\le (c_6)^{\frac{1}{p}}\left(\int_0^\infty (a^{-1}(t))^{(\alpha-1)p}\left(\int_t^\infty f^p(y)y^{p-1}\,dy\right)(a^{-1}(t))'\,dt\right)^{\frac{1}{p}}\\ &= (c_6)^{\frac{1}{p}}\left(\int_0^\infty f^p(y)y^{p-1}\left(\int_0^y (a^{-1}(t))'(a^{-1}(t))^{(\alpha-1)p}\,dt\right)dy\right)^{\frac{1}{p}}.\end{aligned}$$

Since $\alpha > 1 - \frac{1}{p}$ and $a^{-1}(0) = 0$, we have

$$\int_0^y (a^{-1}(t))'(a^{-1}(t))^{(\alpha-1)p}\,dt = \frac{1}{(\alpha-1)p+1}\left[a^{-1}(y)\right]^{(\alpha-1)p+1},$$

consequently,

$$\begin{aligned}\left\|x^\alpha (T_2 f)(x)\right\|_{L_p(0,\infty)} &\le \left(\frac{c_5^{p(1-p)}}{(\alpha-1)p+1}\right)^{\frac{1}{p}}\left(\int_0^\infty f^p(y)y^{p-1}(a^{-1}(y))^{(\alpha-1)p+1}\,dy\right)^{\frac{1}{p}}\\ &= c_7\left\|x^{\frac{1}{p'}}(a^{-1}(x))^{\alpha-\frac{1}{p'}} f(x)\right\|_{L_p(0,\infty)}.\end{aligned}$$ □

Choosing $a(x) = \lambda x$ in Theorem 2.2, where $\lambda > 0$, we obtain the following

Corollary 2. *Let f satisfy the assumptions of Theorem 2.2 and*

$$(S_2 f)(x) = \frac{1}{x} \int_{\lambda x}^{\infty} f(y)\, dy \quad \text{for } x > 0.$$

Then the inequality

$$\left\| x^{\alpha} (S_2 f)(x) \right\|_{L_p(0,\infty)} \le \left(\frac{1}{\lambda}\right)^{\alpha - \frac{1}{p'}} c_7 \left\| x^{\alpha} f(x) \right\|_{L_p(0,\infty)}$$

holds.

Remark 2.3. Taking $\lambda = 1$, we get

$$\left\| x^{\alpha} (H_2 f)(x) \right\|_{L_p(0,\infty)} \le c_7 \| x^{\alpha} f(x) \|_{L_p(0,\infty)}.$$

Now, we have obtained the analogue of Theorem 1.1 for H_2 which is the dual of Hardy averaging operator H_1.

For the next theorem we need the following lemmas.

Lemma 2.2. *Let $0 < p < 1$, $c_8 > 0$ and $a(x)$, $b(x)$ be under the conditions of operator T_3 such that for almost all $x > 0$,*

$$f(x) \le \frac{c_8}{x} \left(\int_{a(x)}^{b(x)} f^p(y) y^{p-1}\, dy \right)^{\frac{1}{p}}. \tag{2.3}$$

Then

$$\left(\int_{a(x)}^{b(x)} f(y)\, dy \right)^{p} \le c_8^{p(1-p)} \int_{a(x)}^{b(x)} f^p(y) y^{p-1}\, dy. \tag{2.4}$$

Proof. The proof is similar to that of Lemma 2.1. □

Lemma 2.3. *Let $0 < p < 1$ and $0 < B < A$, then*

$$A^p - B^p \le (A - B)^p. \tag{2.5}$$

Proof. It is well known that for $0 < B < A$ and $0 < p < 1$,

$$(A + B)^p \le A^p + B^p.$$

Replacing A by $A - B$, we get

$$A^p \le (A - B)^p + B^p.$$ □

For more details, see [1].

Theorem 2.3. *Let $0 < p < 1$, $\alpha > 1 - \frac{1}{p}$ and $c_1 > 0$. If f is a non-negative measurable function on $(0, \infty)$ and satisfies (2.3) for all $x > 0$, then*

$$\left\| x^{\alpha} (T_3 f)(x) \right\|_{L_p(0,\infty)} \le c_9 \left(\left\| x^{\frac{1}{p'}} (a^{-1}(x))^{\alpha - \frac{1}{p'}} f(x) \right\|_{L_p(0,\infty)} - \left\| x^{\frac{1}{p'}} (b^{-1}(x))^{\alpha - \frac{1}{p'}} f(x) \right\|_{L_p(0,\infty)} \right),$$

where

$$c_9 = c_8^{1-p} \big((\alpha - 1)p + 1 \big)^{-\frac{1}{p}}.$$

Proof. Taking into account (2.4), we get

$$\left\|x^{\alpha}(T_3f)(x)\right\|^p_{L_p(0,\infty)} = \int\limits_0^{\infty} x^{(\alpha-1)p}\Big(\int\limits_{a(x)}^{b(x)} f(y)\,dy\Big)^p dx \le c_8^{p(1-p)}\int\limits_0^{\infty} x^{(\alpha-1)p}\Big(\int\limits_{a(x)}^{b(x)} f^p(y)y^{p-1}\,dy\Big)\,dx.$$

Since $a(x) < y < b(x)$, we have $b^{-1}(y) < x < a^{-1}(y)$. Apply Fubini's Theorem, we get

$$\int\limits_0^{\infty} x^{(\alpha-1)p}\Big(\int\limits_{a(x)}^{b(x)} f^p(y)y^{p-1}\,dy\Big)\,dx = \int\limits_0^{\infty} f^p(y)y^{p-1}\Big(\int\limits_{b^{-1}(y)}^{a^{-1}(y)} x^{(\alpha-1)p}\,dx\Big)\,dy.$$

In combination with $\alpha > 1-\frac{1}{p}$ and $0 < a(x) < b(x) < \infty$, this yields

$$\int\limits_{b^{-1}(y)}^{a^{-1}(y)} x^{(\alpha-1)p}\,dx = \frac{1}{(\alpha-1)p+1}\Big((a^{-1}(y))^{(\alpha-1)p+1} - (b^{-1}(y))^{(\alpha-1)p+1}\Big).$$

Consequently,

$$\begin{aligned}&\left\|x^{\alpha}(T_3f)(x)\right\|^p_{L_p(0,\infty)}\\ &\le \frac{c_8^{p(1-p)}}{(\alpha-1)p+1}\Big(\int\limits_0^{\infty} f^p(y)y^{p-1}\Big[(a^{-1}(y))^{(\alpha-1)p+1} - (b^{-1}(y))^{(\alpha-1)p+1}\Big]\,dy\Big)\\ &= \frac{c_8^{p(1-p)}}{(\alpha-1)p+1}\Big(\left\|x^{\frac{1}{p'}}(a^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\right\|^p_{L_p(0,\infty)} - \left\|x^{\frac{1}{p'}}(b^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\right\|^p_{L_p(0,\infty)}\Big).\end{aligned}$$

Using (2.5), we deduce

$$\begin{aligned}&\left\|x^{\alpha}(T_3f)(x)\right\|^p_{L_p(0,\infty)}\\ &\le \frac{c_8^{p(1-p)}}{(\alpha-1)p+1}\Big(\left\|x^{\frac{1}{p'}}(a^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\right\|_{L_p(0,\infty)} - \left\|x^{\frac{1}{p'}}(b^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\right\|_{L_p(0,\infty)}\Big)^p,\end{aligned}$$

hence

$$\begin{aligned}&\left\|x^{\alpha}(T_3f)(x)\right\|_{L_p(0,\infty)}\\ &\le c_8^{1-p}\big((\alpha-1)p+1\big)^{-\frac{1}{p}}\Big(\left\|x^{\frac{1}{p'}}(a^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\right\|_{L_p(0,\infty)} - \left\|x^{\frac{1}{p'}}(b^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\right\|_{L_p(0,\infty)}\Big). \quad \square\end{aligned}$$

Setting $a(x) = \lambda x$ and $b(x) = \beta x$, where $0 < \lambda < \beta < \infty$, in Theorem 2.3 above, leads to the following

Corollary 3. *Let f satisfies the assumptions of Theorem 2.3 and*

$$(S_3f)(x) = \frac{1}{x}\int\limits_{\lambda x}^{\beta x} f(y)\,dy \quad \textit{for} \ \ x > 0,$$

then

$$\left\|x^{\alpha}(S_3f)(x)\right\|_{L_p(0,\infty)} \le c_8\Big(\Big(\frac{1}{\lambda}\Big)^{\alpha-\frac{1}{p'}} - \Big(\frac{1}{\beta}\Big)^{\alpha-\frac{1}{p'}}\Big)\left\|x^{\alpha}f(x)\right\|_{L_p(0,\infty)}.$$

Remark 2.4. Taking $\lambda = \frac{1}{2}$ and $\beta = 1$, we obtain the analogous result for the Pachepatte type operator P:

$$\left\|x^{\alpha}(Pf)(x)\right\|_{L_p(0,\infty)} \le c_8\big(2^{\alpha-\frac{1}{p'}} - 1\big)\left\|x^{\alpha}f(x)\right\|_{L_p(0,\infty)},$$

where

$$(Pf)(x) = \frac{1}{x}\int\limits_{\frac{x}{2}}^{x} f(y)\,dy \quad \text{for} \ \ x > 0.$$

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