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GALERKIN METHOD APPLIED TO $p(\,\cdot\,)$-BI-LAPLACE EQUATION WITH VARIABLE EXPONENT

Abstract. In this article, a Galerkin mixed finite element method is proposed to find the numerical solutions of high order $p(\cdot)$-bi-Laplace equations. The well-posedness of the problem in suitable Lebesgue–Sobolev spaces with variable exponent owing to nonlinear monotone operator theory is investigated. Some a priori error estimates are shown by using the Galerkin orthogonality properties and variable exponent Lebesgue–Sobolev continues embedding.

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1 Introduction

We consider a bounded open domain Ω of $\mathbb{R}^n$ with a Lipschitz-continuous boundary $\partial\Omega$. Our aim is to prove the existence and uniqueness of a weak solution u and some a priori error estimates to the differential $p(x)$-Bilaplace equation

$$\begin{cases} \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big) = f & \text{in } \Omega, \\ u = \Psi, \;\; \nabla u = \nabla\Psi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f and Ψ are the given functions in $L^{q(\cdot)}(\Omega)$ and $W^{2,\infty}(\Omega)$, respectively. Here, $p(\cdot):\Omega\to\mathbb{R}$ denotes the variable exponent which is assumed to be in $L^\infty_+(\Omega)$ such that $1 < p^- \le p(x) \le p^+ < \infty$, where $p^- = \inf\limits_{x\in\overline{\Omega}} p(x)$ and $p^+ = \sup\limits_{x\in\overline{\Omega}} p(x)$ a.e. in Ω. During the last decades, the high-order PDEs with variable exponent has undergone rapid development. From a mathematical point of view, equation (1.1) can be considered as a natural generalization of $p(\cdot)$-bi-Laplace equation

$$\triangle\big(|\triangle u|^{p-2}\triangle u\big) = f,$$

which falls within the framework of nonlinear PDEs, where the exponent p is constant. One of our motivation for studying (1.1) comes from applications in the area of elasticity, more precisely, it can be used in modelling of travelling waves in suspension bridges (see [6,8]). Other interesting applications are related to improve the visual quality of damaged and noisy images if $1 < p^- \le p^+ < 2$ (see, e.g., [14] and the references therein). Note that in the case $p(x) = 2$, problem (1.1) becomes $\triangle^2 u = f$ which models the deformations of a thin homogeneous plate embedded along its beam and subjected to a distribution f of a load normal to the plate (cf. [1]). Among the most recent works concerning the p-Laplace equation, we can review Lazer et al. [8], where the authors tried to demonstrate the existence of periodic solutions for models of nonlinear supported bending beams and periodic flexing in floating beam. In [5], the authors used discontinuous Galekin method to approximate a biharmonic problem. They also gave an a priori analysis of the error in norm L^2. In [11], the author has studied a p-biharmonic problem using discontinuous Galerkin finite element Hessian. An imagery problem caused by a $p(\cdot)$-Laplace operator with $1 \le p(\cdot) \le 2$ has been considered in [14]. To solve the problem, the authors regularized the proposed PDE to be able to use a fixed point iterative method.

The paper is structured as follows. We present in Section 2 some basic notations and material needed for our work. Section 3 is devoted to the existence and uniqueness of a weak solution to the problem under investigation in suitable Lebesgue–Sobolev spaces with variable exponent using the nonlinear monotone operators theory. In Section 4, the Galerkin mixed finite element method and $\inf-\sup$ condition are given. Finally, we show some a priori error estimates with the help of Ritz projection operator and Galerkin orthogonality properties, which are presented in Section 5.

2 Preliminaries

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ as follows:

$$L^{p(\cdot)}(\Omega) = \Big\{ u:\;\; \Omega\to\mathbb{R},\;\; u \;\text{ measurable and }\; \int\limits_\Omega |u(x)|^{p(x)}\,dx < \infty \Big\}.$$

Note that $L^{p(\cdot)}(\Omega)$ equipped with the Luxembourg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{ \gamma > 0,\;\; \int\limits_\Omega \Big|\frac{u(x)}{\gamma}\Big|^{p(x)}\,dx \le 1 \Big\}$$

is a Banach space. Note that all definitions and properties of Lebesgue and Sobelev spaces with variable exponent given below are taken from references [2–4,7,12].

Definition 2.1. Let $u : \Omega \to \mathbb{R}$ be a measurable function, then the expression

$$\rho_{p(\,\cdot\,)}(u) = \int\limits_{\Omega} |u(x)|^{p(x)}\,dx$$

is called modular of u.

Definition 2.2. For some $p \in L^{\infty}_{+}(\Omega)$ and $m \in \mathbb{N} - \{0\}$, we introduce the exponent variable Sobolev space

$$W^{m,p(\,\cdot\,)}(\Omega) = \Big\{u \in L^{p(\,\cdot\,)}(\Omega);\;\; D^{\alpha}u \in L^{p(\,\cdot\,)}(\Omega),\;\; \forall\,\alpha \in \mathbb{N}^n \;\text{ and }\; |\alpha| \le m\Big\}$$

equipped with the norm

$$\|u\|_{m,p(\,\cdot\,)} = \sum_{|\alpha|\le m} \|D^{\alpha}u\|_{L^{p(\,\cdot\,)}(\Omega)}.$$

Remark 2.1.

(1) Let p, q and $r \in L^{\infty}_{+}(\Omega)$, $u \in L^{p(\,\cdot\,)}(\Omega)$, $v \in L^{q(\,\cdot\,)}(\Omega)$ such that

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)}\,.$$

Then

$$\|uv\|_{L^{r(\,\cdot\,)}(\Omega)} \le \Big(\frac{1}{(\frac{p}{r})^{-}} + \frac{1}{(\frac{q}{r})^{-}}\Big)\|u\|_{L^{p(\,\cdot\,)}(\Omega)}\|v\|_{L^{q(\,\cdot\,)}(\Omega)}.$$

(2) Suppose that $p(x) \le q(x)$ a.e. in Ω. Then

$$L^{q(\,\cdot\,)}(\Omega) \hookrightarrow L^{p(\,\cdot\,)}(\Omega).$$

(3)

$$\|u\|_{L^{p(\,\cdot\,)}(\Omega)} = k \Longleftrightarrow \rho_{p(\,\cdot\,)}\Big(\frac{u}{k}\Big) = 1.$$

(4)

$$\Big(\|u_n - u\|_{L^{p(\,\cdot\,)}(\Omega)} \underset{n\to\infty}{\longrightarrow} 0\Big) \Longleftrightarrow \Big(\rho_{p(\,\cdot\,)}(u_n - u) \underset{n\to\infty}{\longrightarrow} 0\Big).$$

(5) Let $p, q \in L^{\infty}_{+}(\Omega)$ and $m \in \mathbb{N}^{*}$ with $p(x) \le q(x)$ a.e. in Ω. Then

$$W^{m,q(\,\cdot\,)}(\Omega) \hookrightarrow W^{m,p(\,\cdot\,)}(\Omega).$$

Definition 2.3 (see [2, Definition 4.1.1, p. 98])**.** A function $\beta : \Omega \to \mathbb{R}$ is locally log-Hölder continuous on Ω if $\exists C > 0$ such that

$$|\beta(x) - \beta(y)| \le \frac{C}{\log(e + \frac{1}{|x-y|})}\,,\quad \forall\, x, y \in \Omega.$$

If

$$|\beta(x) - \beta_{\infty}| \le \frac{C}{\log(e + |x|)}$$

for some $\beta_{\infty} \ge 1$, $c > 0$ and all $x \in \Omega$, then we say that β satisfies the log-Hölder decay condition (at infinity). We denote by $P^{\log}(\Omega)$ the class of variable exponents which are log-Hölder continuous, i.e., which satisfy the local log-Hölder continuity condition and the log-Hölder decay condition.

Definition 2.4 (see [2, Definition 11.2.1])**.** Let $p \in P^{\log}(\Omega)$. We also define

$$W_0^{2,p(\,\cdot\,)}(\Omega) := \overline{C_0^{\infty}(\Omega)}^{W^{2,p(\,\cdot\,)}(\Omega)}$$

Similarly, we define

$$W_{\Psi}^{2,p(\,\cdot\,)}(\Omega) = \Psi + W_0^{2,p(\,\cdot\,)}(\Omega) = \Big\{\varphi \in W^{2,p(\,\cdot\,)}(\Omega); \varphi_{\backslash\partial\Omega} = \Psi \;\text{ and }\; \nabla\varphi_{\backslash\partial\Omega} = \nabla\Psi\Big\}.$$

Remark 2.2.

(i) Note that if $p^- > 1$, then the spaces $W^{2,p(\cdot)}(\Omega)$ and $W_0^{2,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

(ii) (Poincaré inequality) Let $p \in L^\infty(\Omega)$ with $p^- \geq 1$, there exists $C(\Omega, p(\cdot))$ such that

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \quad \forall\, u \in W_0^{1,p(\cdot)}(\Omega)$$

3 Existence and uniqueness of the weak solution to $p(\cdot)$-Bi-Laplacien with variable exponent

Definition 3.1. A function u is a weak solution of problem (1.1) if it satisfies

$$\int_\Omega \big(|\triangle u|^{p(x)-2}\triangle u\big)\triangle v\,dx = \int_\Omega f v\,dx, \quad \forall\, v \in W_0^{2,p(\cdot)}(\Omega).$$

Theorem 3.1. *For $f \in L^{q(\cdot)}(\Omega)$, problem (1.1) admits a unique weak solution u in $W_\Psi^{2,p(\cdot)}(\Omega)$.*

Proof. We prove the theorem in $W_0^{2,p(\cdot)}(\Omega)$ because if $u \in W_\Psi^{2,p(\cdot)}(\Omega)$, then $u - \Psi \in W_0^{2,p(\cdot)}(\Omega)$ and we can take $u - \Psi$ instead of u. We apply the monotone operators theory and prove that

$$\triangle^2_{p(x)} := \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big) : W_0^{2,p(\cdot)}(\Omega) \to (W_0^{2,p(\cdot)}(\Omega))^* \tag{3.1}$$

is a hemicontinuous, coercive and monotone operator.

Let us define the functional A on $W_0^{2,p(\cdot)}(\Omega)$ by

$$A(u) = \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}\,dx.$$

We have

$$\begin{aligned}(A'(u), v) &= \frac{d}{dt}\{A(u+tv)\}_{t=0} = \frac{d}{dt}\Big\{\int_\Omega \frac{1}{p(x)}\,|\triangle(u+tv)|^{p(x)}\,dx\Big\}_{t=0} \\ &= \Big\{\int_\Omega \frac{1}{p(x)}\,\triangle v\cdot p(x)|\triangle(u+tv)|^{p(x)-1}\,dx\Big\}_{t=0} = \int_\Omega \big(|\triangle u|^{p(x)-2}\triangle u\big)\triangle v\,dx \\ &= \int_\Omega \triangle(|\triangle u|^{p(x)-2}\triangle u)v\,dx = \big(\triangle^2_{p(x)}u, v\big), \quad \forall\, v \in W_0^{2,p(\cdot)}(\Omega)\end{aligned} \tag{3.2}$$

which implies that $A(\cdot)$ is differentiable in Gateau sense and $A' = \triangle^2_{p(x)}$. Therefore, $\triangle^2_{p(x)}$ is a hemicontinuous operator.

On the other hand, using Hölder's inequality, we get

$$\begin{aligned}\sup_{\|v\|_{W_0^{2,p(\cdot)}(\Omega)}\leq 1} \big|(\triangle^2_{p(x)}u, v)\big| &= \sup_{\|v\|_{W_0^{2,p(\cdot)}(\Omega)}\leq 1} \Big|\int_\Omega \triangle\big(|\triangle u|^{p(x)-2}\triangle u\big)v\,dx\Big| \\ &\leq \sup_{\|v\|_{W_0^{2,p(\cdot)}(\Omega)}\leq 1} \int_\Omega |\triangle u|^{p(x)-1}|\triangle v|\,dx \leq C^{\frac{p(x)}{q(x)}} \leq C^{\frac{p^+}{q^-}}.\end{aligned} \tag{3.3}$$

This proves that $\triangle^2_{p(\cdot)}$ is bounded on $W_0^{2,p(\cdot)}(\Omega)$. Next, from the inequality (see [10])

$$|b|^{p(\cdot)} \geq |a|^{p(\cdot)} + p|a|^{p(\cdot)-2}a(b-a) + \frac{|b-a|^{p(\cdot)}}{2^{p(\cdot)-1}-1} \quad \text{for } p \geq 2 \text{ and } a, b \in \mathbb{R}^n$$

it follows that

$$\begin{aligned}\big(\triangle^2_{p(x)}(u)-\triangle^2_{p(x)}(v),u-v\big)&=\int\limits_\Omega\big(|\triangle u|^{p(x)-2}\triangle u-|\triangle v|^{p(x)-2}\triangle v\big)\triangle(u-v)\,dx\\&=\int\limits_\Omega|\triangle u|^{p(x)-2}\triangle u(\triangle u-\triangle v)\,dx-\int\limits_\Omega|\triangle v|^{p(x)-2}\triangle v(\triangle u-\triangle v)\,dx\\&\geq\frac{2}{p(x)(2^{p(x)-1}-1)}\int\limits_\Omega|\triangle u-\triangle v|^{p(x)}\,dx\geq\frac{2}{p^+(2^{p^+-1}-1)}\int\limits_\Omega|\triangle u-\triangle v|^{p(x)}\,dx.\end{aligned}\tag{3.4}$$

Now, using Calderon–Zygmund and Poincaré inequalities, we find that the norm $\|\cdot\|_{W_0^{2,p(\cdot)}(\Omega)}$ is equivalent to the semi-norm $\|\triangle(\cdot)\|_{L^{p(\cdot)}(\Omega)}$ over the space $W_0^{2,p(\cdot)}(\Omega)$.

This allows us to write

$$\big(\triangle^2_{p(x)}(u)-\triangle^2_{p(x)}(v),u-v\big)\geq C(p^+)\|u-v\|^{p(x)}_{W_0^{2,p(\cdot)}(\Omega)},$$

from which we conclude the monotonicity of $\triangle^2_{p(x)}$. Similarly,

$$(\triangle^2_{p(x)}(u),u)\geq C(p^+)\|u\|^{p(x)}_{W_0^{2,p(\cdot)}(\Omega)}.$$

This proves the coercivity of $\triangle^2_{p(x)}$. Finally, by Hölder's inequality, we have

$$|(f,v)|=\Big|\int\limits_\Omega fv\,dx\Big|\leq C\|f\|_{q(x)}\|v\|_{p(x)}.$$

Taking into account that $L^{q^+}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ and $W_0^{-2,p(\cdot)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, we arrive at

$$|(f,v)|\leq C\|f\|_{L^{q^+}(\Omega)}\|v\|_{W_0^{2,p(\cdot)}(\Omega)}.$$

Hence $f\in(W_0^{-2,p(\cdot)}(\Omega))^*$. This achieves the proof. □

4 Galerkin mixed formulation

Set $X:=W_\Psi^{2,p(\cdot)}(\Omega)$ and $M:=W_0^{2,p(\cdot)}(\Omega)$. Let us introduce a new variable

$$\varphi=|\triangle u|^{p(x)-2}\triangle u.$$

This allows us to write problem (1.1) as follows:

$$\begin{cases}-\triangle u=|\varphi|^{q(x)-2}\varphi,\\-\triangle\varphi=f.\end{cases}\tag{4.1}$$

The weak formulation associated to (4.1) is: Find $(u,\varphi)\in X\times L^{q(\cdot)}$ satisfying

$$\begin{cases}a(\varphi,v)+c(u,v)=0 & \forall\,v\in X,\\c(\varphi,\mu)=l_M(\mu) & \forall\,\mu\in M,\end{cases}\tag{4.2}$$

where

$$a(\varphi,v):=\int\limits_\Omega|\varphi|^{q(x)-2}\varphi v\,dx,\quad c(\varphi,\mu):=\int\limits_\Omega-\triangle\varphi\mu\,dx,\quad l_M(\mu):=\int\limits_\Omega f\mu\,dx.$$

Proposition 4.1 (inf-sup condition)**.** *There exists* $\gamma > 0$ *such that*

$$\inf_{\mu \in M} \sup_{u \in W_0^{2,p(\cdot)}(\Omega)} \frac{c(u,\mu)}{\|u\|_X \|\mu\|_M} \geq \gamma.$$

Proof. We apply Proposition 2.4 from [11, p. 60]. Our aim is to show that $\forall \mu \in M$, there exists $u_\mu \in X$ such that

$$c(u_\mu, \mu) = \|\mu\|_M^{p(\cdot)} \text{ and } \|u_\mu\|_X \leq \frac{1}{\gamma}\|\mu\|_M.$$

It suffices to find a mapping $\mu \longrightarrow u_\mu$ from $W_0^{2,p(\cdot)}(\Omega)$ to $W_\Psi^{2,p(\cdot)}(\Omega)$ such that

$$(\nabla u_\mu, \nabla \mu) = \|\triangle u_\mu\|_{p(\cdot)}^{p(\cdot)} \text{ and } \|u_\mu\|_X \leq \frac{1}{\gamma}\|\mu\|_M.$$

We can see that with a choice of $u_\mu = |\triangle \mu|^{p(x)-2}\triangle\mu$ we arrive at the desired result. □

5 Discretization

We consider a triangulation Υ_h made of triangles T whose edges are denoted by e. We assume that the intersection of two different elements is either empty, or a vertex, or a whole edge e, and we also assume that this triangulation is regular in Ciarlet sense, i.e.,

$$\exists\, \sigma > 0; \ \frac{h_T}{\rho_T} \leq \sigma, \ \forall\, T \in \Upsilon_h,$$

where h_T is the diameter of T and ρ_T is the diameter of its largest inscribed bull. We define $h = \max_{T \in \Upsilon_h} h_T$. The jump operator for function v across an edge/face at the point x is given by

$$[v(x)]_e = \begin{cases} \lim_{\alpha \to 0^+} v(x+\alpha\eta_e) - v(x+\alpha\eta_e) & \text{if } e \in \zeta_h^{int}, \\ v(x) & \text{if } e \in \zeta_h - \zeta_h^{int}, \end{cases}$$

where ζ_h^{int} is the set of interior edges/faces. Let us define the broken Laplace operator

$$(\triangle_h v)_{\backslash T} := \triangle(v_{\backslash T}), \ \forall\, T \in \Upsilon_h.$$

For $h > 0$, we introduce the following spaces:

$$X^h = \{\phi \in C^0(\overline{\Omega});\ \phi_{\backslash T} \in P^k(T),\ \forall\, T \in \Upsilon_h\},$$
$$X_\Psi^h = \{\phi \in X^h;\ \phi_{\backslash \partial\Omega} = \Pi\Psi\}$$

and the Ritz projection operator Π defined as follows:

$$\int_\Omega \nabla(\Pi v)\nabla\phi\, dx = \int_\Omega \nabla v \nabla \phi\, dx,\ \ \forall\, \phi \in X^h \cap H_0^1(\Omega).$$

Lemma 5.1. *Let* $u \in W^{m+1,q(\cdot)}(\Omega)$*, then for* $m \geq 2$*, we have*

$$\|u - \Pi u\|_{L^{q(\cdot)}(\Omega)} + \|h(\nabla u - \nabla(\Pi u))\|_{L^{q(\cdot)}(\Omega)} + \Big(\sum_{T \in \Upsilon} \|h^2(\triangle u - \triangle(\Pi u))\|_{L^{q(\cdot)}(T)}^{q(x)}\Big)^{\frac{1}{q(x)}} \leq Ch^{m+1}|u|_{m+1,q(\cdot)}.$$

Proof. See [10]. □

The discrete formulation of (4.2) is to seek a solution $(u_h, \varphi_h) \in X_\Psi^h \times X^h$ such that

$$\begin{cases} a(\varphi_h, v) + c_h(u_h, v) = 0, \\ c_h(\varphi_h, \mu) = \int\limits_\Omega f\mu, \quad \forall\, (v, \mu) \in X^h \times X_0^h, \end{cases} \tag{5.1}$$

where c_h is given by

$$c_h(\varphi_h, \mu) = \sum_{T\in\Upsilon_h} \int\limits_T \nabla\varphi_h \nabla\mu\, dx - \int\limits_{\partial\Omega} \nabla\Psi \cdot \eta\mu\, dx = \int\limits_\Omega \nabla\varphi_h \nabla\mu\, dx - \int\limits_{\partial\Omega} \nabla\Psi \cdot \eta\mu\, dx. \tag{5.2}$$

Substituting (5.2) into (5.1), the discrete problem consists in finding $(u_h, \varphi_h) \in X_\Psi^h \times X^h$ satisfying for $(v, \mu) \in X^h \times X_0^h$

$$\int\limits_\Omega |\varphi_h|^{q(x)-2}\varphi_h v\, dx + \int\limits_\Omega \nabla u_h \nabla v\, dx = \int\limits_{\partial\Omega} \nabla\Psi \cdot \eta v\, dx \int\limits_\Omega \nabla\varphi_h \nabla\mu\, dx = \int\limits_\Omega f\mu\, dx.$$

Denote $e_\varphi = \varphi - \varphi_h$ and $e_u = u - u_h$. Now, we are able to announce the following error estimate theorem.

Theorem 5.1. *There exists a constant C such that for $m \geq 2$, we have*

$$\begin{aligned} &\|e_\varphi\|_{L^{q^-}(\Omega)} + \|e_u\|^{p^- -1}_{W_h^{2,p^-}(\Omega)} \\ &\qquad \leq C\Big(h^{\frac{q(x)}{2}(m+1)}|\varphi|^{\frac{q(\,\cdot\,)}{2}}_{m+1,q(\,\cdot\,)} + h^{m+1}|\varphi|_{m+1,q(\,\cdot\,)} + h^{m-1}|u|_{m+1,p(\,\cdot\,)} + h^{m-1}|u|_{m+1,p(\,\cdot\,)}\Big), \end{aligned}$$

where $(u, \varphi) \in W_\Psi^{m+1,p(\,\cdot\,)}(\Omega) \times W^{m+1,q(\,\cdot\,)}(\Omega)$ is the exact solution of (4.2) and $(u_h, \varphi_h) \in X_\Psi^h \times X^h$ is the approximate solution of (5.1).

Proof. It is clear that

$$\|u - u_h\|_{W_h^{2,p(\,\cdot\,)}(\Omega)} \leq \|u_h - \Pi u\|_{W_h^{2,p(\,\cdot\,)}(\Omega)} + \|\Pi u - u\|_{W_h^{2,p(\,\cdot\,)}(\Omega)}. \tag{5.3}$$

Using the discrete form of inf-sup condition and Galerkin orthogonality properties, we get

$$\|u_h - \Pi u\|_{W_h^{2,p(\,\cdot\,)}(\Omega)} \leq \sup_{\mu\in X_0^h(\Omega),\mu\neq 0} \frac{c_h(u - \Pi u_h, \mu)}{\|\mu\|_{L_h^{q(\,\cdot\,)}(\Omega)}} = \sup_{\mu\in X_0^h(\Omega),\mu\neq 0} \frac{a(\varphi_h, \mu) - a(\varphi_h, \mu)}{\|\mu\|_{L_h^{q(\,\cdot\,)}(\Omega)}}.$$

In view of the properties of $a(\,\cdot\,,\,\cdot\,)$ (see [1, Proposition 3.1]) we can write

$$\begin{aligned} \sup_{\mu\in X_0^h(\Omega),\mu\neq 0} \frac{a(\varphi_h, \mu) - a(\varphi_h, \mu)}{\|\mu\|_{L_h^{q(\,\cdot\,)}(\Omega)}} &\leq C\, \frac{\big(\int\limits_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\, |\varphi - \varphi_h|\, dx\big)^{\frac{1}{p(x)}} \|\mu\|_{L^{q(\,\cdot\,)}(\Omega)}}{\|\mu\|_{L_h^{q(\,\cdot\,)}(\Omega)}} \\ &\leq C\Big(\int\limits_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\, |\varphi - \varphi_h|\, dx\Big)^{\frac{1}{p(x)}} \end{aligned}$$

and

$$\begin{aligned} &\int\limits_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\, |\varphi - \varphi_h|\, dx \\ &\qquad\qquad \leq C\Big(\int\limits_\Omega \big||\varphi|^{p(x)-2}\varphi - |\varphi_h|^{p(x)-2}\varphi_h\big|\, |\varphi - \varphi_h|\, dx\Big)^{\frac{1}{p(x)}} \|\varphi - \varphi_h\|_{L^{q(\,\cdot\,)}(\Omega)}. \end{aligned} \tag{5.4}$$

By the ϵ-Young inequality, we obtain that the right-hand side of (5.4) is estimated by

$$\frac{C^{q(x)}}{q(x)\epsilon^{q(x)}}\,\|\varphi-\varphi_h\|^{q(x)}_{L^{q(\cdot)}(\Omega)}+\frac{\epsilon^{p(x)}}{p(x)}\int_\Omega \left||\varphi|^{p(x)-2}\varphi-|\varphi_h|^{p(x)-2}\varphi_h\right|dx. \tag{5.5}$$

Choosing ϵ such that $\frac{\epsilon^{p(x)}}{p(x)}\prec 1$ (for example, we can choose $\epsilon=(\frac{p(x)}{3})^{\frac{1}{p(x)}}$), we find that

$$\int_\Omega \left||\varphi|^{p(x)-2}\varphi-|\varphi_h|^{p(x)-2}\varphi_h\right|\,|\varphi-\varphi_h|\,dx\le C\|\varphi-\varphi_h\|^{q(x)}_{L^{q(\cdot)}(\Omega)}.$$

So, we get

$$\|u_h-\Pi u\|_{W_h^{2,p(\cdot)}(\Omega)}\le C\|\varphi-\varphi_h\|^{\frac{q(x)}{p(x)}}_{L^{q(\cdot)}(\Omega)}\le C\|\varphi-\varphi_h\|^{2\,\frac{q(x)-1}{2}}_{L^{q(\cdot)}(\Omega)}. \tag{5.6}$$

On the other hand, a simple calculation gives

$$a(\varphi,\varphi-\varphi_h)-a(\varphi_h,\varphi-\varphi_h)=a(\varphi,\varphi-v)-a(\varphi_h,\varphi-v)+a(\varphi,v-\varphi_h)-a(\varphi_h,v-\varphi_h). \tag{5.7}$$

Subtracting (5.1) from (4.2), we get

$$\begin{cases} a(\varphi,v)-a(\varphi_h,v)+c_h(u-u_h,v)=0, & \forall\, v\in X^h,\\ c_h(\varphi-\varphi_h,\mu)=0, & \forall\,\mu\in X_0^h.\end{cases}$$

This allows us to rewrite (5.7) as follows:

$$a(\varphi,\varphi-\varphi_h)-a(\varphi_h,\varphi-\varphi_h)=a(\varphi,\varphi-v)-a(\varphi_h,\varphi-v)+c_h(u-u_h,\varphi_h-v)=J_1+J_2,$$

where

$$J_1=a(\varphi,\varphi-v)-a(\varphi_h,\varphi-v)\ \text{ and }\ J_2=c_h(u-u_h,\varphi_h-v).$$

Now, using the properties of $a(\cdot,\cdot)$ (see [1, Proposition 3.1]) once more, the ϵ-Young inequality shows that

$$\frac{C_1}{2}\,\frac{\|\varphi-\varphi_h\|^2_{L^{q(\cdot)}(\Omega)}}{\|\varphi\|^{2-q(x)}_{L^{q(\cdot)}(\Omega)}+\|\varphi_h\|^{2-q(x)}_{L^{q(\cdot)}(\Omega)}}+\frac{C_2}{2}\int_\Omega \left||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\right|\,|\varphi-\varphi_h|\,dx$$
$$\le a(\varphi,\varphi-\varphi_h)-a(\varphi_h,\varphi-\varphi_h)=J_1+J_2 \tag{5.8}$$

and

$$\begin{aligned} J_1&\le C_3\Bigg(\int_\Omega \left||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\right|\,|\varphi-\varphi_h|\,dx\Bigg)^{\frac{1}{p(x)}}\|\varphi-v\|_{L^{q(\cdot)}(\Omega)}\\ &\le \frac{C_3^{q(x)}}{\epsilon^{q(x)}q(x)}\,\|\varphi-v\|^{q(x)}_{L^{q(\cdot)}(\Omega)}+\frac{\epsilon^{p(x)}}{p(x)}\int_\Omega \left||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\right|\,|\varphi-\varphi_h|\,dx,\end{aligned}$$

where C_1, C_2 and C_3 are the same constants as in Proposition 3.1 of [13]. If we choose ϵ such that $\frac{\epsilon^{p(x)}}{p(x)}=\frac{C_2}{2}$, we arrive at

$$J_1\le C(q^-,q^+)\|\varphi-v\|^{q(x)}_{L^{q(\cdot)}(\Omega)}+\frac{C}{2}\int_\Omega \left||\varphi|^{q(x)-2}\varphi-|\varphi_h|^{q(x)-2}\varphi_h\right|\,|\varphi-\varphi_h|\,dx. \tag{5.9}$$

Moreover,

$$J_2=c_h(u-u_h,\varphi_h-v)=c_h(u-\Pi u,\varphi_h-v), \tag{5.10}$$

in view of

$$c_h(\mu,\varphi_h-v)=0,\ \ \forall\,\mu\in X_0^h.$$

The continuity of c_h implies that

$$\begin{aligned} J_2 = c_h(u - \Pi u, \varphi_h - v) &\leq C\|u - \Pi u\|_{W_h^{2,p(x)}(\Omega)}\|\varphi_h - v\|_{L^{q(\cdot)}(\Omega)} \\ &\leq \frac{C\varepsilon^2}{2}\|\varphi_h - v\|^2_{L^{q(\cdot)}(\Omega)} + \frac{C}{2\epsilon^2}\|u - \Pi u\|^2_{W_h^{2,p(x)}(\Omega)} \\ &\leq \frac{C}{2\epsilon^2}\|u - \Pi u\|^2_{W_h^{2,p(x)}(\Omega)} + \frac{C\varepsilon^2}{2}\left(\|\varphi_h - \varphi\|_{L^{q(\cdot)}(\Omega)} + \|\varphi - v\|_{L^{q(\cdot)}(\Omega)}\right)^2 \\ &\leq \frac{C}{2\epsilon^2}\|u - \Pi u\|^2_{W_h^{2,p(x)}(\Omega)} \\ &\quad + \frac{C\varepsilon^2}{2}\left(\|\varphi_h - \varphi\|^2_{L^{q(\cdot)}(\Omega)} + \|\varphi - v\|^2_{L^{q(\cdot)}(\Omega)} + 2\|\varphi_h - \varphi\|_{L^{q(\cdot)}(\Omega)}\|\varphi - v\|_{L^{q(\cdot)}(\Omega)}\right) \\ &\leq \frac{C}{2\epsilon^2}\|u - \Pi u\|^2_{W_h^{2,p(x)}(\Omega)} + C\varepsilon^2\left(\|\varphi_h - \varphi\|^2_{L^{q(\cdot)}(\Omega)} + \|\varphi - v\|^2_{L^{q(\cdot)}(\Omega)}\right). \end{aligned} \tag{5.11}$$

Gathering estimates (5.9)–(5.11), substituting in (5.8) and taking ϵ sufficiently small, we obtain

$$\|\varphi_h - \varphi\|^2_{L^{q(\cdot)}(\Omega)} \leq C\|\varphi - v\|^{q(x)}_{L^{q(\cdot)}(\Omega)} + C\|u - \Pi u\|^2_{W_h^{2,p(x)}(\Omega)} + C\|\varphi - v\|^2_{L^{q(\cdot)}(\Omega)}. \tag{5.12}$$

Using the properties of Π, we obtain the estimate of e_φ. Now, substituting (5.12) into (5.6), taking into account (5.3), Lemma 5.1 and the continuous embedding of $L^{q(\cdot)}(\Omega)$ into $L^{q^-}(\Omega)$, we arrive at the desired estimate for e_u. Thus the proof is completed. □

References

[1] E. Bécache, P. Ciarlet, C. Hazard and E. Lunéville, *La Méthode des Éléments Finis*, Tome II. ENSTA press, November 18, 2010.

[2] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.

[3] X. Fan, S. Wang and D. Zhao, Density of $C^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with discontinuous exponent $p(x)$. *Math. Nachr.* **279** (2006), no. 1-2, 142–149.

[4] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.* **263** (2001), no. 2, 424–446.

[5] E. H. Georgoulis and P. Houston, Discontinuous Galerkin methods for the biharmonic problem. *IMA J. Numer. Anal.* **29** (2009), no. 3, 573–594.

[6] T. Gyulov and G. Moroşanu, On a class of boundary value problems involving the p-biharmonic operator. *J. Math. Anal. Appl.* **367** (2010), no. 1, 43–57.

[7] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.* **41(116)** (1991), no. 4, 592–618.

[8] A. C. Lazer and P. J. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. *SIAM Rev.* **32** (1990), no. 4, 537–578.

[9] H. Li, The W_p^1 stability of the Ritz projection on graded meshes. *Math. Comp.* **86** (2017), no. 303, 49–74.

[10] P. Lindqvist, *Notes on the p-Laplace Equation*. Report. University of Jyväskylä Department of Mathematics and Statistics, 102. University of Jyväskylä, Jyväskylä, 2006.

[11] T. Pryer, Discontinuous Galerkin methods for the p-biharmonic equation from a discrete variational perspective. *Electron. Trans. Numer. Anal.* **41** (2014), 328–349.

[12] S. G. Samko, Convolution type operators in $L^{p(x)}$. *Integral Transform. Spec. Funct.* **7** (1998), no. 1-2, 123–144.

[13] D. Sandri, Sur l'approximation numérique des écoulements quasi-newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau. (French) [On the numerical approximation of quasi-Newtonian flows whose viscosity obeys a power law or the Carreau law] *RAIRO Modél. Math. Anal. Numér.* **27** (1993), no. 2, 131–155.

[14] A. Theljani, Z. Belhachmi M. Moakher, High-order anisotropic diffusion operators in spaces of variable exponents and application to image inpainting and restoration problems. *Nonlinear Anal. Real World Appl.* **47** (2019), 251–271.

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