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Alberto Cabada, Lucía López-Somoza, Mouhcine Yousfi

RELATIONSHIP OF GREEN'S FUNCTIONS RELATED TO HILL'S EQUATION COUPLED TO DIFFERENT BOUNDARY CONDITIONS

Abstract. In this paper, we deduce several properties of Green's functions related to Hill's equation coupled to various boundary value conditions. In particular, the idea is to study Green's functions of the second order differential operator coupled to the Neumann, Dirichlet, periodic and mixed boundary conditions, by expressing Green's function of a given problem as a linear combination of Green's functions of the other problems. This will allow us to compare different Green's functions when their sign is constant. Finally, such properties of Green's function of the linear problem will be fundamental to deduce the existence of solutions to the nonlinear problem. The results are derived from the fixed point theory applied to the related operators defined on suitable cones in Banach spaces.

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1 Introduction

This paper deals with the study of Green's functions related to Hill's equation

$$u''(t) + a(t)\, u(t) = 0.$$

This equation has many applications in several fields as it models a large set of physical problems. Some examples of such applications are the inverted pendulum, Airy's equation or Mathieu's equation, which can be found in [3, 10, 12, 14, 15, 18].

Furthermore, it is important to note that the results obtained for Hill's equation can be easily extended (with a suitable change of variable, see [14]) to a general second order linear differential equation of the form

$$u''(t) + a_1(t)\, u'(t) + a_0\, u(t) = 0$$

provided that the functions a_0 and a_1 have sufficient regularity.

Moreover, the nonhomogeneous problem related to Hill's equation

$$u''(t) + a(t)\, u(t) = \sigma(t)$$

has also been extensively studied (see [1, 3, 7, 9, 11, 13, 15–20] and the references therein), especially coupled to periodic conditions. In this sense, a particularly interesting case happens when σ has constant sign, which can be interpreted as the action of an external force acting over the system in a certain direction (positive or negative). In such a case, the solutions of constant sign of the equation can be interpreted as situations in which the deviation caused by the force is produced only in one direction (that is, the object oscillates only above or below the equilibrium point of the system).

It is in this context when the study of Green's functions gains importance, since the existence of solutions of differential equations with constant sign is directly related to the constant sign of Green's functions. In particular, the fact that Green's function related to a differential problem does not change its sign, allows the application of several topological and iterative methods to deduce the existence results for suitable nonlinear problems.

Having this idea in mind, in [6], the authors developed a method which allows to write Green's functions related to the Neumann, Dirichlet and mixed problems defined on the interval $[0, T]$ as a linear combination of Green's functions of some extended periodic problem (that is, the periodic problem was considered either on the interval $[0, 2T]$ or on $[0, 4T]$ and the potentials for these problems were the even extension $\widetilde{a}$ to $[0, 2T]$ of the potential $a(t)$ considered on $[0, T]$ and the even extension of $\widetilde{a}$ to $[0, 4T]$, respectively). As a consequence of such decomposition, the authors were able to deduce some comparison results between the solutions of the aforementioned problems.Moreover, they were able to relate the constant sign of the corresponding Green's functions.

This paper can be regarded then as a continuation of the work developed in [6], since our main objective will also be the decomposition of some Green's functions in terms of others. However, the techniques used in this paper are completely different from those mentioned in [6]. More concretely, we will consider two different ways of making the decomposition of Green's functions. The first one will be based on the superposition property of the solutions of a differential problem. On the other hand, the second one will make use of a general formula proved in [8], which allows to relate two different Green's functions as long as the boundary value conditions of one of them can be rewritten in terms of the other and both problems are nonresonant.

This way, we will consider periodic, Neumann, Dirichlet and mixed conditions and relate their corresponding Green's functions pairwise. One of the differences between this approach and the one considered in [6] is the fact that here we are able to find a relation between any pair of the aforementioned Green's functions, not only between any of them and the periodic one. Another difference is that in the present paper we are able to connect Green's function related to the periodic problem on $[0, T]$ with Green's function related to any of the other cited boundary condition on $[0, T]$, which was not possible with the techniques used in [6].

As a consequence of the expressions relating Green's functions, we are able to find some connections between their constant signs. Some of the results were already proved in [6] (although, the proof was different) and some others are, as far as we know, new in the literature.

The paper is divided into 5 sections. In Section 2, we compile some preliminary results from [8]. Sections 3 and 4 include the decomposition of Green's functions using the two different approaches mentioned before. Finally, Section 5 includes an application to ensure the existence and find some bounds for the solution of nonlinear problems.

2 Preliminaries

Consider the second order linear operator

$$L\,u(t) := u''(t) + a(t)u(t), \;\; t \in I,$$

with $I \equiv [0,1]$, $a : I \to \mathbb{R}$, $a \in L^1(I)$, and

$$B_i(u) := \sum_{j=0}^{1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \;\; i = 1, 2,$$

where α_j^i, β_j^i are real constants for $i = 1, 2$, $j = 0, 1$.

We will work on the space

$$W^{2,1}(I) = \{u \in C(I) : u' \in AC(I)\},$$

where $AC(I)$ is the set of absolutely continuous functions on I. In particular, we will work with a Banach space $X \subset W^{2,1}(I)$ in which the operator L is nonresonant, that is, the homogeneous equation

$$u''(t) + a(t)u(t) = 0 \;\text{ a.e. } \; t \in I, \quad u \in X,$$

has as a unique solution the trivial one. In such a case, it occurs that for every $\sigma \in L^1(I)$, the non-homogeneous problem

$$u''(t) + a(t)u(t) = \sigma(t) \;\text{ a.e. } \; t \in I, \quad u \in X,$$

has a unique solution given by

$$u(t) = \int_0^1 G(t,s)\sigma(s)\,ds, \;\; \forall\, t \in I,$$

where G denotes the corresponding Green's function which is the unique function that satisfies the following properties (see [4] for details)

Definition 2.1. We say that $G\colon I \times I \to \mathbb{R}$ is Green's function for the problem

$$Lu(t) = \sigma(t), \;\; \text{a.e. } \; t \in I, \quad B_1(u) = h_1, \quad B_2(u) = h_2,$$

being $\sigma \in L^1(I)$ and $h_1, h_2 \in \mathbb{R}$, if it satisfies the following properties:

- $G \in C(I \times I) \cap C^2((I \times I) \setminus \{(s,s), \; s \in I\})$.
- For each $s \in (0,1)$, $G(\,\cdot\,,s)$ solves the differential equation $Ly(t) = 0$ on $[0,s) \cup (s,1]$ and satisfies the boundary conditions $B_1(G(\,\cdot\,,s)) = B_2(G(\,\cdot\,,s)) = 0$.
- For each $t \in (0,1)$, there exist the lateral limits

  $$\frac{\partial}{\partial t}\,G(t^-,t) = \frac{\partial}{\partial t}\,G(t,t^+) \;\text{ and }\; \frac{\partial}{\partial t}\,G(t,t^-) = \frac{\partial}{\partial t}\,G(t^+,t)$$

  and, moreover,

  $$\frac{\partial}{\partial t}\,G(t^+,t) - \frac{\partial}{\partial t}\,G(t^-,t) = \frac{\partial}{\partial t}\,G(t,t^-) - \frac{\partial}{\partial t}\,G(t,t^+) = 1.$$

We compile now some properties of Green's functions related to operator L. The following result is an adaptation of [8, Lemma 3.1] to the problem considered in this paper.

Lemma 2.1. *The problem*

$$L\,u(t) = \sigma(t), \quad a.e.\ t \in I, \quad B_1(u) = B_2(u) = 0, \tag{2.1}$$

has a unique Green's function if and only if following two problems

$$\begin{aligned} L\,u(t) = 0, \quad a.e.\ t \in I, \quad B_1(u) = 1, \quad B_2(u) = 0,\\ L\,u(t) = 0, \quad a.e.\ t \in I, \quad B_1(u) = 0, \quad B_2(u) = 1, \end{aligned}$$

have a unique solution that we denote as ω_1 and ω_2, respectively.

In such a case, for any $\sigma \in L^1(I)$, the problem

$$L\,u(t) = \sigma(t), \quad a.e.\ t \in I, \quad B_1(u) = \lambda_1, \quad B_2(u) = \lambda_2,$$

has a unique solution given by

$$u(t) = \int_0^1 g(t,s)\,\sigma(s)\,ds + \lambda_1\,\omega_1(t) + \lambda_2\,\omega_2(t).$$

Here, by considering $C_1, C_2 : C^1(I) \to \mathbb{R}$, two linear and continuous operators, we formulate the following result for general second order non-local boundary value problems. This result is an adaptation of [8, Theorem 3.2] to the second order problem. The general result (which proves an analogous formula for the arbitrary n-th order problem) can be found in [8].

Theorem 2.1. *Let us suppose that the homogeneous problem of* (2.1) *($\sigma \equiv 0$) has a unique solution ($u \equiv 0$) and let g be its related Green's function. Let $\sigma \in L^1(I)$, and δ_1, δ_2 be such that*

$$\det(I - A) \neq 0,$$

with I the identity matrix of order 2 and $A = (a_{ij})_{2\times 2} \in \mathcal{M}_{2\times 2}$ given by

$$a_{ij} = \delta_j\,C_i(\omega_j), \quad i,j \in \{1,2\}.$$

Then the problem

$$L\,u(t) = \sigma(t), \quad a.e.\ t \in I, \quad B_1(u) = \delta_1\,C_1(u), \quad B_2(u) = \delta_2\,C_2(u), \tag{2.2}$$

has a unique solution $u \in C^2(I)$ given by the expression

$$u(t) = \int_0^1 G(t,s,\delta_1,\delta_2)\,\sigma(s)\,ds,$$

where

$$G(t,s,\delta_1,\delta_2) := g(t,s) + \sum_{i=1}^{2}\sum_{j=1}^{2} \delta_i\,b_{ij}\,\omega_i(t)\,C_j(g(\,\cdot\,,s)), \quad t,s \in I, \tag{2.3}$$

with $B = (b_{ij})_{2\times 2} = (I - A)^{-1}$.

For any $\lambda \in \mathbb{R}$, consider the operator $L[\lambda]$ defined as follows:

$$L[\lambda]\,u(t) \equiv u''(t) + (a(t) + \lambda)\,u(t), \quad t \in I.$$

When working with this operator, to empasize the dependence of Green's function on the parameter λ, we denote by $G[\lambda]$ Green's function related to $L[\lambda]$.

In this paper, we deal with some problems related to the operator $L[\lambda]$, which will be described in the sequel:

- Neumann problem:

$$L[\lambda]\, u(t) = \sigma(t), \;\; \text{a.e. } \; t \in I, \quad u \in X_N = \big\{u \in W^{2,1}(I) : \; u'(0) = u'(1) = 0\big\}. \tag{2.4}$$

- Dirichlet problem:

$$L[\lambda]\, u(t) = \sigma(t), \;\; \text{a.e. } \; t \in I, \quad u \in X_D = \{u \in W^{2,1}(I) : \; u(0) = u(1) = 0\big\}. \tag{2.5}$$

- Mixed problem 1:

$$L[\lambda]\, u(t) = \sigma(t), \;\; \text{a.e. } \; t \in I, \quad u \in X_{M_1} = \big\{u \in W^{2,1}(I) : \; u'(0) = u(1) = 0\big\}. \tag{2.6}$$

- Mixed problem 2:

$$L[\lambda]\, u(t) = \sigma(t), \;\; \text{a.e. } \; t \in I, \quad u \in X_{M_2} = \big\{u \in W^{2,1}(I) : \; u(0) = u'(1) = 0\big\}. \tag{2.7}$$

- Periodic problem:

$$L[\lambda]\, u(t) \; = \sigma(t), \;\; \text{a.e. } \; t \in I, \quad u \in X_P = \big\{u \in W^{2,1}(I) : \; u(0) = u(1), \; u'(0) = u'(1)\big\}. \tag{2.8}$$

We denote by $G_D[\lambda]$, $G_P[\lambda]$, $G_N[\lambda]$, $G_{M_1}[\lambda]$ and $G_{M_2}[\lambda]$ Green's function related to the Dirichlet, Periodic, Neumann, Mixed 1 and Mixed 2 problems, respectively. Moreover, we denote by u_D, u_P, u_N, u_{M_1} and u_{M_2} the solutions of the corresponding problems and by λ_0^D, λ_0^P, λ_0^N, $\lambda_0^{M_1}$ and $\lambda_0^{M_2}$ the first eigenvalues of each problem.

Now, let us consider the following first order differential 2-dimensional linear system:

$$x'(t) = A(t)x(t) + f(t), \;\; \text{a.e. } \; t \in I, \tag{2.9}$$

subject to the two-point boundary value condition

$$B\, x(0) + C\, x(1) = 0, \tag{2.10}$$

being $A \in L^1(I, M_{2\times 2})$, $f \in L^1(I, \mathbb{R}^2)$, $B, C \in \mathcal{M}_{2\times 2}$, and $x \in AC(I, \mathbb{R}^2)$.

From [4, pp. 9 and 15], we know that there is a unique Green's function related to (2.9), (2.10), denoted by g, if and only if $\det(M_\phi) \neq 0$, being

$$M_\phi := B\, \phi(0) + C\, \phi(1)$$

and ϕ any fundamental matrix related to (2.9) (in [4, Remark 1.2.6], it is shown that such a property is independent of the choice of ϕ).

In such a case, the expression of the Green's function g does not depend on the election of the fundamental matrix ϕ and is given by

$$g(t,s) = \begin{cases} -\,\phi(t)\, M_\phi^{-1}\, C\, \phi(1)\, \phi^{-1}(s) + \phi(t)\, \phi^{-1}(s), & 0 \le s < t \le 1, \\ -\,\phi(t)\, M_\phi^{-1}\, C\, \phi(1)\, \phi^{-1}(s), & 0 \le t < s \le 1. \end{cases} \tag{2.11}$$

Let

$$R(t) := g(t,0) = -\phi(t)\, M_\phi^{-1}\, C\, \phi(1)\, \phi^{-1}(0) + \phi(t)\, \phi^{-1}(0), \;\; t \in (0,1].$$

We extend with continuity the function R to the interval I as $R(0) = \lim_{t \to 0^+} R(t)$.

By definition, it is immediate to verify that

$$R'(t) = A(t)\, R(t), \;\; \text{a.e. } \; t \in I.$$

Let us see that

$$B\, R(0) + C\, R(1) = B. \tag{2.12}$$

Indeed, using expression (2.11) we have that

$$\begin{aligned} B\,R(0)+C\,R(1) &= -B\,\phi(0)\,M_\phi^{-1}\,C\,\phi(1)\,\phi^{-1}(0)+B-C\,\phi(1)\,M_\phi^{-1}\,C\,\phi(1)\,\phi^{-1}(0)+C\,\phi(1)\,\phi^{-1}(0)\\ &= -M_\phi\,M_\phi^{-1}\,C\,\phi(1)\,\phi^{-1}(0)+B+C\,\phi(1)\,\phi^{-1}(0)=B. \end{aligned}$$

Now, defining

$$S(t):=g(t,1)=-\phi(t)\,M_\phi^{-1}\,C\,\phi(1)\,\phi^{-1}(1),\quad t\in[0,1),$$

and extending it to I, by $S(1)=\lim_{t\to 1^-} S(t)$, we have that

$$S'(t)=A(t)\,S(t),\quad \text{a.e. } t\in I.$$

Now we verify that

$$B\,S(0)+C\,S(1)=-C. \tag{2.13}$$

Again, using the expression (2.11) we have that

$$\begin{aligned} B\,S(0)+C\,S(1) &= -\,B\,\phi(0)\,M_\phi^{-1}\,C\,\phi(1)\,\phi^{-1}(1)-C\,\phi(1)\,M_\phi^{-1}\,C\,\phi(1)\,\phi^{-1}(1)\\ &= -\,(B\,\phi(0)+C\,\phi(1))\,M_\phi^{-1}\,C=-C. \end{aligned}$$

Now, we observe that the equation

$$L[\lambda]\,u(t)=\sigma(t),\quad \text{a.e. } t\in I, \tag{2.14}$$

can be rewritten as a system of type (2.9) as follows:

$$\begin{pmatrix} u(t)\\ u'(t)\end{pmatrix}'=\begin{pmatrix} 0 & 1\\ -a(t)-\lambda & 0\end{pmatrix}\begin{pmatrix} u(t)\\ u'(t)\end{pmatrix}+\begin{pmatrix} 0\\ \sigma(t)\end{pmatrix}. \tag{2.15}$$

In this case, we have

$$A(t)=\begin{pmatrix} 0 & 1\\ -a(t)-\lambda & 0\end{pmatrix}\quad\text{and}\quad f(t)=\begin{pmatrix} 0\\ \sigma(t)\end{pmatrix}.$$

Now, we give here the expression of different problems related to the operator $L[\lambda]$ mentioned above based on equation (2.10), by giving the corresponding matrices B and C in each case:

- Neumann problem:
$$B=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}.$$
- Dirichlet problem:
$$B=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}.$$
- Mixed problem 1:
$$B=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}.$$
- Mixed problem 2:
$$B=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}.$$
- Periodic problem:
$$B=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}.$$

Remark 2.1. The matrices B and C are not unique, since we can take as B and C a multiple $k\,B$ and $k\,C$ with k a nonzero real number. We can also swap the rows of the two matrices B and C.

Using [5, p. 11], we know that the matrix function

$$g[\lambda](t,s)=\begin{pmatrix} -\dfrac{\partial}{\partial s}\,G[\lambda](t,s) & G[\lambda](t,s) \\ -\dfrac{\partial^2}{\partial s\partial t}\,G[\lambda](t,s) & \dfrac{\partial}{\partial t}\,G[\lambda](t,s) \end{pmatrix}$$

is Green's function related to system (2.15) associated with the differential equation (2.14), coupled to the boundary conditions (2.10), where $G[\lambda]$ is Green's function of the linear equation (2.14) coupled to the boundary conditions (2.10) under the notation $x=\begin{pmatrix} u \\ u' \end{pmatrix}$.

Now, we introduce some auxiliary functions that we are going to use throughout this paper to relate the different problems that we have defined above.

Let us define $r_1[\lambda]$ as the unique solution to the problem

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u(0)=1,\ \ u(1)=0, \tag{2.16}$$

$r_2[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u(0)=0,\ \ u(1)=1, \tag{2.17}$$

$r_3[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u(0)-u(1)=1,\ \ u'(0)-u'(1)=0,$$

$r_4[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u(0)-u(1)=0,\ \ u'(0)-u'(1)=1, \tag{2.18}$$

$r_5[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u'(0)=1,\ \ u'(1)=0,$$

$r_6[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u'(0)=0,\ \ u'(1)=1,$$

$r_7[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u(0)=1,\ \ u'(1)=0,$$

$r_8[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u(0)=0,\ \ u'(1)=1,$$

$r_9[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u'(0)=1,\ \ u(1)=0,$$

$r_{10}[\lambda]$ as the unique solution of the problem

$$L[\lambda]\,u(t)=0,\ \text{ a.e. }\ t\in I,\quad u'(0)=0,\ \ u(1)=1.$$

Now, using equalities (2.12) and (2.13), we will find the expression of $r_1[\lambda]$ as a function of Green's function of the Dirichlet problem.

For the Dirichlet problem, equation (2.13) becomes the following equality:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\dfrac{\partial}{\partial s}\,G_D[\lambda](0,0) & G_D[\lambda](0,0) \\ -\dfrac{\partial^2}{\partial s\partial t}\,G_D[\lambda](0,0) & \dfrac{\partial}{\partial t}\,G_D[\lambda](0,0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\dfrac{\partial}{\partial s}\,G_D[\lambda](1,0) & G_D[\lambda](1,0) \\ -\dfrac{\partial^2}{\partial s\partial t}\,G_D[\lambda](1,0) & \dfrac{\partial}{\partial t}\,G_D[\lambda](1,0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$-\frac{\partial}{\partial s}\,G_D[\lambda](0,0)=1,\quad \frac{\partial}{\partial s}\,G_D[\lambda](1,0)=0,$$
$$G_D[\lambda](0,0)=0,\quad G_D[\lambda](1,0)=0.$$

By the uniqueness of the function $r_1[\lambda]$, it follows that

$$r_1[\lambda](t)=-\frac{\partial}{\partial s}\,G_D[\lambda](t,0).$$

Making similar arguments, we can deduce that

$$r_2[\lambda](t)=\frac{\partial}{\partial s}\,G_D[\lambda](t,1),\quad r_3[\lambda](t)=-\frac{\partial}{\partial s}\,G_P[\lambda](t,0),\quad r_4[\lambda](t)=G_P[\lambda](t,0),$$
$$r_5[\lambda](t)=G_N[\lambda](t,0),\quad r_6[\lambda](t)=-G_N[\lambda](t,1),\quad r_7[\lambda](t)=-\frac{\partial}{\partial s}\,G_{M_2}[\lambda](t,0),$$
$$r_8[\lambda](t)=-G_{M_2}[\lambda](t,1),\quad r_9[\lambda](t)=G_{M_1}[\lambda](t,0),\quad r_{10}[\lambda](t)=-\frac{\partial}{\partial s}\,G_{M_1}[\lambda](t,1).$$

3 Decomposing Green's functions

This section is devoted to the study of the relationships between the expressions of Green's functions related to problems (2.4), (2.5), (2.6), (2.7) and (2.8).

Toward this end, we compare different expressions by putting each boundary condition as a combination of the others.

Such expressions will be deduced from Lemma 2.1. We pay special attention to the fact that in this case we are considering the potential $a(t)$ and the definition on the interval $[0,1]$. So, we make a different approach to the one given in [6], where the expressions are obtained for the corresponding extensions of the potential $a(t)$ to the intervals $[0,2]$ and $[0,4]$.

3.1 Dirichlet and Periodic problems

In this subsection, we study the relation between Green's functions of the Dirichlet and Periodic problems.

Theorem 3.1. *If the operator $L[\lambda]$ is nonresonant both in X_D and X_P, then*

$$\begin{aligned} G_P[\lambda](t,s)&=G_D[\lambda](t,s)-\big(r_1[\lambda](t)+r_2[\lambda](t)\big)\,G_P[\lambda](1,s)\\ &=G_D[\lambda](t,s)+\Big(\frac{\partial}{\partial s}\,G_D[\lambda](t,1)-\frac{\partial}{\partial s}\,G_D[\lambda](t,0)\Big)\,G_P[\lambda](1,s),\quad \forall\,(t,s)\in I\times I. \end{aligned} \tag{3.1}$$

Proof. We express the Green's function related to the Periodic problem (2.8) as a function of the Dirichlet one (2.5) as follows:

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in I,\quad u(0)=u(1),\ \ u(1)=u(1)+u'(0)-u'(1). \tag{3.2}$$

Then, using Lemma 2.1, we find that the solution of problem (3.2) is given by the following expression:

$$\begin{aligned} u_P(t)&=\int_0^1 G_P[\lambda](t,s)\,\sigma(s)\,ds\\ &=\int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds+r_1[\lambda](t)u_P(1)+r_2[\lambda](t)\big(u_P(1)+u_P'(0)-u_P'(1)\big)\\ &=\int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds+r_1[\lambda](t)\int_0^1 G_P[\lambda](1,s)\,\sigma(s)\,ds \end{aligned}$$

$$+ r_2[\lambda](t) \int_0^1 \left[G_P[\lambda](1,s) + \frac{\partial}{\partial t} G_P[\lambda](0,s) - \frac{\partial}{\partial t} G_P[\lambda](1,s) \right] \sigma(s)\, ds$$

$$= \int_0^1 \left[G_D[\lambda](t,s) + \big(r_1[\lambda](t) + r_2[\lambda](t)\big) G_P[\lambda](1,s) \right] \sigma(s)\, ds,$$

where the last equality follows from Definition 2.1, condition ($G6$):

$$\frac{\partial}{\partial t} G_P[\lambda](0,s) = \frac{\partial}{\partial t} G_P[\lambda](1,s), \quad \forall\, s \in (0,1).$$

Since previous equalities hold for every $\sigma \in L^1(I)$, we obtain (3.1). □

Remark 3.1. We point out that, as a direct consequence of Lemma 2.1, we have that both $r_1[\lambda]$ and $r_2[\lambda]$ are uniquely determined. In fact, with the notation used in Lemma 2.1, we have $B_1(u) = u(0)$, $B_2(u) = u(1)$, $\omega_1 = r_1[\lambda]$ and $\omega_2 = r_2[\lambda]$.

Next, we study the oscillation of the functions $r_1[\lambda]$ and $r_2[\lambda]$ by using the Sturm–Liouville theory of eigenvalues. Let $\{\lambda_n^D\}_{n=0}^{\infty}$ be the sequence of eigenvalues of the Dirichlet problem

$$(D_\lambda) \qquad L[\lambda]\, u(t) = 0, \;\text{ a.e. } t \in I, \quad u(0) = u(1) = 0.$$

It is well-known that $\lim\limits_{n\to\infty} \lambda_n^D = \infty$ (see [21, Theorem 4.3.1]) and that any of the eigenvalues has a single associated eigenvector v_n such that

$$(D_n) \qquad L[\lambda_n^D]\, v_n(t) = 0, \;\text{ a.e. } t \in I, \quad v_n(0) = v_n(1) = 0,$$

with exactly n zeros in $(0,1)$.

Moreover, this eigenfunction satisfies the condition $v_n'(0) \neq 0$.

Lemma 3.1. *Problem* (2.16) *has a unique solution if and only if* $\lambda \neq \lambda_n^D$, $n = 0, 1, \dots$.

Lemma 3.2. *The unique solution* $r_1[\lambda]$ *of problem* (2.16) *has exactly* n *zeros in* $(0,1)$ *if and only if* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \dots$, *and* $r_1[\lambda] > 0$ *on* $[0,1)$ *if and only if* $\lambda < \lambda_0^D$. *In addition,* $(-1)^n r_1'(1) < 0$ *for all* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \dots$, *and* $r_1'[\lambda](1) < 0$, *for all* $\lambda < \lambda_0^D$.

Lemma 3.3. *Problem* (2.17) *has a unique solution if and only if* $\lambda \neq \lambda_n^D$, $n = 0, 1, \dots$.

Lemma 3.4. *The unique solution of problem* (2.17) $r_2[\lambda]$ *has exactly* n *zeros in* $(0,1)$ *if and only if* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \dots$ *and* $r_2[\lambda] > 0$ *on* $(0,1]$ *if and only if* $\lambda < \lambda_0^D$. *In addition,* $(-1)^n r_2'[\lambda](0) > 0$ *for all* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \dots$ *and* $r_2'[\lambda](0) > 0$, *for all* $\lambda < \lambda_0^D$.

Remark 3.2. Lemmas 3.1 and 3.3 are corollaries of Lemma 2.1. Lemmas 3.2 and 3.4 follow from Sturm's comparison theorem.

As a direct consequence of equality (3.1), we deduce the following comparison between the values of Green's functions related to the Dirichlet and Periodic problems.

Theorem 3.2. *The following inequality holds:*

$$G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \;\; \forall\,(t,s) \in (0,1) \times (0,1), \;\; \forall\, \lambda < \lambda_0^P. \tag{3.3}$$

Proof. It is immediately can be verified that the function $r[\lambda](t) := r_1[\lambda](t) + r_2[\lambda](t)$ solves the following problem:

$$L[\lambda]\, r[\lambda](t) = 0, \;\text{ a.e. } t \in I, \quad r[\lambda](0) = r[\lambda](1) = 1.$$

From Lemmas 3.2 and 3.4, it is obvious that if $\lambda < \lambda_0^D$, then $r[\lambda](t) > 0$ for all $t \in I$.

Moreover, we know that $G_P[\lambda]$ is negative on $I \times I$ for all $\lambda < \lambda_0^P$ and $G_D[\lambda]$ is negative on $(0,1) \times (0,1)$ for all $\lambda < \lambda_0^D$ (see [6, Lemma 2.9]). In addition, $\lambda_0^P < \lambda_0^D$ [7, p. 44].

As $r[\lambda](t) > 0$ for all $t \in I$ when $\lambda < \lambda_0^D$, using (3.1), we obtain the result. □

Remark 3.3. From equality (3.1) it follows that

$$\frac{G_D[\lambda](t,s)-G_P[\lambda](t,s)}{G_P[\lambda](1,s)}=-r[\lambda](t),$$

if λ is not an eigenvalue of the Dirichlet and Periodic problems.

Deriving the above equality with respect to s, we obtain the following identity

$$\begin{aligned}\frac{\partial}{\partial s}\big(G_D[\lambda](t,s)-G_P[\lambda](t,s)\big)G_P[\lambda](1,s)&\\ =\big(G_D[\lambda](t,s)-G_P[\lambda](t,s)\big)\,\frac{\partial}{\partial s}\,G_P[\lambda](1,s),\ \ \forall\,(t,s)\in I\times I.&\end{aligned}$$

In the sequel, we will carryout an alternative study to the one done in Theorem 3.1. In this case, we consider the Dirichlet conditions as a combination of the periodic ones.

Let us write the Dirichlet problem as a function of the periodic problem as follows:

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in I,\quad u(0)-u(1)=-u(1),\ \ u'(0)-u'(1)=u'(0)-u'(1)+u(1).$$

Taking into account that $r_4[\lambda](t)=G_P[\lambda](t,0)$ solves (2.18), performing the calculations in an analogous way as before, using Lemma 2.1, the following result is attained.

Theorem 3.3. *Assume that the operator $L[\lambda]$ is nonresonant both in X_D and X_P, then there holds:*

$$\begin{aligned}G_D[\lambda](t,s)&=G_P[\lambda](t,s)+r_4[\lambda](t)\left(\frac{\partial}{\partial t}\,G_D[\lambda](0,s)-\frac{\partial}{\partial t}\,G_D[\lambda](1,s)\right)\\&=G_P[\lambda](t,s)+G_P[\lambda](t,0)\left(\frac{\partial}{\partial t}\,G_D[\lambda](0,s)-\frac{\partial}{\partial t}\,G_D[\lambda](1,s)\right),\ \ \forall\,(t,s)\in I\times I.\end{aligned}\tag{3.4}$$

Remark 3.4. Notice that if $\lambda<\lambda_0^D$, we have $G_D[\lambda]<0$ on $(0,1)\times(0,1)$ and, as a consequence,

$$\frac{\partial}{\partial t}\,G_D[\lambda](0,s)<0<\frac{\partial}{\partial t}\,G_D[\lambda](1,s),\ \ s\in(0,1).$$

Moreover, if $\lambda<\lambda_0^D$, then $G_P[\lambda]<0$ on $I\times I$. So, from (3.4) and the fact that $\lambda_0^P<\lambda_0^D$, we deduce inequality (3.3) again.

3.2 Dirichlet and Neumann problems

In this section, we continue the work done in the previous section. In this case, we will consider the Dirichlet and Neumann problems. We will obtain some expressions that allow us to connect both Green's functions.

Theorem 3.4. *Assume that the operator $L[\lambda]$ is nonresonant in the spaces X_D and X_N. Then the following equality is satisfied:*

$$\begin{aligned}G_N[\lambda](t,s)&=G_D[\lambda](t,s)+r_1[\lambda](t)\,G_N[\lambda](0,s)+r_2[\lambda](t)\,G_N[\lambda](1,s)\\&=G_D[\lambda](t,s)-\frac{\partial}{\partial s}\,G_D[\lambda](t,0)\,G_N[\lambda](0,s)\\&\qquad+\frac{\partial}{\partial s}\,G_D[\lambda](t,1)\,G_N[\lambda](1,s),\ \ \forall\,(t,s)\in I\times I.\end{aligned}\tag{3.5}$$

Proof. Let us rewrite the Neumann problem in the following way:

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in I,\quad u(0)=u(0)+u'(0),\ \ u(1)=u(1)+u'(1).$$

Using Lemma 2.1, the solution to the above problem is

$$\begin{aligned} u_N(t) &= \int_0^1 G_N[\lambda](t,s)\,\sigma(s)\,ds = \int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds + r_1[\lambda](t)u_N(0) + r_2[\lambda](t)u_N(1) \\ &= \int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds + r_1[\lambda](t)\int_0^1 G_N[\lambda](0,s)\,\sigma(s)\,ds + r_2[\lambda](t)\int_0^1 G_N[\lambda](0,s)\,\sigma(s)\,ds. \end{aligned}$$

Therefore, since the previous equalities hold for every $\sigma \in L^1(I)$, we obtain (3.5). □

Corollary 3.1. *The following inequality holds:*

$$G_N[\lambda](t,s) < G_D[\lambda](t,s) < 0, \;\; \forall\,(t,s) \in (0,1)\times(0,1), \;\; \forall\,\lambda < \lambda_0^N. \tag{3.6}$$

Proof. We know that, from Lemmas 3.2 and 3.4, $r_1[\lambda]$ and $r_2[\lambda]$ are positive on $(0,1)$ for all $\lambda < \lambda_0^D$. In addition, $\lambda_0^N \le \lambda_0^P < \lambda_0^D$ (see [7, p. 44]), $G_N[\lambda] < 0$ on $I \times I$ for all $\lambda < \lambda_0^N$ (see [6, Corollary 4.5]) and $G_D[\lambda] < 0$ on $(0,1)\times(0,1)$ for all $\lambda < \lambda_0^D$ (see [6, Lemma 2.9]). Then for all $\lambda < \lambda_0^N$, $r_1[\lambda]$ and $r_2[\lambda]$ are positive on $(0,1)$. Hence, using (3.5), we obtain the result. □

Remark 3.5. The above result can be deduced from [6, Corollaries 4.5, 4.8 and 4.10], but in a different way than that we have explained here. In such reference, the argument used is based on considering the even extension of the solution to the interval $[0,2]$. In any case, expression (3.5) relating $G_N[\lambda]$ and $G_D[\lambda]$ is different from the one obtained in that article.

For the reverse process, by writing the Dirichlet problem as a function of Neumann problem as

$$L[\lambda]\,u(t) = \sigma(t), \;\; \text{a.e. } t \in I, \quad u'(0) = u(0) + u'(0), \;\; u'(1) = u(1) + u'(1),$$

we arrive at the next result as a consequence of Lemma 2.1.

Theorem 3.5. *Assume that the operator $L[\lambda]$ is nonresonant in the spaces X_D and X_N. Then the following equalities are satisfied:*

$$\begin{aligned} G_D[\lambda](t,s) &= G_N[\lambda](t,s) + r_5[\lambda](t)\,\frac{\partial}{\partial t}\,G_D[\lambda](0,s) + r_6[\lambda](t)\,\frac{\partial}{\partial t}\,G_D[\lambda](1,s) \\ &= G_N[\lambda](t,s) + G_N[\lambda](t,0)\,\frac{\partial}{\partial t}\,G_D[\lambda](0,s) \\ &\quad - G_N[\lambda](t,1)\,\frac{\partial}{\partial t}\,G_D[\lambda](1,s), \;\; \forall\,(t,s) \in I \times I. \end{aligned} \tag{3.7}$$

Remark 3.6. Since for $\lambda < \lambda_0^N$, we have $G_N[\lambda] < 0$ on $I \times I$ and $G_D[\lambda] < 0$ on $(0,1)\times(0,1)$, we conclude from (3.7) that inequality (3.6) is valid again.

3.3 Dirichlet and Mixed problems

In this case, we carry out an analysis of the relationship between Green's functions of the Dirichlet and Mixed problems. Following the same steps as before in the previous subsection, we get the next result.

Theorem 3.6. *Assume that $L[\lambda]$ is nonresonant both in X_D and X_{M_1}, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_D[\lambda](t,s) + r_1[\lambda](t)\,G_{M_1}[\lambda](0,s) \\ &= G_D[\lambda](t,s) - \frac{\partial}{\partial s}\,G_D[\lambda](t,0)\,G_{M_1}[\lambda](0,s), \;\; \forall\,(t,s) \in I \times I. \end{aligned} \tag{3.8}$$

As a consequence, we deduce the following result.

Corollary 3.2. *The following inequality holds:*

$$G_{M_1}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \;\; \forall\,(t,s) \in (0,1) \times (0,1), \;\; \forall\,\lambda < \lambda_0^{M_1}. \tag{3.9}$$

Proof. The inequality $\lambda_0^{M_1} < \lambda_0^D$ is provided in [6, Remark 4.19]. In addition, we have that $G_{M_1}[\lambda] < 0$ on $[0,1) \times [0,1)$ if and only if $\lambda < \lambda_0^{M_1}$ (see [6, Corollary 4.7]) and $G_D[\lambda] < 0$ on $(0,1) \times (0,1)$ if and only if $\lambda < \lambda_0^D$, which implies that $\frac{\partial}{\partial s} G_D(t,0) < 0$ for all $\lambda < \lambda_0^D$ and $t \in (0,1)$.

Therefore, using (3.8), we deduce the inequality. □

Similarly, for Mixed 2 problem, we arrive at the following results.

Theorem 3.7. *If the operator $L[\lambda]$ is nonresonant in X_D and X_{M_2}, then the following equality holds:*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_D[\lambda](t,s) + r_2[\lambda](t)\, G_{M_2}[\lambda](1,s) \\ &= G_D[\lambda](t,s) + \frac{\partial}{\partial s} G_D[\lambda](t,1)\, G_{M_2}[\lambda](1,s), \;\; \forall\,(t,s) \in I \times I, \end{aligned} \tag{3.10}$$

Corollary 3.3. *The following inequality holds:*

$$G_{M_2}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \;\; \forall\,(t,s) \in (0,1) \times (0,1), \;\; \forall\,\lambda < \lambda_0^{M_2}.$$

Remark 3.7. The above inequality between $G_{M_2}[\lambda]$ and $G_D[\lambda]$ can be deduced from [6, Corollaries 4.7, 4.8, 4.13]. Moreover, expression (3.10) relating $G_{M_2}[\lambda]$ and $G_D[\lambda]$ is different from the one obtained in that reference.

However, as far as we know, there is no expression in the literature that relate G_{M_1} and G_D and, as a consequence, equality (3.8) and inequality (3.9) are new.

Analogously to previous sections, we can relate expressions of Green's function of the Dirichlet problem and the ones of the Mixed problems.

Theorem 3.8. *If the operator $L[\lambda]$ is nonresonant in X_D and X_{M_2}, then*

$$\begin{aligned} G_D[\lambda](t,s) &= G_{M_2}[\lambda](t,s) + r_8[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) \\ &= G_{M_2}[\lambda](t,s) - G_{M_2}[\lambda](t,1)\, \frac{\partial}{\partial t} G_D[\lambda](1,s), \;\; t,s \in I. \end{aligned}$$

Theorem 3.9. *If the operator $L[\lambda]$ is nonresonant in X_D and X_{M_1}, then*

$$\begin{aligned} G_D[\lambda](t,s) &= G_{M_1}[\lambda](t,s) + r_9[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &= G_{M_1}[\lambda](t,s) + G_{M_1}[\lambda](t,0)\, \frac{\partial}{\partial t} G_D[\lambda](0,s), \;\; t,s \in I. \end{aligned}$$

Remark 3.8. Notice that from two previous results we can deduce Corollaries 3.2 and 3.3.

3.4 Neumann and Mixed problems

In this section, arguing in a similar manner as in the previous ones, we can relate the expression of Green's functions of the Neumann problem and the ones of the corresponding Mixed ones.

Theorem 3.10. *Assume that the operator $L[\lambda]$ is nonresonant in X_N and X_{M_1}. Then*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_N[\lambda](t,s) + r_5[\lambda](t)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) \\ &= G_N[\lambda](t,s) + G_N[\lambda](t,0)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s), \;\; \forall\,(t,s) \in I \times I. \end{aligned} \tag{3.11}$$

Corollary 3.4. *The following inequality holds:*

$$G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s) < 0, \;\; \forall\,(t,s) \in (0,1] \times (0,1], \;\; \forall\,\lambda < \lambda_0^N. \tag{3.12}$$

Proof. We know that $G_{M_2}[\lambda](t,s) < 0$ for all $(t,s) \in (0,1] \times (0,1]$ if and only if $\lambda < \lambda_0^{M_2}$ (see [6, Corollary 4.6]). Since $G_{M_2}[\lambda](0,s) = 0$, we deduce that $\frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) < 0$ for such λ. In addition, $\lambda_0^N < \lambda_0^{M_1}$ (see [6, Remark 4.19]). Therefore, using equality (3.11), we obtain the result. □

Analogously, for Mixed 1 problem, we have the following results.

Theorem 3.11. *Assume that $L[\lambda]$ is nonresonant in X_{M_1} and X_N, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_N[\lambda](t,s) + r_6[\lambda](t) \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s) \\ &= G_N[\lambda](t,s) - G_N[\lambda](t,1) \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned} \tag{3.13}$$

Corollary 3.5. *The following equality is fulfilled:*

$$G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0, \quad \forall\, (t,s) \in [0,1) \times [0,1), \quad \forall\, \lambda < \lambda_0^N. \tag{3.14}$$

Remark 3.9. Inequality (3.14) can be deduced from [6, Corollaries 4.5, 4.8, 4.13]. Identities (3.11) and (3.13) together with inequality (3.12) are new.

By the reciprocal process, we can obtain additional relations between Green's function of the Neumann problem and the ones of the Mixed problems as follows.

Theorem 3.12. *If the operator $L[\lambda]$ is nonresonant in X_N and X_{M_2}, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_{M_2}[\lambda](t,s) + r_7[\lambda](t)\, G_N[\lambda](0,s) \\ &= G_{M_2}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_2}[\lambda](t,0)\, G_N[\lambda](0,s), \quad t,s \in I. \end{aligned}$$

Theorem 3.13. *If the operator $L[\lambda]$ is nonresonant in X_N and X_{M_1}, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_{M_1}[\lambda](t,s) + r_{10}[\lambda](t)\, G_N[\lambda](1,s) \\ &= G_{M_1}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)\, G_N[\lambda](1,s), \quad t,s \in I. \end{aligned}$$

Remark 3.10. Notice that Corollaries 3.4 and 3.5 can be deduced from Theorem 3.12 and 3.13, respectively.

3.5 Periodic and Neumann problems

Concerning the Neumann and Periodic problems and arguing as before, we arrive at the next theorem.

Theorem 3.14. *If the operator $L[\lambda]$ is nonresonant both in X_N and X_P, the following equality is fulfilled:*

$$\begin{aligned} G_P[\lambda](t,s) &= G_N[\lambda](t,s) + \big(r_5[\lambda](t) + r_6[\lambda](t)\big) \frac{\partial}{\partial t} G_P[\lambda](0,s) \\ &= G_N[\lambda](t,s) + \big(G_N[\lambda](t,0) - G_N[\lambda](t,1)\big) \frac{\partial}{\partial t} G_P[\lambda](0,s), \quad \forall\, (t,s) \in I \times I. \end{aligned} \tag{3.15}$$

Remark 3.11. From (3.15) and due to the symmetry of $G_P[\lambda]$ and $G_N[\lambda]$, we deduce that

$$\begin{aligned} &\big(G_N[\lambda](t,0) - G_N[\lambda](t,1)\big) \frac{\partial}{\partial t} G_P[\lambda](0,s) \\ &\qquad = \big(G_N[\lambda](s,0) - G_N[\lambda](s,1)\big) \frac{\partial}{\partial t} G_P[\lambda](0,t), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

If $\frac{\partial}{\partial t} G_P[\lambda](0,t) \neq 0$ and $\frac{\partial}{\partial t} G_P[\lambda](0,s) \neq 0$, then

$$\frac{G_N[\lambda](t,0) - G_N[\lambda](t,1)}{\frac{\partial}{\partial t} G_P[\lambda](0,t)} = \frac{G_N[\lambda](s,0) - G_N[\lambda](s,1)}{\frac{\partial}{\partial t} G_P[\lambda](0,s)} = c_1 \in \mathbb{R}.$$

We know that

$$\frac{\partial}{\partial t} G_P[\lambda](t,s) = \frac{\partial}{\partial s} G_P[\lambda](s,t) \text{ and } \frac{\partial}{\partial s} G_P[\lambda](t,s) = \frac{\partial}{\partial t} G_P[\lambda](s,t).$$

Then

$$G_N(t,0) - G_N(t,1) = c_1 \frac{\partial}{\partial t} G_P[\lambda](0,t) = c_1 \frac{\partial}{\partial s} G_P[\lambda](t,0).$$

With the reverse process we arrive at the following result.

Theorem 3.15. *Assume that $L[\lambda]$ is nonresonant in X_N and X_P, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_P[\lambda](t,s) + r_3[\lambda](t)\,(G_N[\lambda](0,s) - G_N[\lambda](1,s)) \\ &= G_P[\lambda](t,s) - \frac{\partial}{\partial s} G_P[\lambda](t,0)\,(G_N[\lambda](0,s) - G_N[\lambda](1,s)), \quad \forall\,(t,s) \in I \times I. \end{aligned}$$

3.6 Periodic and Mixed problems

The same arguments as in the previous subsections are applicable to the Periodic and Mixed 1 problems. We omit the proof, which is analogous to those of previous cases.

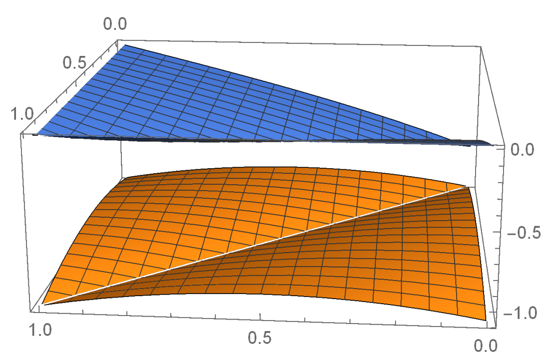
Theorem 3.16. *Assume that $L[\lambda]$ is nonresonant in X_P and X_{M_1}. Then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_P[\lambda](t,s) + r_3[\lambda](t)\, G_{M_1}[\lambda](0,s) - r_4[\lambda](t) \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s) \\ &= G_P[\lambda](t,s) - \frac{\partial}{\partial s} G_P[\lambda](t,0)\, G_{M_1}[\lambda](0,s) \\ &\qquad - G_P[\lambda](t,0) \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s), \quad \forall\,(t,s) \in I \times I. \end{aligned}$$

Next example shows that, in general, Green's functions of Periodic and Mixed 1 problems are not comparable.

Example 3.1. We consider the differential equation $u''(t) - m^2\,u(t) = 0$, $t \in I$ and $m \in (0,\infty)$. In this case, $a(t) = -m^2$, $t \in I$, $\lambda = 0$ and $m \in (0,\infty)$.

Green's functions G_P and G_{M_1} are comparable for small values of m. Figure 3.1 represents Green's functions G_P and G_{M_1} for $m = 1$ (in which case $G_P < G_{M_1}$) and for $m = 2$ (which are not comparable).



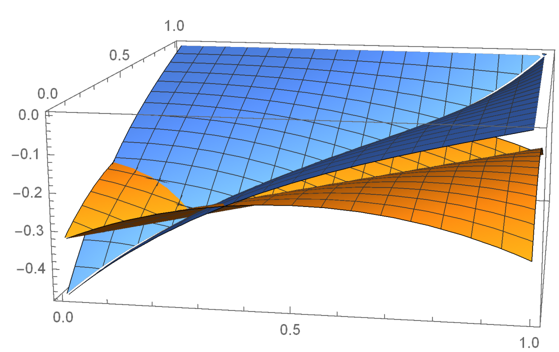


Figure 3.1: The blue graph corresponds to function G_{M_1} and the orange graph represents the function G_P on $I \times I$. The figure on the left is the case $m = 1$ and the figure on the right is the case $m = 2$.

Analogously, we study the relationship between Green's functions of Periodic and Mixed 2 problems.

Theorem 3.17. *Assume that $L[\lambda]$ is nonresonant in X_P and X_{M_2}, then*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_P[\lambda](t,s) - r_3[\lambda](t)\, G_{M_2}[\lambda](1,s) + r_4[\lambda](t)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) \\ &= G_P[\lambda](t,s) + \frac{\partial}{\partial s} G_P[\lambda](t,0)\, G_{M_2}[\lambda](1,s) \\ &\quad + G_P[\lambda](t,0)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

The above equation is analogous to the equation relating Periodic to Mixed 1 problems. So, in general, Green's functions of the Periodic and Mixed 2 problems will not be comparable either.

Example 3.2. In this example, we use the same equation as in Example 3.1.

Green's functions G_P and G_{M_2} are comparable for small values of m. Figure 3.2 represents Green's functions G_P and G_{M_2} for $m = 1$ (in which case $G_P < G_{M_2}$) and for $m = 3$ (which are not comparable).

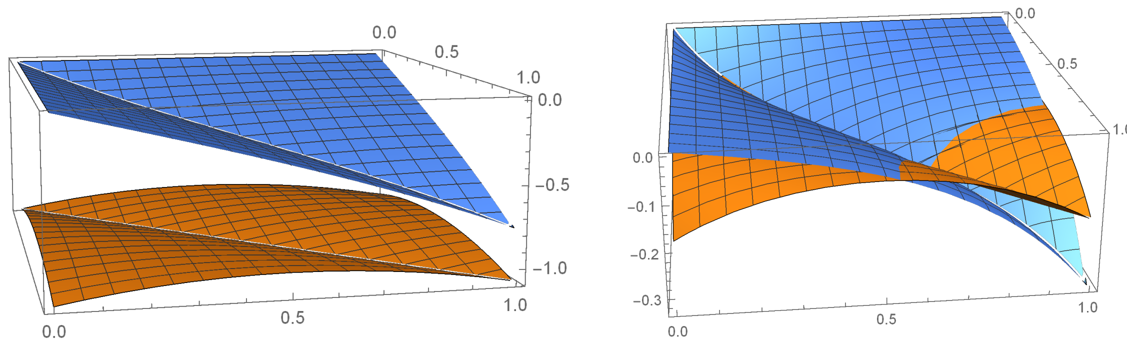


Figure 3.2: The blue graph corresponds to the function G_{M_2} and the orange graph represents the function G_P on $I \times I$. The figure on the left is the case $m = 1$ and the figure on the right is the case $m = 3$.

Finally, for the reverse process, we can obtain additional relations for Green's function of the Periodic and the ones related to Mixed problems.

Theorem 3.18. *If the operator $L[\lambda]$ is nonresonant both in X_P and X_{M_2}, then*

$$\begin{aligned} G_P[\lambda](t,s) &= G_{M_2}[\lambda](t,s) + r_7[\lambda](t)\, G_P[\lambda](1,s) + r_8[\lambda](t)\, \frac{\partial}{\partial t} G_P[\lambda](0,s) \\ &= G_{M_2}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_2}[\lambda](t,0)\, G_P[\lambda](1,s) - G_{M_2}[\lambda](t,1)\, \frac{\partial}{\partial t} G_P[\lambda](0,s), \quad t,s \in I. \end{aligned}$$

Theorem 3.19. *If the operator $L[\lambda]$ is nonresonant both in X_P and X_{M_1}, then*

$$\begin{aligned} G_P[\lambda](t,s) &= G_{M_1}[\lambda](t,s) + r_9[\lambda](t)\, \frac{\partial}{\partial t} G_P[\lambda](1,s) + r_{10}[\lambda](t)\, G_P[\lambda](0,s) \\ &= G_{M_1}[\lambda](t,s) + G_{M_1}[\lambda](t,0)\, \frac{\partial}{\partial t} G_P[\lambda](1,s) - \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)\, G_P[\lambda](0,s), \quad (t,s) \in I \times I. \end{aligned}$$

4 Alternative decomposition of Green's functions

This section is devoted to the derivation of additional relationships between the expressions of Green's functions related to different boundary value conditions studied in the previous section. The main difference consists in the fact that in this case, instead of Lemma 2.1 as in the previous section, we use Theorem 2.1.

It is important to point out that in this situation, as an application of equality (2.3), we are able to express any considered Green's function explicitly from any other one.

The obtained expressions will be different to the ones deduced in the previous section.

4.1 Dirichlet and Mixed problems

We start this subsection by expressing Green's function of Mixed 2 problem in terms of Green's function of Dirichlet problem.

Theorem 4.1. *If the operator $L[\lambda]$ is nonresonant in X_D and $r_2'[\lambda](1) \neq 0$, then the following equality holds:*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_D[\lambda](t,s) - \frac{r_2[\lambda](t)}{r_2'[\lambda](1)} \frac{\partial}{\partial t} G_D[\lambda](1,s) \\ &= G_D[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_D\lambda](t,1)}{\frac{\partial^2}{\partial s \partial t} G_D[\lambda](1,1)} \frac{\partial}{\partial t} G_D[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned} \tag{4.1}$$

Proof. We write Mixed 2 problem based on the Dirichlet problem as follows:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in I, \quad u(0) = 0, \quad u(1) = u(1) + u'(1). \tag{4.2}$$

Using the notation of Theorem 2.1, we have that in this case $C_1(u) = 0$, $C_2(u) = u(1) + u'(1)$ and $\delta_1 = \delta_2 = 1$. Moreover, $\omega_1(t) = r_1[\lambda](t)$, $\omega_2(t) = r_2[\lambda](t)$ and the matrix A_D^1 in this case is

$$A_D^1 = \begin{pmatrix} 0 & 0 \\ r_1'[\lambda](1) & 1 + r_2'[\lambda](1) \end{pmatrix}$$

and $|I - A_D^1[\lambda]| = -r_2'[\lambda](1) \neq 0$. So,

$$b_D^1 = (I - A_D^1[\lambda])^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{r_1'[\lambda](1)}{r_2'[\lambda](1)} & -\dfrac{1}{r_2'[\lambda](1)} \end{pmatrix}.$$

In consequence, as a direct application of equality (2.3), we obtain the result. □

Corollary 4.1. *For all $\lambda < \lambda_0^{M_2}$, we infer that $r_2'[\lambda](1) > 0$.*

Proof. From Corollary 3.3, we have $G_{M_2}[\lambda] < G_D[\lambda] < 0$ for all $\lambda < \lambda_0^{M_2}$ and, as a direct consequence, $\frac{\partial}{\partial t} G_D[\lambda](1,s) > 0$. Lemma 3.4 says us that $r_2[\lambda] > 0$ on $(0,1]$ for all $\lambda < \lambda_0^D$. Since (see $\lambda_0^{M_2} < \lambda_0^D$ [7, p. 108]), we deduce from equality (4.1) that $r_2'[\lambda](1) > 0$. □

Similarly, we study Mixed 1 problem as a function of the Dirichlet one.

Theorem 4.2. *If the operator $L[\lambda]$ is nonresonant in X_D and $r_1'[\lambda](0) \neq 0$, then the following equality holds:*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_D[\lambda](t,s) - \frac{r_1[\lambda](t)}{r_1'[\lambda](0)} \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &= G_D[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_D\lambda](t,0)}{\frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,0)} \frac{\partial}{\partial t} G_D[\lambda](0,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

Proof. Let us rewrite Mixed 1 problem in the following way:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in I, \quad u(0) = u(0) + u'(0), \quad u(1) = 0. \tag{4.3}$$

In this case, we have that $C_1(u) = u(0) + u'(0)$, $C_2(u) = 0$ and $\delta_1 = \delta_2 = 1$. Moreover, $\omega_1(t) = r_1[\lambda](t)$, $\omega_2(t) = r_2[\lambda](t)$ and the matrix $A_D^2[\lambda]$ is

$$A_D^2[\lambda] = \begin{pmatrix} 1 + r_1'[\lambda](0) & r_2'[\lambda](0) \\ 0 & 0 \end{pmatrix}$$

and $|I - A_D^2[\lambda]| = -r_1'[\lambda](0) \neq 0$. So,

$$b_D^2 = (I - A_D^2[\lambda])^{-1} = \begin{pmatrix} -\dfrac{1}{r_1'[\lambda](0)} & -\dfrac{r_2'[\lambda](0)}{r_1'[\lambda](0)} \\ 0 & 1 \end{pmatrix}.$$

Therefore, using (2.3), we deduce the equality. □

Corollary 4.2. *For all $\lambda < \lambda_0^{M_1}$, we infer that $r_1'[\lambda](0) < 0$.*

Proof. From Lemma 3.2, we know that $r_1[\lambda] > 0$ on $[0, 1)$ for all $\lambda < \lambda_0^D$. Corollary 3.2 ensures that $G_{M_1}[\lambda] < G_D[\lambda]$ for all $\lambda < \lambda_0^{M_1}$. Since $\lambda_0^{M_1} < \lambda_0^D$ (see [7, p. 108]), we arrive at the result. □

Remark 4.1. In problem (4.2), we can perform the calculations in a simpler way by taking $C_1(u) = u(0) + u'(0)$, $C_2(u) = 0$, $\delta_1 = 1$ and $\delta_2 = 0$. The same can be done with problem (4.3) by taking $C_1(u) = C_2(u) = u(1) + u'(1)$, $\delta_1 = 0$ and $\delta_2 = 1$.

We now carryout the process backwards by writing the Dirichlet problem based on the Mixed ones. We arrive at the following results.

Theorem 4.3. *If the operator $L[\lambda]$ is nonresonant in X_{M_2} and $r_8[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_D[\lambda](t,s) &= G_{M_2}[\lambda](t,s) - \frac{r_8[\lambda](t)}{r_8[\lambda](1)}\, G_{M_2}[\lambda](1,s) \\ &= G_{M_2}[\lambda](t,s) - \frac{G_{M_2}[\lambda](t,1)}{G_{M_2}[\lambda](1,1)}\, G_{M_2}[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

Theorem 4.4. *If the operator $L[\lambda]$ is nonresonant in X_{M_1} and $r_9[\lambda](0) \neq 0$, then*

$$\begin{aligned} G_D[\lambda](t,s) &= G_{M_1}[\lambda](t,s) - \frac{r_9[\lambda](t)}{r_9[\lambda](0)}\, G_{M_1}[\lambda](0,s) \\ &= G_{M_1}[\lambda](t,s) - \frac{G_{M_1}[\lambda](t,0)}{G_{M_1}[\lambda](0,0)}\, G_{M_1}[\lambda](0,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

4.2 Neumann and Dirichlet problems

In this case, we study the relationships between Green's function of the Neumann and Dirichlet problems. Reasoning as in the previous subsection, we have the next result.

Theorem 4.5. *If the operator $L[\lambda]$ is nonresonant in X_D and*

$$|I - A_D^3[\lambda]| := r_1'[\lambda](0)\, r_2'[\lambda](1) - r_2'[\lambda](0)\, r_1'[\lambda](1) \neq 0,$$

then

$$\begin{aligned} G_N[\lambda](t,s) = G_D[\lambda](t,s) &- \frac{r_2'[\lambda](1)}{|I - A_D^3[\lambda]|}\, r_1[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) + \frac{r_1'[\lambda](1)}{|I - A_D^3[\lambda]|}\, r_2[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &+ \frac{r_2'[\lambda](0)}{|I - A_D^3[\lambda]|}\, r_1[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) - \frac{r_1'[\lambda](0)}{|I - A_D^3[\lambda]|}\, r_2[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) \\ = G_D[\lambda](t,s) &+ \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](1,1)\, \frac{\partial}{\partial s} G_D[\lambda](t,0)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &- \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](1,0)\, \frac{\partial}{\partial s} G_D[\lambda](t,1)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &- \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,1)\, \frac{\partial}{\partial s} G_D[\lambda](t,0)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) \\ &+ \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,0)\, \frac{\partial}{\partial s} G_D[\lambda](t,1)\, \frac{\partial}{\partial t} G_D[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

We now reverse the process by studying the Dirichlet problem as a function of the Neumann one and apply analogous calculations.

Theorem 4.6. *Assume that $L[\lambda]$ is nonresonant in X_N and*

$$|I - A_N^1[\lambda]| := r_5[\lambda](0)\, r_6[\lambda](1) - r_6[\lambda](0)\, r_5[\lambda](1) \neq 0,$$

then

$$\begin{aligned} G_D[\lambda](t,s) = G_N[\lambda](t,s) - \frac{1}{|I - A_N^1[\lambda]|} \Big(r_5[\lambda](t) \Big(- r_6[\lambda](1)\, G_N[\lambda](0,s) + r_6[\lambda](0)\, G_N[\lambda](1,s) \Big) \\ + r_6[\lambda](t) \Big(r_5[\lambda](1)\, G_N[\lambda](0,s) - r_5[\lambda](0)\, G_N[\lambda](1,s) \Big) \Big) \\ = G_N[\lambda](t,s) - \frac{1}{|I - A_N^1[\lambda]|} \Big(G_N[\lambda](t,0) \Big(G_N[\lambda](1,1)\, G_N[\lambda](0,s) - G_N[\lambda](0,1)\, G_N[\lambda](1,s) \Big) \\ - G_N[\lambda](t,1) \Big(G_N[\lambda](1,0)\, G_N[\lambda](0,s) - G_N[\lambda](0,0)\, G_N[\lambda](1,s) \Big) \Big), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

4.3 Periodic and Dirichlet problems

In this section, we give a relationship between $G_P[\lambda]$ and $G_D[\lambda]$ following the same steps as in the previous sections.

Theorem 4.7. *Assume that $L[\lambda]$ is nonresonant in X_D and*

$$|I - A_D^4[\lambda]| := 2r_1'[\lambda](1) + r_2'[\lambda](1) - r_1'[\lambda](0) \neq 0,$$

then

$$\begin{aligned} G_P[\lambda](t,s) = G_D[\lambda](t,s) + \frac{(r_1[\lambda](t) + r_2[\lambda](t))}{|I - A_D^4[\lambda]|} \Big(\frac{\partial}{\partial t}\, G_D[\lambda](0,s) - \frac{\partial}{\partial t}\, G_D[\lambda](1,s) \Big) \\ = G_D[\lambda](t,s) + \frac{(\frac{\partial}{\partial s}\, G_D[\lambda](t,1) - \frac{\partial}{\partial s}\, G_D[\lambda](t,0))}{|I - A_D^4[\lambda]|} \Big(\frac{\partial}{\partial t}\, G_D[\lambda](0,s) - \frac{\partial}{\partial t}\, G_D[\lambda](1,s) \Big), \\ \forall\, (t,s) \in I \times I. \end{aligned}$$

Remark 4.2. Notice that from (3.1), using the last equality, we have

$$G_P[\lambda](1,s) = \frac{1}{|I - A_D^4[\lambda]|} \Big[\frac{\partial}{\partial t}\, G_D[\lambda](0,s) - \frac{\partial}{\partial t}\, G_D[\lambda](1,s) \Big].$$

Finally, carrying out the process backwards by studying the Dirichlet problem as a function of the Periodic one, we obtain the next theorem.

Theorem 4.8. *If the operator $L[\lambda]$ is nonresonant in X_P and $r_4[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_D[\lambda](t,s) = G_P[\lambda](t,s) - \frac{r_4[\lambda](t)}{r_4[\lambda](1)}\, G_P[\lambda](1,s) \\ = G_P[\lambda](t,s) - \frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\, G_P[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

Remark 4.3. From Theorem 4.8, we deduce Theorem 3.2:

$$G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\, (t,s) \in (0,1) \times (0,1), \quad \forall\, \lambda < \lambda_0^P.$$

4.4 Neumann and Mixed problems

We operate in the same way as before to study the relationship between Green's functions of Neumann and Mixed problems 1 and 2.

Theorem 4.9. *Assume that $L[\lambda]$ is nonresonant in X_N and $r_5[\lambda](0) \neq 0$, then*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_N[\lambda](t,s) - \frac{r_5[\lambda](t)}{r_5[\lambda](0)}\, G_N[\lambda](0,s) \\ &= G_N[\lambda](t,s) - \frac{G_N[\lambda](t,0)}{G_N[\lambda](0,0)}\, G_N[\lambda](0,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

Using the previous expression, we have another proof of Corollary 3.4. Indeed, we know that $G_N[\lambda] < 0$ for all $\lambda < \lambda_0^N$ and $r_5[\lambda](t) = G_N[\lambda](t,0) < 0$, using the above equality, we deduce for all $\lambda < \lambda_0^N$ that

$$G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s) \ \text{ for all } \ (t,s) \in I \times I.$$

Remark 4.4. As a consequence of the last equality, we give a proof of Corollary 3.5. Taking into account that $G_N[\lambda] < 0$ for all $\lambda < \lambda_0^N$ and $r_6[\lambda](t) > 0$, $t \in I$, it follows that for all $\lambda < \lambda_0^N$,

$$G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0 \ \text{ for all } \ (t,s) \in [0,1) \times [0,1).$$

Performing the calculations analogously for the Mixed 1 problem as a function of Neumann problem, we have the relationship between Green's functions given in the next theorem.

Theorem 4.10. *Assume that $L[\lambda]$ is nonresonant in X_N and $r_6[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_N[\lambda](t,s) - \frac{r_6[\lambda](t)}{r_6[\lambda](1)}\, G_N[\lambda](1,s) \\ &= G_N[\lambda](t,s) - \frac{G_N[\lambda](t,1)}{G_N[\lambda](1,1)}\, G_N[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

We now carry out the process backwards by writing the Neumann problem based on the Mixed problems.

Performing the calculations in a similar way, we arrive at the next theorems.

Theorem 4.11. *Assume that $L[\lambda]$ is nonresonant in X_{M_1} and $r'_{10}[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_{M_1}[\lambda](t,s) - \frac{r_{10}[\lambda](t)}{r'_{10}[\lambda](1)}\, \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s) \\ &= G_{M_1}[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)}{\frac{\partial^2}{\partial s \partial t} G_{M_1}[\lambda](1,1)}\, \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

Theorem 4.12. *Assume that $L[\lambda]$ is nonresonant in X_{M_2} and $r_7[\lambda](0) \neq 0$, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_{M_2}[\lambda](t,s) - \frac{r_7[\lambda](t)}{r'_7[\lambda](0)}\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) \\ &= G_{M_2}[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_{M_2}[\lambda](t,0)}{\frac{\partial^2}{\partial s \partial t} G_{M_2}[\lambda](0,0)}\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s), \quad \forall\, (t,s) \in I \times I. \end{aligned}$$

4.5 Periodic and Neumann problems

In this section, we look for a relationship between Green's functions $G_P[\lambda]$ and $G_N[\lambda]$ following the same steps as in the previous sections.

Theorem 4.13. *Assume that $L[\lambda]$ is nonresonant in X_P and $r_3'[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_P[\lambda](t,s) - \frac{r_3[\lambda](t)}{r_3'[\lambda](1)} \frac{\partial}{\partial t} G_P[\lambda](1,s) \\ &= G_P[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_P[\lambda](t,0)}{\frac{\partial^2}{\partial s \partial t} G_P[\lambda](1,0)} \frac{\partial}{\partial t} G_P[\lambda](1,s), \;\; \forall\, (t,s) \in I \times I. \end{aligned}$$

Finally carrying out the reverse process by studying the Periodic problem as a function of the Neumann one, we deduce the following result.

Theorem 4.14. *If the operator $L[\lambda]$ is nonresonant in X_N and*

$$|I - A_N^2[\lambda]| := r_5[\lambda](1) - r_5[\lambda](0) + r_6[\lambda](1) - r_6[\lambda](0) \neq 0,$$

then the following equality is fulfilled:

$$\begin{aligned} G_P[\lambda](t,s) &= G_N[\lambda](t,s) + \frac{1}{|I - A_N^2[\lambda]|} \Big(r_5[\lambda](t) + r_6[\lambda](t) \Big) \Big(G_N[\lambda](0,s) - G_N[\lambda](1,s) \Big) \\ &= G_N[\lambda](t,s) + \frac{1}{|I - A_N^2[\lambda]|} \Big(G_N[\lambda](t,0) - G_N[\lambda](t,1) \Big) \Big(G_N[\lambda](0,s) - G_N[\lambda](1,s) \Big), \\ &\qquad \forall\, (t,s) \in I \times I. \end{aligned}$$

4.6 Periodic and Mixed problems

The same arguments of the previous subsections are applicable to the Periodic and Mixed problems.

Theorem 4.15. *If the operator $L[\lambda]$ is nonresonant in X_P and*

$$|I - A_P^2[\lambda]| := \big(1 - r_3[\lambda](0)\big)\big(1 + r_4'[\lambda](1)\big) + r_4[\lambda](0)\, r_3'[\lambda](1) \neq 0,$$

then the following equality is fulfilled:

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_P[\lambda](t,s) + \frac{r_3[\lambda](t)}{|I - A_P^2[\lambda]|} \Big(\big(1 + r_4'[\lambda](1)\big) G_P[\lambda](0,s) - r_4[\lambda](0) \frac{\partial}{\partial t} G_P[\lambda](1,s) \Big) \\ &\quad - \frac{r_4[\lambda](t)}{|I - A_P^2[\lambda]|} \Big(r_3'[\lambda](1)\, G_P[\lambda](0,s) + \big(1 - r_3[\lambda](0)\big) \frac{\partial}{\partial t} G_P[\lambda](1,s) \Big) \\ &= G_P[\lambda](t,s) - \frac{1}{|I - A_P^2[\lambda]|} \Big(1 + \frac{\partial}{\partial t} G_P[\lambda](1,0) \Big) \frac{\partial}{\partial s} G_P[\lambda](t,0)\, G_P[\lambda](0,s) \\ &\quad + \frac{G_P[\lambda](0,0)}{|I - A_P^2[\lambda]|} \frac{\partial}{\partial s} G_P[\lambda](t,0) \frac{\partial}{\partial t} G_P[\lambda](1,s) \\ &\quad + \frac{1}{|I - A_P^2[\lambda]|} \frac{\partial^2}{\partial s \partial t} G_P(1,0)\, G_P[\lambda](t,0)\, G_P[\lambda](0,s) \\ &\quad - \frac{1}{|I - A_P^2[\lambda]|} \Big(1 + \frac{\partial}{\partial s} G_P[\lambda](0,0) \Big) G_P[\lambda](t,0) \frac{\partial}{\partial t} G_P[\lambda](1,s), \;\; \forall\, (t,s) \in I \times I. \end{aligned}$$

Similarly, we study Mixed 2 problem as a function of the Periodic problem.

Theorem 4.16. *If the operator $L[\lambda]$ is nonresonant in X_P and*

$$|I - A_P^3[\lambda]| = \big(1 + r_3[\lambda](1)\big)\big(1 - r_4'[\lambda](0)\big) + r_3'[\lambda](0)\, r_4[\lambda](1) \neq 0,$$

then the following equality is fulfilled:

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_P[\lambda](t,s) - \frac{r_3[\lambda](t)}{|I-A_P^3[\lambda]|}\Big(\big(1-r_4'[\lambda](0)\big)\, G_P[\lambda](1,s) + r_4[\lambda](1)\,\frac{\partial}{\partial t}\, G_P[\lambda](0,s)\Big) \\ &\quad - \frac{r_4[\lambda](t)}{|I-A_P^3[\lambda]|}\Big(r_3'[\lambda](0)\, G_P[\lambda](1,s) - \big(1+r_3[\lambda](1)\big)\,\frac{\partial}{\partial t}\, G_P[\lambda](0,s)\Big) \\ &= G_P[\lambda](t,s) + \frac{1}{|I-A_P^3[\lambda]|}\Big(1-\frac{\partial}{\partial t}\, G_P[\lambda](0,0)\Big)\,\frac{\partial}{\partial s}\, G_P[\lambda](t,0)\, G_P[\lambda](1,s) \\ &\quad + \frac{G_P[\lambda](1,0)}{|I-A_P^3[\lambda]|}\,\frac{\partial}{\partial s}\, G_P[\lambda](t,0)\,\frac{\partial}{\partial t}\, G_P[\lambda](0,s) \\ &\quad + \frac{1}{|I-A_P^3[\lambda]|}\,\frac{\partial^2}{\partial s\partial t}\, G_P[\lambda](0,0)\, G_P[\lambda](t,0)\, G_P[\lambda](1,s) \\ &\quad + \frac{1}{|I-A_P^3[\lambda]|}\Big(1-\frac{\partial}{\partial s}\, G_P[\lambda](1,0)\Big) G_P[\lambda](t,0)\,\frac{\partial}{\partial t}\, G_P[\lambda](0,s), \quad \forall\,(t,s)\in I\times I. \end{aligned}$$

Now we do the process backwards by writing the Periodic problem based on the Mixed problems. Performing the calculations analogously to the previous subsections, we deduce the next theorems.

Theorem 4.17. *Assume that $L[\lambda]$ is nonresonant in X_{M_2} and*

$$|I-A_{M_2}| := \big(1-r_7[\lambda](1)\big)\big(1-r_8'(0)\big) - r_8[\lambda](1)\, r_7'[\lambda](0) \neq 0,$$

then

$$\begin{aligned} G_P[\lambda](t,s) &= G_{M_2}[\lambda](t,s) + \frac{1}{|I-A_{M_2}|}\Big(\big(1-r_8'[\lambda](0)\big)\, r_7[\lambda](t)\, G_{M_2}[\lambda](1,s) \\ &\qquad + r_8[\lambda](1)\, r_7[\lambda](t)\,\frac{\partial}{\partial t}\, G_{M_2}[\lambda](0,s) + r_7'[\lambda](0)\, r_8[\lambda](t)\, G_{M_2}(1,s) \\ &\qquad + \big(1-r_7[\lambda](1)\big)\, r_8[\lambda](t)\,\frac{\partial}{\partial t}\, G_{M_2}[\lambda](0,s)\Big) \\ &= G_{M_2}[\lambda](t,s) + \frac{1}{|I-A_{M_2}|}\Big(-\Big(1+\frac{\partial}{\partial t}\, G_{M_2}[\lambda](0,1)\Big)\,\frac{\partial}{\partial s}\, G_{M_2}[\lambda](t,0)\, G_{M_2}[\lambda](1,s) \\ &\qquad + G_{M_2}[\lambda](1,1)\,\frac{\partial}{\partial s}\, G_{M_2}[\lambda](t,0)\,\frac{\partial}{\partial t}\, G_{M_2}[\lambda](0,s) \\ &\qquad + \frac{\partial^2}{\partial s\partial t}\, G_{M_2}[\lambda](0,0)\, G_{M_2}[\lambda](t,1)\, G_{M_2}(1,s) \\ &\qquad - \Big(1+\frac{\partial}{\partial s}\, G_{M_2}[\lambda](1,0)\Big)\, G_{M_2}[\lambda](t,1)\,\frac{\partial}{\partial t}\, G_{M_2}[\lambda](0,s)\Big), \quad \forall\,(t,s)\in I\times I. \end{aligned}$$

Theorem 4.18. *Assume that $L[\lambda]$ is nonresonant in X_{M_1} and*

$$|I-A_{M_1}[\lambda]| := \Big(1-r_9'[\lambda](1)\Big)\Big(1-r_{10}[\lambda](0)\Big) - r_9[\lambda](0)\, r_{10}'[\lambda](1) \neq 0,$$

then

$$\begin{aligned} G_P[\lambda](t,s) &= G_{M_1}[\lambda](t,s) + \frac{r_9[\lambda](t)}{|I-A_{M_1}[\lambda]|}\Big(\big(1-r_{10}[\lambda](0)\big)\,\frac{\partial}{\partial t}\, G_{M_1}[\lambda](1,s) + r_{10}'[\lambda](1)\, G_{M_1}[\lambda](0,s)\Big) \\ &\quad + \frac{r_{10}[\lambda](t)}{|I-A_{M_1}[\lambda]|}\Big(r_9[\lambda](0)\,\frac{\partial}{\partial t}\, G_{M_1}[\lambda](1,s) + \big(1-r_9'[\lambda](1)\big)\, G_{M_1}[\lambda](0,s)\Big) \\ &= G_{M_1}[\lambda](t,s) + \frac{1}{|I-A_{M_1}[\lambda]|}\Big(1+\frac{\partial}{\partial s}\, G_{M_1}[\lambda](0,1)\Big)\, G_{M_1}[\lambda](t,0)\,\frac{\partial}{\partial t}\, G_{M_1}[\lambda](1,s) \\ &\quad - \frac{1}{|I-A_{M_1}[\lambda]|}\,\frac{\partial^2}{\partial s\partial t}\, G_{M_1}[\lambda](1,1)\, G_{M_1}[\lambda](t,0)\, G_{M_1}[\lambda](0,s) \end{aligned}$$

$$- \frac{G_{M_1}[\lambda](0,0)}{|I - A_{M_1}[\lambda]|} \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1) \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s)$$
$$- \frac{1}{|I - A_{M_1}[\lambda]|} \left(1 - \frac{\partial}{\partial t} G_{M_1}[\lambda](1,0)\right) \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)\, G_{M_1}[\lambda](0,s), \quad \forall\, (t,s) \in I \times I.$$

5 Nonlinear problem

In this section, we study the existence of solutions of the nonlinear problem

$$\begin{cases} L_n\, u(t) = f(t, u(t)), & \text{a.e. } t \in I, \\ B_i(u) = \delta_i\, C_i(u), & i = 1, \dots, n, \end{cases} \tag{5.1}$$

with

$$L_n u(t) := u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_n(t)\, u(t)$$

the general n-th order linear operator.

The existence results will be deduced by applying Schaefer's fixed point theorem of integral operators defined in the Banach spaces.

We also consider the homogeneous particular problem

$$\begin{cases} L_n\, u(t) = f(t, u(t)), & \text{a.e. } t \in I, \\ B_i(u) = 0, & i = 1, \dots, n. \end{cases} \tag{5.2}$$

We assume that the nonlinear part of problem (5.1) satisfies the following regularity conditions:

(H_1) For $n \geq 2$, the function $f : I \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function, that is,

- $f(\,\cdot\,, x)$ is measurable for all $x \in \mathbb{R}$.
- $f(t, \,\cdot\,)$ is continuous for a.e. $t \in I$.
- For every $R > 0$, there exists $\phi_R \in L^1(\mathbb{R})$ such that

$$|f(t,x)| \leq \phi_R(t),$$

for all $x \in [-R, R]$ and a.e. $t \in I$.

For $n = 1$, the function $f : I \times \mathbb{R} \to \mathbb{R}$ is L^∞-Carathéodory function, that is,

- $f(\,\cdot\,, x)$ is measurable for all $x \in \mathbb{R}$.
- $f(t, \,\cdot\,)$ is continuous for a.e. $t \in I$.
- For every $r > 0$, there exists $h_r \in L^\infty(\mathbb{R})$ such that

$$|f(t,x)| \leq h_r(t),$$

for all $x \in [-r, r]$ and a.e. $t \in I$.

(H_2) $\exists\, K \in L^1(I)$, $K \geq 0$ such that

$$|f(t,x) - f(t,y)| \leq K(t)\, |x - y| \ \text{ for all } x, y \in \mathbb{R} \ \text{ and } \ t \in I.$$

Let us define $X \equiv (C(I), \|\cdot\|_\infty)$, the real Banach space endowed with the supremum norm

$$\|u\|_\infty = \sup_{t \in I} |u(t)|, \quad \text{for all } u \in X.$$

We denote by u_A and u_B the solutions of problems (5.1) and (5.2), respectively. We know that these solutions are given by the following expressions:

$$u_A(t) = \int_0^1 G(t,s,\delta_1,\dots,\delta_n)\, f(s,u_A(s))\, ds,$$

$$u_B(t) = \int_0^1 g(t,s)\, f(s,u_B(s))\, ds,$$

where G and g are Green's functions related to the linear problems obtained from (5.1) and (5.2), respectively. In particular, for $n=2$, these problems are (2.2) and (2.1) and, for $n \neq 2$, they are formulated in an analogous way, with obvious notations. Furthermore, they are linked by the generalization of formula (2.3) to arbitrary order:

$$G(t,s,\delta_1,\dots,\delta_n) := g(t,s) + \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j(g(\,\cdot\,,s)), \quad t,\ s \in I.$$

As we can see, this formula is totally analogous to (2.3), with obvious notations, and for its proof one can consult in [8].

Let us define

$$K^1 = \max_{t\in I} \int_0^1 |g(t,s)|\, K(s)\, ds,$$

$$K^2_{ij} = \max_{t\in I} |\omega_i(t)| \int_0^1 |C_j(g(\,\cdot\,,s))|\, K(s)\, ds, \quad \forall\, i,j = 1,\dots,n,$$

$$K^3_{ij} = \max_{t\in I} |\omega_i(t)| \int_0^1 |C_j(g(\,\cdot\,,s))\, f(s,0)|\, ds, \quad \forall\, i,j = 1,\dots,n,$$

$$P = \max_{t\in I} \int_0^1 |G(t,s,\delta_1,\dots,\delta_n)|\, K(s)\, ds,$$

$$Q = \max_{t\in I} \int_0^1 |G(t,s,\delta_1,\dots,\delta_n)\, f(s,0)|\, ds.$$

We assume that the following condition is fulfilled:

(H_3) $K^1 < 1$.

Theorem 5.1. *If conditions (H_2) and (H_3) hold, then the following inequality is fulfilled:*

$$\|u_B - u_A\|_\infty \le \frac{1}{1-K^1} \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |\delta_i\, b_{ij}| K^2_{ij}\, \|u_A\|_\infty + \sum_{i=1}^{n}\sum_{j=1}^{n} |\delta_i\, b_{ij}|\, K^3_{ij} \Big). \tag{5.3}$$

Proof. Using (2.3), we have

$$\begin{aligned} u_B(t) - u_A(t) &= \int_0^1 g(t,s)\, f(s,u_B(s))\, ds - \int_0^1 G(t,s,\delta_1,\dots,\delta_n)\, f(s,u_A(s))\, ds \\ &= \int_0^1 g(t,s)\, \big(f(s,u_B(s)) - f(s,u_A(s))\big)\, ds \end{aligned}$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \, b_{ij} \, \omega_i(t) \int_0^1 C_j(g(\,\cdot\,,s))\big(f(s,u_A(s)) - f(s,0)\big)\, ds$$
$$- \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \, b_{ij} \, \omega_i(t) \int_0^1 C_j(g(\,\cdot\,,s))\, f(s,0)\, ds.$$

Then, for all $t \in I$, from (H_2), we infer that

$$|u_B(t) - u_A(t)| \leq \|u_B - u_A\|_\infty \int_0^1 |g(t,s)|\, K(s)\, ds$$
$$+ \|u_A\|_\infty \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, |\omega_i(t)| \int_0^1 |C_j(g(\,\cdot\,,s))|\, K(s)\, ds$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, |\omega_i(t)| \int_0^1 |C_j(g(\,\cdot\,,s))\, f(s,0)|\, ds.$$

Therefore,

$$\|u_B - u_A\|_\infty \leq K^1 \, \|u_B - u_A\|_\infty + \|u_A\|_\infty \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, K^2_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, K^3_{ij},$$

that is, using (H_3),

$$\|u_B - u_A\|_\infty \leq \frac{1}{1 - K^1} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}| K^2_{ij} \, \|u_A\|_\infty + \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, K^3_{ij} \Big). \qquad \square$$

Corollary 5.1. *If conditions (H_2) and (H_3) hold, then the following inequalities are fulfilled:*

$$\|u_B\|_\infty \leq \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}| K^2_{ij} - K^1 + 1}{1 - K^1} \, \|u_A\|_\infty + \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, K^3_{ij}}{1 - K^1},$$
$$\|u_B\|_\infty \geq \frac{1 - K^1 - \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}| K^2_{ij}}{1 - K^1} \, \|u_A\|_\infty - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_i \, b_{ij}|\, K^3_{ij}}{1 - K^1}.$$

Proof. The proof is an immediate consequence of (5.3) and the inequality

$$\big| \|u_B\|_\infty - \|u_A\|_\infty \big| \leq \|u_B - u_A\|_\infty. \qquad \square$$

Next we state Scheafer's fixed-point theorem (see [2]) that will be applied to the operator $T : X \to X$ given by

$$T\, u(t) := \int_0^1 G(t,s,\delta_1,\ldots,\delta_n)\, f(s,u(s))\, ds, \quad t \in I, \tag{5.4}$$

to guarantee the existence of a solution of problem (5.1).

Theorem 5.2 (Schaefer)**.** *Let $T : X \to X$ be a continuous and compact mapping of a Banach space X such that the set*

$$\big\{ x \in X : \; x = \mu \, T\, x \;\; \textit{for some} \;\; 0 \leq \mu \leq 1 \big\}$$

is bounded. Then T has a fixed point.

Now, we use Schaefer's theorem to ensure the existence of solutions of the nonlinear problem (5.1).

Theorem 5.3. *Assume that* (H_1) *and* (H_2) *hold and* $P < 1$. *Then problem* (5.1) *has at least one solution* $u \in X$.

Proof. First, note that the fixed points of the operator T defined in (5.4) coincide with the solutions of problem (5.1).

Now, we show that the operator T is compact. Since $G(t,s,\delta_1,\dots,\delta_n)$ is continuous and f is Carathéodory, we have that the operator T is continuous too.

Next, we prove that T maps the bounded sets into relatively compact sets. Let $H \subset X$ be a bounded set. Since H is bounded, there exists $r \in \mathbb{R}$, $r > 0$ such that $\|u\|_\infty \le r$ for all $u \in H$. Then

$$\begin{aligned}|T\,u(t)| &\le \int_0^1 \big|G(t,s,\delta_1,\dots,\delta_n)\big|\,\big|f(s,u(s)) - f(s,0)\big|\,ds + \int_0^1 \big|G(t,s,\delta_1,\dots,\delta_n)\big|\,|f(s,0)|\,ds\\ &\le \|u\|_\infty \int_0^1 \big|G(t,s,\delta_1,\dots,\delta_n)\big|\,K(s)\,ds + \int_0^1 \big|G(t,s,\delta_1,\dots,\delta_n)\big|\,|f(s,0)|\,ds.\end{aligned}$$

So, for all $u \in H$, we have

$$\|T\,u\|_\infty \le r\,P + Q, \tag{5.5}$$

that is, $T(H)$ is bounded.

Let us show now the equicontinuity of T. For all $t \in I$ and $u \in H$, we have

$$\begin{aligned}|(T\,u)'(t)| &= \left|\int_0^1 \frac{\partial}{\partial t} G(t,s,\delta_1,\dots,\delta_n)\,f(s,u(s))\,ds\right| \le \int_0^1 \left|\frac{\partial}{\partial t} G(t,s,\delta_1,\dots,\delta_n)\right|\,|f(s,u(s))|\,ds\\ &\le \int_0^1 \left|\frac{\partial}{\partial t} G(t,s,\delta_1,\dots,\delta_n)\right|\,\phi_r(s)\,ds.\end{aligned}$$

If $n \ge 2$, then the regularity of Green's function $G(t,s,\delta_1,\dots,\delta_n)$ allows us to guarantee that there exists $M \in \mathbb{R}$, $M > 0$ such that $|\frac{\partial}{\partial t} G(t,s,\delta_1,\dots,\delta_n)| \le M$. Therefore,

$$\int_0^1 \left|\frac{\partial}{\partial t} G(t,s,\delta_1,\dots,\delta_n)\right|\,\phi_r(s)\,ds \le M \int_0^1 \phi_r(s)\,ds.$$

So, for all $t_1, t_2 \in I$, $t_1 < t_2$, we infer that

$$\big|(T\,u)(t_2) - (T\,u)(t_1)\big| = \left|\int_{t_1}^{t_2} (T\,u)'(s)\,ds\right| \le \int_{t_1}^{t_2} |(T\,u)'(s)|ds \le N(t_2 - t_1).$$

If $n = 1$, then the regularity of Green's function $G(t,s,\delta_1)$ allows us to ensure that there exists $\widetilde{N} \in \mathbb{R}$, $\widetilde{N} > 0$ such that

$$\int_0^1 |G(t,s,\delta_1)|\,\phi_r(s)\,ds \le \widetilde{N}.$$

Therefore,

$$\int_0^1 \left|\frac{\partial}{\partial t} G(t,s,\delta_1)\right|\,\phi_r(s)\,ds = \int_0^1 |a_1(t)|\,|G(t,s,\delta_1)|\,\phi_r(s)\,ds \le \widetilde{N}\,|a_1(t)|.$$

Then, for all $t_1, t_2 \in I,\ t_1 < t_2$, we have

$$\big|(T\,u)(t_2) - (T\,u)(t_1)\big| = \left|\int_{t_1}^{t_2} (T\,u)'(s)\,ds\right| \le \int_{t_1}^{t_2} |(T\,u)'(s)|\,ds \le \widetilde{N} \int_{t_1}^{t_2} |a_1(s)|\,ds.$$

Thus, $T(H)$ is an equicontinuous set in X. By the Arzelà-Ascoli Theorem, we deduce that $T(H)$ is relatively compact, that is, T is a compact operator.

Let $u \in X$ be such that $u = \mu\,T\,u$ for some $0 \le \mu \le 1$. Then, using (5.5), we have

$$\|u\|_\infty = \mu \|T\,u\|_\infty \le \|T\,u\|_\infty \le \|u\|_\infty\,P + Q.$$

Thus

$$\|u\|_\infty \le \frac{Q}{1-P}\,.$$

Therefore, applying Schaefer's Theorem, we conclude that problem (5.1) has at least one solution $u \in X$. □

Remark 5.1. We note that by the definition of X, $(T\,u)$ is not necessarily derivable. However, $(T\,u)'$ always exists because of the regularity of Green's function.

Next, we apply the above results to the particular case of the nonlinear second order Dirichlet problem.

Suppose there exists u_D, a solution of the nonlinear Dirichlet problem

$$L[\lambda]\,u(t) = f(t,u(t)),\ \text{ a.e. } t \in I, \quad u(0) = u(1) = 0, \tag{5.6}$$

and u_P, a solution of the nonlinear Periodic problem

$$L[\lambda]\,u(t) = f(t,u(t)),\ \text{ a.e. } t \in I, \quad u(0) - u(1) = u'(0) - u'(1) = 0.$$

By the definition of Green's functions, we have

$$u_D(t) = \int_0^1 G_D[\lambda](t,s)\,f(s,u_D(s))\,ds$$

and

$$u_P(t) = \int_0^1 G_P[\lambda](t,s)\,f(s,u_P(s))\,ds.$$

We know from Theorem 4.8 that

$$G_D[\lambda](t,s) = G_P[\lambda](t,s) - \frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\,G_P[\lambda](1,s), \ \ \forall\,(t,s) \in I \times I.$$

Let us define

$$K_1 = \max_{t\in I} \int_0^1 |G_P[\lambda](t,s)|\,K(s)\,ds,$$

$$K_2 = \max_{t\in I} \left|\frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\right| \int_0^1 |G_P[\lambda](1,s))|\,K(s)\,ds,$$

$$K_3 = \max_{t\in I} \left|\frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\right| \int_0^1 \big|G_P[\lambda](1,s)\,f(s,0)\big|\,ds,$$

$$P_D = \max_{t\in I} \int_0^1 |G_D[\lambda](t,s)|\, K(s)\, ds,$$

$$Q_D = \max_{t\in I} \int_0^1 \big|G_D[\lambda](t,s)\, f(s,0)\big|\, ds.$$

As a direct consequence of Theorem 5.1 and Corollary 5.1 we arrive at the follow results.

Theorem 5.4. *Suppose that* (H_2) *holds and* $K_1 < 1$*, then the following inequality is fulfilled:*

$$\|u_D - u_P\|_\infty \le \frac{1}{1-K_1}\big(K_2\, \|u_D\|_\infty + K_3\big). \tag{5.7}$$

Corollary 5.2. *Assume that* (H_2) *holds and* $K_1 < 1$*. Then the following inequalities are fulfilled:*

$$\|u_P\|_\infty \le \frac{K_2 - K_1 + 1}{1-K_1}\, \|u_D\|_\infty + \frac{K_3}{1-K_1}\,,$$
$$\|u_P\|_\infty \ge \frac{1 - K_1 - K_2}{1-K_1}\, \|u_D\|_\infty - \frac{K_3}{1-K_1}\,.$$

Theorem 5.5. *Assume that* (H_1) *and* (H_2) *hold and* $P_D < 1$*. Then the Dirichlet problem* (5.6) *has at least one solution.*

Remark 5.2. The same previous arguments can be applied to the rest of the problems discussed in this article using the formulas that relate Green's functions obtained in the previous section.

In the sequel, we present an example to illustrate our results.

Example 5.1. Consider the following equation:

$$u''(t) - u(t) = \frac{c}{\sqrt{t}}\, e^{-u^2(t)}, \quad \text{a.e. } t \in I, \quad \text{and} \quad c > 0.$$

In this case, $f(t,u) = \frac{c}{\sqrt{t}}\, e^{-u^2}$ is an L^1-Carathéodory function and $f(t,0) = \frac{c}{\sqrt{t}} \neq 0$ for all $t \in (0,1]$. Moreover, it is immediately obvious that f satisfies condition (H_2) with $K(t) = c\sqrt{\frac{2}{e\,t}}$ for a.e. $t \in [0,1]$.

We have that Green's function of the periodic problem is given by

$$G_P(t,s) = \begin{cases} \dfrac{e^{s-t+1} + e^{t-s}}{2(1-e)}\,, & 0 \le s \le t \le 1, \\[2mm] \dfrac{e^{t-s+1} + e^{s-t}}{2(1-e)}\,, & 0 \le t < s \le 1, \end{cases}$$

and that of the Dirichlet problem is

$$G_D(t,s) = \begin{cases} -\dfrac{(e^{2s}-1)(e^2 - e^{2t})e^{-(s+t)}}{2(e^2-1)}\,, & 0 \le s \le t \le 1, \\[2mm] \dfrac{(e^{2(s-1)}-1)(e^{2t}-1)e^{-(s+t-2)}}{2(e^2-1)}\,, & 0 \le t < s \le 1. \end{cases}$$

With the notation used in Theorem 5.4 and by a numerical approach, it can be seen that

$$K_1 = \max_{t\in I} \int_0^1 |G_P(t,s)|\, K(s)\, ds \approx 1.7472\, c,$$

$$K_2 = \max_{t\in I} \Big| \frac{G_P(t,0)}{G_P(1,0)} \Big| \int_0^1 |G_P(1,s))|\, K(s)\, ds \approx 1.744\, c,$$

$$K_3 = \max_{t\in I} \left| \frac{G_P(t,0)}{G_P(1,0)} \int_0^1 G_P[\lambda](1,s)\, f(s,0)\, ds \right| \approx 2.033\, c,$$

$$P_P = \max_{t\in I} \int_0^1 |G_P(t,s)|\, K(s)\, ds \approx 1.7472\, c,$$

$$Q_P = \max_{t\in I} \int_0^1 \big| G_D(t,s)\, f(s,0) \big|\, ds \approx 2.0369\, c,$$

$$P_D = \max_{t\in I} \int_0^1 |G_D(t,s)|\, K(s)\, ds \approx 0.1651\, c,$$

$$Q_D = \max_{t\in I} \int_0^1 \big| G_D(t,s)\, f(s,0) \big|\, ds \approx 0.179\, c.$$

Then the conditions $K_1 < 1$ and $P_D < 1$ are fulfilled if and only if

$$0 < c < \min\left\{ \frac{1}{0.1651}\, , \frac{1}{1.7472} \right\} \approx 0.572344.$$

Therefore, if $0 < c < 0.572344$, then, by Theorem 5.5, there is at least one solution u_D of the Dirichlet problem

$$u''(t) - \, u(t) = \frac{c}{\sqrt{t}}\, e^{-u^2(t)}, \ \text{ a.e. } \ t \in I, \quad u(0) = u(1) = 0.$$

By the proof of Theorem 5.3, we have

$$\|u_P\|_\infty \le \frac{Q_P}{1 - P_P} \approx \frac{2.0369\, c}{1 - 1.7472\, c}$$

and

$$\|u_D\|_\infty \le \frac{Q_D}{1 - P_D} \approx \frac{0.179\, c}{1 - 0.1651\, c}\, .$$

As a consequence, we deduce that

$$\begin{aligned} \|u_P - u_D\|_\infty &\le \|u_P\|_\infty + \|u_D\|_\infty \\ &\le \frac{2.0369\, c}{1 - 1.7472\, c} + \frac{0.179\, c}{1 - 0.1651\, c} = \frac{c\,(7.68176 - 2.25\, c)}{c^2 - 6.62928\, c + 3.46665} := \gamma(c). \end{aligned} \tag{5.8}$$

On the other hand, if $0 < c < 0.572344$, applying inequality (5.7), we obtain the following estimate of the distance between the solutions:

$$\begin{aligned} \|u_P - u_D\|_\infty &\le \frac{1}{1 - K_1} \left(K_2\, \|u_D\|_\infty + K_3 \right) \\ &\le \frac{1}{1 - K_1} \left(K_2\, \frac{0.179\, c}{1 - 0.1651\, c} + K_3 \right) \\ &\approx \frac{1}{1 - 1.7472\, c} \left(1.744\, c\, \frac{0.179\, c}{1 - 0.1651\, c} + 2.033\, c \right) = \frac{c\,(7.0477 - 0.0813703\, c)}{c^2 - 6.62928\, c + 3.46665} := \psi(c). \end{aligned} \tag{5.9}$$

Comparing (5.8) and (5.9) (see Figure 5.1), we have that estimate (5.9) is better than (5.8) for $0 < c < 0.2878$ and worse for $0.2878 < c < 0.572344$.

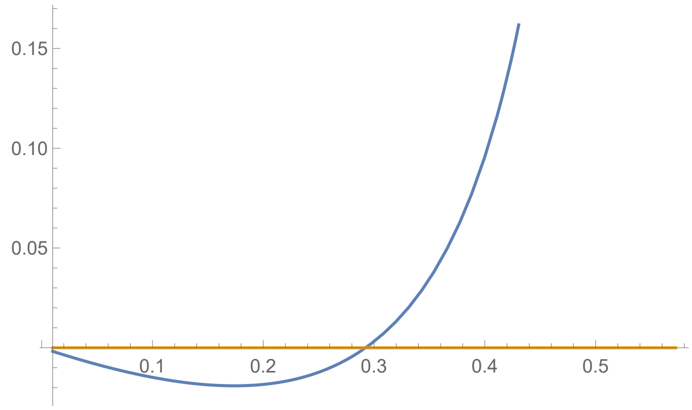


Figure 5.1: Representation of the function $\psi - \gamma$ on the interval $(0, 0.572344)$.

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Authors' addresses:

Alberto Cabada

CITMAga, 15782, Santiago de Compostela, Galicia, Departamento de Estatística, Análise Matemática e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain.

E-mail: alberto.cabada@usc.es

Lucía López-Somoza

CITMAga, 15782, Santiago de Compostela, Galicia, Departamento de Estatística, Análise Matemática e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain.

E-mail: lucia.lopez.somoza@usc.es

Mouhcine Yousfi

CITMAga, 15782, Santiago de Compostela, Galicia, Departamento de Estatística, Análise Matemática e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain.

E-mail: yousfi.mouhcine@usc.es