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MEASURE OF WEAK NONCOMPACTNESS AND INTEGRO-DIFFERENTIAL EQUATIONS WITH FINITE DELAY

Abstract. This paper discusses the existence of solutions for integro-differential equations via resolvent operators in Banach space. Our approach is based on a Mönch's type fixed point theorem with measure of weak noncompactness. An example is given to show the application of our result.

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1 Introduction

The aim of this work is to study the existence of solutions for the following integrodifferential equation with delay:

$$\begin{cases} \phi'(\delta) = \mathfrak{K}(\delta)\phi(\delta) + \aleph(\delta, \phi_\delta) + \int\limits_0^\delta \gamma(\delta, \varrho)\phi(\varrho)\, d\varrho & \text{if } \delta \in \mathfrak{Q} = [0, \varkappa], \\ \phi(\delta) = \Phi(\delta) & \text{if } \delta \in [-\zeta, 0]. \end{cases} \tag{1.1}$$

Assume that $\Im$ is a Banach space with norm $\|\cdot\|_\Im$, $\aleph : \mathfrak{Q} \times C([-\zeta, 0], \Im) \to \Im$ is a continuous function, $\Phi \in C([-\zeta, 0], \Im)$, $\mathfrak{K}(\delta) : D(\mathfrak{K}(\delta)) \subset \Im \to \Im$ is a closed linear operator on $\Im$, with dense domain $D(\mathfrak{K}(\delta))$, and $\gamma(\delta, \varrho)$ is a family of linear operators in $\Im$ with $D(\mathfrak{K}) \subset D(\gamma(\delta, \varrho))$ for $\delta \geq 0$ and of bounded linear operators from $D(\mathfrak{K})$ into $\Im$, $C([-\zeta, 0], \Im)$ is the space of all continuous functions on $[-\zeta, 0]$ with values in $\Im$ and with the uniform norm topology $\|\cdot\|_\infty$. For $\phi \in C([-\zeta, \varkappa], \Im)$ and for every $\delta \geq 0$, $\phi_\delta \in C([-\zeta, 0], \Im)$ is given by

$$\phi_\delta(\vartheta) = \phi(\delta + \vartheta) \text{ for } \vartheta \in [-\zeta, 0].$$

Integro-differential equations are effective mathematical models for explaining a wide range of phenomena in physics, biology and engineering, as demonstrated in [1, 8, 9, 13, 15]. The analysis of such equations presents major difficulties. Because of the existence of both differential and integral variables, sophisticated dynamics and complex behavior of solution result. The discovery and analysis of fixed points, which provide significant insights into the long-term behavior of the system, is a vital part of researching integro-differential equations. (For further information, read [4, 6, 7, 14, 25, 26]).

The resolvent operator plays a crucial role in addressing equation (1.1) in both weak and strong senses, serving as a replacement for the C_0-semigroup in evolution equations. Numerous researchers have devoted significant efforts in recent years to explore diverse aspects, including the existence and regularity, stability, (almost) periodicity of solutions, and control problems associated with semi-linear integro-differential evolution equations. This exploration is conducted through the utilization of the theory of the resolvent operator, as highlighted in works such as [10, 16, 21] and the references provided therein.

In particular, Dos Santos *et al.* [17] studied neutral integro-differential equations with infinite delay of the following form:

$$\begin{cases} \dfrac{d}{d\delta}\left[\beta(\delta) + \int\limits_{-\infty}^\delta N(\delta - \varrho)\beta(\varrho)\, d\varrho\right] = \mathfrak{K}\beta(\delta) + \aleph(\delta, \beta_\delta) + \int\limits_{-\infty}^\delta \gamma(\delta - \varrho)\beta(\varrho)\, d\varrho & \text{if } \delta \in [0, \varkappa], \\ \beta_0 = \jmath & \text{if } \delta \in [-\infty, 0], \end{cases}$$

where $\mathfrak{K}$ and $\gamma(\delta)$, $\delta \geq 0$, are closed linear operators on a common domain $D(\mathfrak{K})$ which is dense in Ψ, and $N(\delta), \delta \geq 0$, is a bounded linear operator on a Banach space Ψ, the history $\beta_\delta : (-\infty, 0] \to \Psi$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically.

In [5], Benchohra *et al.* studied the fractional differential equations with boundary nonlinear integral conditions in reflexive Banach spaces of the following form:

$$\begin{cases} {}^C D^\zeta \beta(\delta) = \aleph(\delta, \beta(\delta)) \text{ a.e } \delta \in \mathfrak{Q} := [0, \varkappa], \\ \beta(0) - \beta'(0) = \int\limits_0^\varkappa \aleph_1(\varrho, \beta(\varrho))\, d\varrho, \\ \beta(\varkappa) + \beta'(\varkappa) = \int\limits_0^\varkappa \aleph_2(\varrho, \beta(\varrho))\, d\varrho, \end{cases}$$

where ${}^C D^\zeta$, $1 < \zeta \leq 2$, is the Caputo fractional derivative, $\aleph$, $\aleph_1$ and $\aleph_2 : \mathfrak{Q} \times \Im \to \Im$ are the given functions, and $\Im$ is a reflexive Banach space endowed with the weak topology.

The methodology involving the measure of weak noncompactness and the fixed point theorem of Mönch type originated primarily from the monograph authored by Bana's and Goebel [2]. Subsequently, it underwent further development and application in numerous publications. Notable examples include the works of Banaś and Sadarangani [3], Guo *et al.* [20], Krzyska and Kubiaczyk [22], Lakshmikantham and Leela [24], Mönch [27,28], O'Regan [29,30], Szufla and Szukala [34], along with the references provided therein. Recent papers on the existence results of boundary value problems of fractional order involving the Pettis integral can be found in [32,33] and the references therein.

This paper is organized as follows. In Section 2, some necessary concepts and important definitions and lemmas are given. In Section 3, we show the existence of weak solutions for problem (1.1) by applying the Mönch fixed point theorem combined with the technique of weakly measure of noncompactness. An example is also given in Section 4 to illustrate the theory of the abstract main result.

2 Preliminaries

Let $\Im$ be a real Banach space with the norm $\|\cdot\|_{\Im}$ and dual space $\Im^*$, let $C(\mathfrak{Q},\Im)$ be the space of all continuous functions from $\mathfrak{Q}$ into $\Im$ and $\mathcal{A}(\Im)$ be the space of all bounded linear operators from $\Im$ into $\Im$ with

$$\|\aleph\|_{\mathcal{A}(\Im)} = \sup_{\|\phi\|=1} \|\aleph(\phi)\|_{\Im}.$$

A measurable function $\phi : \mathfrak{Q} \to \Im$ is Bochner integrable if and only if $\|\phi\|_{\Im}$ is Lebesgue integrable. Let $L^1(\mathfrak{Q},\Im)$ denote the Banach space of measurable functions $\phi : \mathfrak{Q} \to \Im$, which are Bochner integrable, with

$$\|\phi\|_{L^1} = \int_0^{\varkappa} \|\phi(\delta)\|_{\Im}\, d\delta.$$

By $C(\mathfrak{Q},\Im)$ we denote the Banach space of all continuous functions from $\mathfrak{Q}$ into $\Im$ with

$$\|\phi\| = \sup_{\delta\in\mathfrak{Q}} \|\phi(\delta)\|_{\Im},$$

and by $C([-\zeta,0],\Im)$ we denote the Banach space of all continuous functions from $[-\zeta,0]$ into $\Im$ with

$$\|\phi\|_{\infty} = \sup_{\delta\in[-\zeta,0]} \|\phi(\delta)\|_{\Im}.$$

2.1 The weak topology $\varpi(\Psi,\Psi^*)$

In this section, Ψ is a normed space and Ψ^* is its dual.

Definition 2.1 ([11])**.** The weak topology on Ψ is the topology $\varpi(\Psi,(\aleph)_{\aleph\in\Psi^*})$. For convenience, it is simply denoted by $\varpi(\Psi,\Psi^*)$.

Theorem 2.1 ([11])**.** *The topology $\varpi(\Psi,\Psi^*)$ is Hausdorff.*

Proposition 2.1 ([11])**.** *The weak topology and the strong topology on Ψ coincide if and only if Ψ is finite-dimensional.*

Theorem 2.2 ([11])**.** *Let C be a nonempty convex set in Ψ. Then C is strongly closed if and only if it is weakly closed.*

Let $\Im_\omega = (\Im, \varpi(\Im,\Im^*))$ be the Banach space $\Im$ with its weak topology.

Definition 2.2 ([30])**.** The mapping β is said to be weakly continuous (measurable) at $\delta_0 \in \mathfrak{Q}$ if for every $\jmath \in \Im^*$, $\jmath(\beta)$ is continuous (measurable) at δ_0. We denote by $C(\mathfrak{Q},\Im_\omega)$ the space of weakly continuous functions defined on $\mathfrak{Q}$ with values in the Banach space $\Im_\omega$.

Definition 2.3 ([30])**.** Let $\aleph : \mathfrak{Q} \times \Im \to \Im$. Then $\aleph(\delta, \beta)$ is said to be weakly continuous at (δ_0, β_0) if for the given $\epsilon > 0$ and $\jmath \in \Im^*$, there exist a constant $\kappa > 0$ and a weakly open set U containing β_0 such that

$$\left|\jmath(\aleph(\delta, \beta) - \aleph(\delta_0, \beta_0))\right| < \epsilon \ \text{ for } \ |\delta - \delta_0| < \kappa \ \text{ and } \ \beta \in U.$$

Definition 2.4 ([30])**.** A function $\aleph : \Im \to \Im$ is said to be weakly sequentially continuous if $\aleph$ takes each weakly convergent sequence in $\Im$ to weakly convergent sequence in $\Im$ (i.e., for any $\{\beta_n\}$ in $\Im$ with $\{\beta_n\} \to \beta$ in $\Im_\omega$, there is $\aleph(\beta_n) \to \aleph(\beta)$ in $\Im_\omega$).

Definition 2.5 ([31])**.** The function $\beta : \mathfrak{Q} \to \Im$ is said to be Pettis integrable on $\mathfrak{Q}$ if and only if there is an element $\beta_I \in \Im$ corresponding to each $I \subset \mathfrak{Q}$ such that

$$\jmath(\beta) = \int\limits_I \jmath(\beta(\varrho))\, d\varrho \ \text{ for all } \ \jmath \in \Im^*,$$

where the integral on the right-hand side is supposed to exist in the sense of Lebesgue. By definition,

$$\beta_I = \int\limits_I \beta(\varrho)\, d\varrho.$$

We denote the space of Pettis integrable functions by $P[\mathfrak{Q}, \Im]$.

The immediate consequence of the definition of the Pettis integral is that if $\beta : \mathfrak{Q} \to \Im$ is Pettis integrable on $\mathfrak{Q}$, then $\jmath(\beta(\cdot))$ is Lebesgue integrable on $\mathfrak{Q}$ for every $\jmath \in \Im^*$. We point out that a bounded weakly measurable function $\beta : \mathfrak{Q} \to \Im$ need not be Pettis integrable even $\Im$ is reflexive.

Lemma 2.1 ([31])**.** *In reflexive Banach space $\Im$, the weakly measurable function $\beta : \mathfrak{Q} \to \Im$ is Pettis integrable if and only if $\jmath(\beta(\cdot))$ is Lebesgue integrable on $\mathfrak{Q}$ for every $\jmath \in \Im^*$.*

2.2 Measure of weak noncompactness

Given a Banach space $\Im$, $\mathcal{A}(\Im)$ will denote the collection of all nonempty bounded subsets of $\Im$ and $\mathcal{W}(\Im)$ is the subset of $\mathcal{A}(\Im)$ consisting of all compact subsets of $\Im$.

The first measure of noncompactness was first introduced by Kuratowski [23] in 1930. For any subset $\Omega \in \mathcal{A}(\Im)$, the Kuratowski measure of noncompactness is defined by

$$\alpha(\Omega) = \inf\Big\{\epsilon > 0 : \ \ \Omega \ \text{ can be covered by finitely many sets of diameter } \ \leq \epsilon\Big\}.$$

The second important example of measure of noncompactness in the Banach space endowed with the weak topology has been defined by De Blasi [12] in 1977 as follows: for a subset $\Omega \in \mathcal{A}(\Im)$ and all $\Psi \in \mathcal{A}(\Im)$,

$$\chi(\Psi) = \inf\Big\{\epsilon > 0 : \ \text{ there exists a weakly compact subset } \ \widehat{\Omega} \in \mathcal{W}(\Im) : \Psi \subset \widehat{\Omega} + \epsilon\Omega\Big\}.$$

χ is called the De Blasi measure of weak noncompactness. We can check that χ verifies the following properties (see [12] for more details).

Lemma 2.2 ([27])**.** *Let $\widehat{\Omega}, \Omega \in \mathcal{A}(\Im)$. We have:*

(a) $\chi(\widehat{\Omega}) \leq \chi(\Omega)$ *whenever* $\widehat{\Omega} \subseteq \Omega$*;*

(b) $\chi(\widehat{\Omega}) = 0$ *if and only if* $\widehat{\Omega}$ *is weakly relatively compact;*

(c) $\chi(\widehat{\Omega} \cup \Omega) = \max\{\chi(\widehat{\Omega}), \chi(\Omega)\}$*;*

(d) $\chi(\overline{\widehat{\Omega}}^{\omega}) = \chi(\widehat{\Omega})$*, where* $\overline{\widehat{\Omega}}^{\omega}$ *denotes the weak closure of* $\widehat{\Omega}$*;*

(e) $\chi(\widehat{\Omega} + \Omega) \leq \chi(\widehat{\Omega}) + \chi(\Omega)$*;*

(f) $\chi(\lambda\widehat{\Omega}) = |\lambda|\chi(\widehat{\Omega})$, $\lambda \in \mathbb{R}$,

(g) $\chi(\mathrm{conv}(\widehat{\Omega})) = \chi(\widehat{\Omega})$*, where* $\mathrm{conv}(\widehat{\Omega})$ *refers to the convex hull of* $\widehat{\Omega}$*.*

2.3 Resolvent operators

We consider the following linear Cauchy problem:

$$\begin{cases} \phi'(\delta) = \mathfrak{K}\phi(\delta) + \int\limits_0^\delta \gamma(\delta-\varrho)\phi(\varrho)\,d\varrho \quad \text{for} \ \ \delta \geq 0, \\ \phi(0) = \phi_0 \in \Im. \end{cases} \tag{2.1}$$

The existence and properties of a resolvent operator have been discussed in [18].

Definition 2.6 ([19])**.** A resolvent operator for the Cauchy problem (2.1) is a bounded linear operator-valued function $\Theta \in \mathcal{A}(\Im)$ for $\delta \geq 0$ satisfying the following conditions:

(1) $\Theta(0) = I$ (the identity map of $\Im$) and $\|\Theta(\delta)\|_{\mathcal{A}(\Im)} \leq Me^{\eta\delta}$ for some constants $M > 0$ and $\eta \in \mathbb{R}$.

(2) For each $\phi \in \Im$, $\delta \to \Theta(\delta)\phi$ is strongly continuous for $\delta \geq 0$.

(3) $\Theta \in \mathcal{A}(\Im)$ for $\delta \geq 0$. For $\phi \in \Im$, $\Theta(\,\cdot\,)\phi \in C^1(\mathbb{R}_+, \Im) \cap C(\mathbb{R}_+, \Im)$ and

$$\Theta'(\delta)\phi = \mathfrak{K}\Theta(\delta)\phi + \int\limits_0^\delta \gamma(\delta-\varrho)\Theta(\varrho)\phi\,d\varrho = \Theta(\delta)\mathfrak{K}\phi + \int\limits_0^\delta \Theta(\delta-\varrho)\gamma(\varrho)\phi\,d\varrho$$

for $\delta \geq 0$.

From now on, we assume that:

($P1$) The operator $\mathfrak{K}$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(\delta)\}_{\delta>0}$.

($P2$) For all $\delta \geq 0$, $\gamma(\delta)$ is a closed linear operator from $D(\mathfrak{K})$ to $\Im$ and $\gamma(\delta) \in \mathcal{A}(\Im)$. For any $\phi \in \Im$, the map $\delta \to \gamma(\delta)\phi$ is bounded, differentiable and the derivative $\delta \to \gamma'(\delta)\phi$ is bounded uniformly continuous on $\mathbb{R}^+$.

The following theorem gives a satisfactory answer to the problem of existence of a solution.

Theorem 2.3 ([19])**.** *Assume that* ($P1$)–($P2$) *hold, then there exists a unique resolvent operator for the Cauchy problem* (2.1).

3 Main result

In this section, we prove the existence of weak solutions of problem (1.1).

Definition 3.1. We say that a continuous function $\phi(\,\cdot\,) : \mathfrak{Q} \to \Im$ is a weak solution of problem (1.1) if ϕ satisfies the following integral equation:

$$\phi(\delta) = \begin{cases} \Theta(\delta,0)\Phi(0) + \int\limits_0^\delta \Theta(\delta,\varrho)\aleph(\varrho,\phi_\varrho)\,d\varrho & \text{if} \ \ \delta \in \mathfrak{Q}, \\ \Phi(\delta) & \text{if} \ \ \delta \in [-\zeta, 0]. \end{cases}$$

The hypotheses:

($H1$) The function $u \to \aleph(\delta, u)$ is weakly sequentially continuous for each $u \in C([-\zeta, 0], \Im)$, and there exist $p_\aleph \in L^1(\mathfrak{Q}, \mathbb{R}^+)$ and a nondecreasing continuous function $\psi : [0, +\infty) \to (0, +\infty)$ such that

$$\|\aleph(\delta, u)\|_\Im \leq p_\aleph(\delta)\psi(\|u\|_\infty) \ \ \text{for} \ \ u \in C([-\zeta, 0], \Im) \ \ \text{and for a.e.} \ \ \delta \in \mathfrak{Q}.$$

($H2$) The function $\delta \to \aleph(\delta, u_\delta)$ is Pettis integrable for each $u \in C([-\zeta, \varkappa], \Im)$.

$(H3)$ There exist $\Xi \geq 1$ and $\varsigma \geq 0$ such that

$$\|\Theta(\delta,\varrho)\|_{\mathcal{A}(\Im)} \leq \Xi e^{-\varsigma\delta}.$$

$(H4)$ There exists $\widehat{\zeta} > 0$ such that

$$\Xi\widehat{\zeta} + \Xi\psi(\widehat{\zeta})\|p_{\aleph}\|_{L^1} \leq \widehat{\zeta}.$$

Theorem 3.1. *Assume that the conditions $(H1)$–$(H4)$ are satisfied. Then system (1.1) has at least one weak solution.*

Proof. We consider the set

$$\Bbbk = \Bigg\{\phi \in C([-\zeta,\varkappa],\Im) : \ \|\phi\|_{\Bbbk} \leq \widehat{\zeta} \ \text{ and } \ \|\phi(\delta_2)-\phi(\delta_1)\| \leq \|\Theta(\delta_2,0)-\Theta(\delta_1,0)\|_{\mathcal{A}(\Im)}\widehat{\zeta}$$
$$+ \|p_{\aleph}\|_{L^1}\psi(\widehat{\zeta})|\delta_2-\delta_1|\left[\Xi + \int_0^{\delta_1}\|\Theta(\delta_2,\varrho)-\Theta(\delta_1,\varrho)\|_{\mathcal{A}(\Im)}\,d\varrho\right] \ \text{ for } \ \delta_1,\delta_2 \in [-\zeta,\varkappa]\Bigg\}.$$

We note that $\Bbbk$ is closed, convex and equicontinuous, with the norm

$$\|\phi\|_{\Bbbk} = \sup_{\delta\in[-\zeta,\varkappa]}\|\phi(\delta)\|_{\Im}.$$

Firstly, transform problem (1.1) into a fixed point problem and define the operator

$$\wp\phi(\delta) = \begin{cases} \Theta(\delta,0)\Phi(0) + \displaystyle\int_0^{\delta}\Theta(\delta,\varrho)\aleph(\varrho,\phi_\varrho)\,d\varrho & \text{if } \ \delta \in \mathfrak{Q}, \\ \Phi(\delta) & \text{if } \ \delta \in [-\zeta,0]. \end{cases}$$

Step 1: $\wp$ maps $\Bbbk$ into itself.

Take $\phi \in \Bbbk$ and assume that $\wp\phi(\delta) \neq 0$. Then there exists $\jmath \in \Im^*$ such that $\|\wp\phi(\delta)\|_{\Im} = \jmath(\wp\phi(\delta))$. Thus if $\delta \in [-\zeta,0]$, we have

$$\|\wp\phi(\delta)\|_{\Im} = \jmath(\wp\phi(\delta)) = \jmath(\Phi(\delta)) = \|\Phi(\delta)\|_{\infty} \leq \|\Phi(\delta)\|_{\Bbbk} \leq \widehat{\zeta}.$$

If $\delta \in \mathfrak{Q}$, we have

$$\begin{aligned}
\|\wp\phi(\delta)\|_{\Im} &= \jmath(\wp\phi(\delta)) \\
&= \jmath\Big(\Theta(\delta,0)\Phi(0) + \int_0^{\delta}\Theta(\delta,\varrho)\aleph(\varrho,\phi_\varrho)\,d\varrho\Big) \leq \jmath(\Theta(\delta,0)\Phi(0)) + \int_0^{\delta}\jmath\big(\Theta(\delta,\varrho)\aleph(\varrho,\phi_\varrho)\big)\,d\varrho \\
&\leq \|\Theta(\delta,0)\|_{\mathcal{A}(\Im)}\|\Phi(0)\|_{\infty} + \int_0^{\delta}\|\Theta(\delta,\varrho)\|_{\mathcal{A}(\Im)}\|\aleph(\varrho,\phi_\varrho)\|_{\Im}\,d\varrho \\
&\leq \Xi\|\Phi(0)\|_{\Bbbk} + \Xi\int_0^{\delta}p_{\aleph}(\varrho)\psi(\|\phi\|_{\Bbbk})\,d\varrho \leq \Xi\widehat{\zeta} + \Xi\int_0^{\delta}p_{\aleph}(\varrho)\psi(\widehat{\zeta})\,d\varrho \\
&\leq \Xi\widehat{\zeta} + \Xi\psi(\widehat{\zeta})\int_0^{\delta}p_{\aleph}(\varrho)\,d\varrho \leq \Xi\widehat{\zeta} + \Xi\psi(\widehat{\zeta})\|p_{\aleph}\|_{L^1} \leq \widehat{\zeta}.
\end{aligned}$$

Then

$$\|\wp(\phi(\delta))\|_{\Bbbk} \leq \widehat{\zeta} \ \text{ for } \ \delta \in [-\zeta,\varkappa].$$

Let $\delta_1,\delta_2 \in [-\zeta,\varkappa]$, $\delta_1 < \delta_2$, $\phi \in \Bbbk$, so $\wp\phi(\delta_2) - \wp\phi(\delta_1) \neq 0$, then there exists $\jmath \in \Im^*$ such that:

If $\delta_1, \delta_2 \in [-\zeta, 0]$, then

$$\begin{aligned}\|\wp\phi(\delta_2) - \wp\phi(\delta_1)\|_{\Im} &= \jmath(\wp\phi(\delta_2) - \wp\phi(\delta_1)) \\ &= \jmath(\xi(\delta_2) - \xi(\delta_1)) = \|\xi(\delta_2) - \xi(\delta_1)\|_{\infty} \to 0 \text{ as } \delta_1 \to \delta_2.\end{aligned}$$

If $\delta_1, \delta_2 \in \mathfrak{Q}$, then

$$\begin{aligned}\|\wp\phi(\delta_2) - \wp\phi(\delta_1)\|_{\Im} &= \jmath(\wp\phi(\delta_2) - \wp\phi(\delta_1)) \\ &\leq \jmath\Bigg((\Theta(\delta_2, 0) - \Theta(\delta_1, 0))\xi(0) + \int_{\delta_1}^{\delta_2} \Theta(\delta_2, \varrho)\aleph(\varrho, \phi_\varrho)\, d\varrho \\ &\qquad + \int_0^{\delta_1} (\Theta(\delta_2, \varrho) - \Theta(\delta_1, \varrho))\aleph(\varrho, \phi_\varrho)\, d\varrho\Bigg) \\ &\leq \jmath\big((\Theta(\delta_2, 0) - \Theta(\delta_1, 0))\xi(0)\big) + \int_{\delta_1}^{\delta_2} \jmath(\Theta(\delta_2, \varrho)\aleph(\varrho, \phi_\varrho))\, d\varrho \\ &\qquad + \int_0^{\delta_1} \jmath((\Theta(\delta_2, \varrho) - \Theta(\delta_1, \varrho))\aleph(\varrho, \phi_\varrho))\, d\varrho \\ &\leq \|\Theta(\delta_2, 0) - \Theta(\delta_1, 0)\|_{\mathcal{A}(\Im)}\|\xi(0)\|_{\infty} + \int_{\delta_1}^{\delta_2} \|\Theta(\delta_2, \varrho)\|_{\mathcal{A}(\Im)}\|\aleph(\varrho, \phi_\varrho)\|_{\Im}\, d\varrho \\ &\qquad + \int_0^{\delta_1} \|\Theta(\delta_2, \varrho) - \Theta(\delta_1, \varrho)\|_{\mathcal{A}(\Im)}\|\aleph(\varrho, \phi_\varrho)\|_{\Im}\, d\varrho \\ &\leq \|\Theta(\delta_2, 0) - \Theta(\delta_1, 0)\|_{\mathcal{A}(\Im)}\|\xi(0)\|_{\Bbbk} + \int_{\delta_1}^{\delta_2} \|\Theta(\delta_2, \varrho)\|_{\mathcal{A}(\Im)} p_\aleph(\varrho)\psi(\|\phi\|_{\Bbbk})\, d\varrho \\ &\qquad + \int_0^{\delta_1} \|\Theta(\delta_2, \varrho) - \Theta(\delta_1, \varrho)\|_{\mathcal{A}(\Im)} p_\aleph(\varrho)\psi(\|\phi\|_{\Bbbk})\, d\varrho \\ &\leq \|\Theta(\delta_2, 0) - \Theta(\delta_1, 0)\|_{\mathcal{A}(\Im)}\widehat{\zeta} + \Xi\|p_\aleph\|_{L^1}\psi(\widehat{\zeta})|\delta_2 - \delta_1| \\ &\qquad + \|p_\aleph\|_{L^1}\psi(\widehat{\zeta}) \int_0^{\delta_1} \|\Theta(\delta_2, \varrho) - \Theta(\delta_1, \varrho)\|_{\mathcal{A}(\Im)}\, d\varrho \to 0 \text{ as } \delta_1 \to \delta_2.\end{aligned}$$

Thus $\wp(\Bbbk) \subset \Bbbk$.

Step 2: $\wp$ is weakly sequentially continuous.

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $\Bbbk$ and let $\phi_n(\delta) \to \phi(\delta)$ in $\Im_\omega$ for each $\delta \in \mathfrak{Q}$. Fix $\delta \in \mathfrak{Q}$. Since $\aleph$ satisfies assumption (H1), we have $\aleph(\delta, \phi_{\delta n})$ converging weakly uniformly to $\aleph(\delta, \phi_\delta)$. Hence the Lebesgue dominated convergence theorem for the Pettis integral implies $\wp\phi_n(\delta)$ converging weakly uniformly to $\wp\phi(\delta)$ in $\Im_\omega$. We do it for each $\delta \in \mathfrak{Q}$, so $\wp\phi_n(\delta) \to \wp\phi$. Then $\wp : \Bbbk \to \Bbbk$ is weakly sequentially continuous.

Step 3:

Now, let $\mathfrak{Z}$ be a subset of $\Bbbk$ such that $\mathfrak{Z} = \mathrm{conv}(\wp(\mathfrak{Z}) \cup 0)$. Obviously, $\mathfrak{Z}(\delta) \subset \mathrm{conv}(\wp(\mathfrak{Z}) \cup 0), \forall\delta \in \mathfrak{Q}$. $\wp\mathfrak{Z}(\delta) \subset \wp\Bbbk(\delta)$, $\delta \in \mathfrak{Q}$, is bounded in $\Im$. So, $\wp\mathfrak{Z}(\delta)$ is weakly relatively compact since a subset of a reflexive Banach space is weakly relatively compact if and only if it is bounded in the norm topology. So,

$$\chi(\mathfrak{Z}(\delta)) \leq \chi(\text{conv}(\wp(\mathfrak{Z})) \cup 0) \leq \chi(\wp\mathfrak{Z}(\delta)) = 0.$$

Thus, $\mathfrak{Z}$ is relatively weakly compact. Applying now the Mönch type fixed point theorem [30], we conclude that $\wp$ has a fixed point which is a solution of problem (1.1). □

Remark 3.1. In Theorem 3.1, we have presented an existence result of weak solutions for problem (1.1) in the case when the Banach space $\Im$ is reflexive.

However, in the nonreflexive case, hypotheses $(H1)$–$(H4)$ are not sufficient for the application of the Mönch type fixed point theorem [30].

Let us introduce the following conditions:

(C1) For $\Bbbk \subset \Im$ and each $\delta \in \mathfrak{Q}$, the set $\aleph(\delta, \Bbbk)$ is weakly relatively compact in $\Im$.

(C2) For $\Bbbk \subset \Im$ and each $\delta \in \mathfrak{Q}$,

$$\chi(\aleph(\delta, \Bbbk)) \leq p_{\aleph}(\delta)\chi(\Bbbk).$$

We then have the following results.

Theorem 3.2. *Let $\Im$ be a Banach space, and assume that assumptions $(H1)$–$(H4)$, $(C1)$ are satisfied, then problem (1.1) has at least one solution.*

Theorem 3.3. *Let $\Im$ be a Banach space, and assume that assumptions $(H1)$–$(H4)$, $(C2)$ are satisfied, then problem (1.1) has at least one solution.*

4 An example

Consider the following class of partial integro-differential system:

$$\begin{cases} \dfrac{\partial}{\partial \delta} z(\delta, \beta) = -\dfrac{\partial}{\partial \beta} z(\delta, \beta) - \pi z(\delta, \beta) - \displaystyle\int_0^{\delta} \Gamma(\delta - \varrho)\Big(\dfrac{\partial}{\partial \beta} z(\varrho, \beta) + \pi z(\varrho, \beta)\Big)\, d\varrho \\ \qquad\qquad +(e^{\frac{1}{\delta}} + 1)\Big(\dfrac{1}{(\delta^2+1)^2} + \arctan\big(|z(\delta, \beta)|\big)\Big) \ \text{ if } \ \delta \in \mathfrak{Q} = [0,1], \ \ \beta \in (0,1), \\ z(\delta, 0) = z(\delta, 1) = 0 \ \text{ for } \ \delta \in \mathfrak{Q}, \\ z(\vartheta, \beta) = \Phi(\vartheta, \beta) \ \text{ if } \ \vartheta \in [-\zeta, 0] \ \text{ and } \ \beta \in (0,1), \end{cases} \tag{4.1}$$

Let $\mathfrak{K}$ be defined by

$$(\mathfrak{K}z)(\beta) = \frac{\partial}{\partial \beta} z(\delta, \beta) + \pi z(\delta, \beta),$$

and

$$D(\mathfrak{K}) = \Big\{ z \in L^2(0,1) : \ \ z, \frac{\partial}{\partial \beta} z \in L^2(0,1), \ \ z(0) = z(1) = 0 \Big\}.$$

The operator $\mathfrak{K}$ is the infinitesimal generator of a C_0-semigroup on $L^2(0,1)$ with domain $D(\mathfrak{K})$, and with more appropriate conditions on the operator $\gamma(\,\cdot\,) = \Gamma(\,\cdot\,)\mathfrak{K}$, problem (4.1) has a resolvent operator $(\Theta(\delta))_{\delta \geq 0}$ on $L^2(0,1)$, which is norm continuous.

Now, define

$$\phi(\delta)(\beta) = z(\delta, \beta),$$

and $\aleph : \mathfrak{Q} \times L^2(0,1) \to L^2(0,1)$ given by

$$\aleph(\delta, z)(\beta) = (e^{\frac{1}{\delta}} + 1)\Big(\frac{1}{(\delta^2+1)^2} + \arctan\big(|z(\delta, \beta)|\big)\Big) \ \text{ for } \ \delta \in \mathfrak{Q}, \beta \in (0,1).$$

Now, for $\delta \in \mathfrak{Q}$ and $\jmath \in L^2(0,1)$ and by $(H1), (H2)$, we have

$$\|\aleph(\delta, z_\delta)\| = \jmath(\aleph(\delta, z_\delta)) = \jmath\Big((e^{\frac{1}{\delta}}+1)\Big(\frac{1}{(\delta^2+1)^2} + \arctan\big(|z(\delta,\beta)|\big)\Big)\Big)$$
$$\leq (e^{\frac{1}{\delta}}+1)\big(1+\|z(\delta,\beta)\|\big) = p_\aleph(\delta)\psi\big(\|z(\delta,\beta)\|\big).$$

Therefore, assumption $(H1)$ is satisfied with

$$p_\aleph(\delta) = e^{\frac{1}{\delta}}+1, \;\; \delta \in \mathfrak{Q} \;\text{ and }\; \psi(\beta) = 1+\beta, \;\; \beta \in (0,1).$$

Now, we will check that condition of $(H4)$ is satisfied. Indeed, we have

$$\widehat{\zeta} \geq \Xi\widehat{\zeta} + 2\Xi(1+\widehat{\zeta}).$$

Thus

$$\widehat{\zeta} \geq \frac{2\Xi}{1-3\Xi}\,.$$

Then $\widehat{\zeta}$ can be any positive constant.

Consequently, all the hypotheses of Theorem 3.1 are satisfied and we conclude that problem (4.1) has at least one solution $\phi \in C(\mathfrak{Q}, L^2(0,1))$.

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