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Bouharket Benaissa, Noureddine Azzouz

A NEW FRACTIONAL VERSION OF BULLEN INEQUALITY FOR h-CONVEX FUNCTIONS

Abstract. In this study, the Bullen inequalities for h-convex functions involving Riemann–Liouville fractional operators are established, where h is a B-function. In addition, new results are presented that generalize various inequalities known in the literature.

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1 Introduction

For the convex function f, the well-known Hermite–Hadamard inequality reads as follows [10]:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}\,. \tag{1.1}$$

In [4], Bullen improved the right-hand side of (1.1) by using the following inequality, known as Bullen's inequality:

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\Big[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\Big] \leq \frac{f(a)+f(b)}{2}\,.$$

In 2016, the authors presented an estimate of Bullen-type inequalities for functions whose absolute values of first derivatives are convex [12, Remark 4.2]:

$$\left|\frac{1}{2}\Big[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\Big] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)\,[\,|f'(a)|+|f'(b)|\,]}{16}\,. \tag{1.2}$$

Bullen's inequalities provide an estimate of the average value of a function that is convex on both sides, while simultaneously ensuring that the function is integrable. This inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [5,6,8,9,11,12,16]).

The analysis of fractional calculations is a generalization of classical analysis, and it advanced rapidly thanks to the exciting concept of convexity. Its extensive applications in functional analysis and optimization theory have made it a very popular research area. The author in [17] introduced a novel class of functions called h-convex functions.

Definition 1.1. Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function if f is non-negative and for all $x, y \in I$, $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{1.3}$$

If inequality (1.3) is reversed, then f is said to be h-concave.

Setting

- $h(\lambda) = \lambda$, Definition 1.1 reduces to convex function [14].
- $h(\lambda) = 1$, Definition 1.1 reduces to P-functions [7,15].
- $h(\lambda) = \lambda^s$, Definition 1.1 reduces to s-convex functions [3].
- $h(\lambda) = \frac{1}{n}\sum_{k=1}^{n} \lambda^{\frac{1}{k}}$, Definition 1.1 reduces to polynomial n-fractional convex functions [13].

In recent works [1,2], the authors introduced a novel class of functions termed B-functions, defined as follows.

Definition 1.2. Let $g : [0,\infty) \to \mathbb{R}$ be a non-negative function. The function g is called a B-function if

$$g(x-a) + g(b-x) \leq 2g\Big(\frac{a+b}{2}\Big), \tag{1.4}$$

where $a < x < b$ with $a, b \in [0,\infty)$.

If inequality (1.4) is reversed, g is called A-function, or g belongs to the class $A(a,b)$. If we have equality in (1.4), g is called AB-function, or g belongs to the class $AB(a,b)$.

Corollary 1.1. *Let $h:(0,1)\to\mathbb{R}$ be a non-negative function. The function h is a B-function if for all $\lambda\in(0,1)$, we have*

$$h(\lambda)+h(1-\lambda)\le 2\,h\Big(\frac{1}{2}\Big). \tag{1.5}$$

- *The functions $h(\lambda)=\lambda$ and $h(\lambda)=1$ are AB-function, B-function and A-function.*
- *The function $h(\lambda)=\lambda^s$, $s\in(0,1]$, is a B-function.*
- *The function $h(\lambda)=\frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$, $n,k\in\mathbb{N}$, is a B-function.*

Let $f\in L[a,b]$. The left- and right-sided Riemann–Liouville fractional operators of order $\alpha>0$ are defined as follows:

$$\mathfrak{I}^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt,\quad x>a,$$

$$\mathfrak{I}^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt,\quad x<b.$$

Based on earlier research, we developed an additional version of Bullen inequality for h-convex functions using Riemann–Liouville integral operators.

2 Bullen inequalities

Lemma 2.1. *If $\alpha>0$ and $f:[a,b]\to\mathbb{R}$ is a differentiable mapping such that $f'\in L_1([a,b])$, then the following identity holds:*

$$\begin{aligned}\frac{1}{2}\Big[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\Big]&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[\mathfrak{I}^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^{\alpha}_{b^-}f\Big(\frac{a+b}{2}\Big)\Big]\\&=\frac{(b-a)}{8}\int_0^1(1-2t^{\alpha})\Big[f'\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)-f'\Big(\Big(\frac{1-t}{2}\Big)a+\Big(\frac{1+t}{2}\Big)b\Big)\Big]\,dt. \end{aligned}\tag{2.1}$$

Proof. Using the integration by parts, we deduce

$$\begin{aligned}J_1&=\int_0^1(1-2t^{\alpha})f'\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)\,dt\\&=-\Big(\frac{2}{b-a}\Big)(1-2t^{\alpha})f\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)\Big|_0^1-\Big(\frac{4\alpha}{b-a}\Big)\int_0^1 t^{\alpha-1}f\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)\,dt\\&=\Big(\frac{2}{b-a}\Big)\Big[f(a)+f\Big(\frac{a+b}{2}\Big)\Big]-\Big(\frac{2}{b-a}\Big)^{\alpha+1}2\Gamma(\alpha+1)\mathfrak{I}^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big),\end{aligned}$$

where we apply $\tau=(\frac{1+t}{2})a+(\frac{1-t}{2})b$, then

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)\,dt\\&\qquad=\Big(\frac{2}{b-a}\Big)^{\alpha+1}\int_a^{\frac{a+b}{2}}\Big(\frac{a+b}{2}-\tau\Big)^{\alpha-1}f(\tau)\,d\tau=\Big(\frac{2}{b-a}\Big)^{\alpha}\Gamma(\alpha)\mathfrak{I}^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big).\end{aligned}$$

Similarly,

$$J_2 = \int_0^1 \Big(1-2t^\alpha\Big) f'\Big(\Big(\frac{1-t}{2}\Big)a + \Big(\frac{1+t}{2}\Big)b\Big)\,dt$$

$$= \Big(\frac{2}{b-a}\Big)(1-2t^\alpha) f\Big(\Big(\frac{1-t}{2}\Big)a + \Big(\frac{1+t}{2}\Big)b\Big)\Big|_0^1 + \Big(\frac{4\,\alpha}{b-a}\Big)\int_0^1 t^{\alpha-1} f\Big(\Big(\frac{1-t}{2}\Big)a + \Big(\frac{1+t}{2}\Big)b\Big)\,dt$$

$$= -\Big(\frac{2}{b-a}\Big)\Big[f(b) + f\Big(\frac{a+b}{2}\Big)\Big] + \Big(\frac{2}{b-a}\Big)^{\alpha+1} 2\Gamma(\alpha+1)\mathfrak{I}^\alpha_{b-} f\Big(\frac{a+b}{2}\Big),$$

where we apply $\tau = (\frac{1-t}{2})a + (\frac{1+t}{2})b$, then

$$\int_0^1 t^{\alpha-1} f\Big(\Big(\frac{1-t}{2}\Big)a + \Big(\frac{1+t}{2}\Big)b\Big)\,dt$$

$$= \Big(\frac{2}{b-a}\Big)^{\alpha+1}\int_{\frac{a+b}{2}}^{b} \Big(\tau - \frac{a+b}{2}\Big)^{\alpha-1} f(\tau)\,d\tau = \Big(\frac{2}{b-a}\Big)^{\alpha}\Gamma(\alpha)\mathfrak{I}^\alpha_{b-} f\Big(\frac{a+b}{2}\Big).$$

As a result,

$$\frac{b-a}{8}\,(J_1 - J_2) = \frac{1}{2}\Big[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\Big] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[\mathfrak{I}^\alpha_{a+} f\Big(\frac{a+b}{2}\Big) + \mathfrak{I}^\alpha_{b-} f\Big(\frac{a+b}{2}\Big)\Big].$$

This gives us the desired result. □

We now present the first results on the estimation of the Bullen inequality.

Theorem 2.1. *Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is a h-convex mapping on $[a,b]$, then the following Bullen inequality for Riemann–Liouville fractional operators holds:*

$$\Big|\frac{1}{2}\Big[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\Big] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[\mathfrak{I}^\alpha_{a+} f\Big(\frac{a+b}{2}\Big) + \mathfrak{I}^\alpha_{b-} f\Big(\frac{a+b}{2}\Big)\Big]\Big|$$
$$\leq \frac{b-a}{4}\,h\Big(\frac{1}{2}\Big)\,C_\alpha\big[\,|f'(a)| + |f'(b)|\,\big],$$

where

$$C_\alpha = \Big(\frac{1}{2}\Big)^\alpha\Big(\frac{2\alpha}{\alpha+1}\Big) + \Big(\frac{1-\alpha}{\alpha+1}\Big). \tag{2.2}$$

Proof. Using the absolute value of identity (2.1) and the h-convexity of the function $|f'|$, we deduce

$$\Big|\frac{1}{2}\Big[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\Big] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[\mathfrak{I}^\alpha_{a+} f\Big(\frac{a+b}{2}\Big) + \mathfrak{I}^\alpha_{b-} f\Big(\frac{a+b}{2}\Big)\Big]\Big|$$

$$\leq \frac{b-a}{8}\int_0^1 |1-2t^\alpha|\Big[\Big|f'\Big(\Big(\frac{1+t}{2}\Big)a + \Big(\frac{1-t}{2}\Big)b\Big)\Big| + \Big|f'\Big(\Big(1-\frac{t}{2}\Big)a + \Big(\frac{t}{2}\Big)b\Big)\Big|\Big]\,dt$$

$$\leq \frac{b-a}{8}\int_0^1 |1-2t^\alpha|\Big[h\Big(\frac{1+t}{2}\Big)|f'(a)| + h\Big(\frac{1-t}{2}\Big)|f'(b)| + h\Big(\frac{1-t}{2}\Big)|f'(a)| + h\Big(\frac{1+t}{2}\Big)|f'(b)|\Big]\,dt$$

$$= \frac{b-a}{8}\big[\,|f'(a)| + |f'(b)|\,\big]\int_0^1 |1-2t^\alpha|\Big[h\Big(\frac{1+t}{2}\Big) + h\Big(\frac{1-t}{2}\Big)\Big]\,dt.$$

Since h is a B-function, applying inequality (1.5) for $\lambda = \frac{1+t}{2}$ yields the following inequality:

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{a^+} f\Big(\frac{a+b}{2}\Big) + \mathfrak{I}^\alpha_{b^-} f\Big(\frac{a+b}{2}\Big)\right]\right|$$
$$\leq \frac{b-a}{4}\, h\Big(\frac{1}{2}\Big)\big[\,|f'(a)| + |f'(b)|\,\big]\int_0^1 |1-2t^\alpha|\, dt.$$

Given

$$\Big|1-2t^\alpha\Big| = \begin{cases} 1-2t^\alpha & t \in \Big(0, \big(\frac{1}{2}\big)^{\frac{1}{\alpha}}\Big), \\ 2t^\alpha - 1 & t \in \Big(\big(\frac{1}{2}\big)^{\frac{1}{\alpha}}, 1\Big), \end{cases}$$

we have

$$\int_0^1 |1-2t^\alpha|\, dt = \int_0^{(\frac{1}{2})^{\frac{1}{\alpha}}} (1-2t^\alpha)\, dt + \int_{(\frac{1}{2})^{\frac{1}{\alpha}}}^1 (2t^\alpha - 1)\, dt = \Big(\frac{1}{2}\Big)^\alpha \Big(\frac{2\alpha}{\alpha+1}\Big) + \Big(\frac{1-\alpha}{\alpha+1}\Big). \qquad \square$$

Taking $\alpha = 1$, we obtain the following Bullen inequalities via the Riemann integral for h-convex function.

Corollary 2.1. *Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is a h-convex mapping on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \leq \frac{b-a}{8}\, h\Big(\frac{1}{2}\Big)\big[\,|f'(a)| + |f'(b)|\,\big].$$

Next, consider some particular cases on h-convexity.

1. Putting $h(t) = t^s$ with $s \in (0,1]$ in Theorem 2.1 and Corollary 2.1, we deduce the following result.

Corollary 2.2. *Assume α and f are defined according to Theorem 2.1. If $|f'|$ is a s-convex function on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{a^+} f\Big(\frac{a+b}{2}\Big) + \mathfrak{I}^\alpha_{b^-} f\Big(\frac{a+b}{2}\Big)\right]\right|$$
$$\leq \frac{b-a}{4}\Big(\frac{1}{2}\Big)^s C_\alpha\big[\,|f'(a)| + |f'(b)|\,\big], \quad (2.3)$$

where C_α is defined by (2.2).

For $\alpha = 1$,

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \leq \frac{b-a}{8}\Big(\frac{1}{2}\Big)^s \big[\,|f'(a)| + |f'(b)|\,\big]. \qquad (2.4)$$

Putting $s = 1$ in inequality (2.4), we get the Bullen inequality via the Riemann integral for the convex function in (1.2).

2. Setting $h(\lambda) = 1$ in Theorem 2.1 and Corollary 2.1, we obtain the following new result for the class of P-function. This also corresponds to the cases $s \to 0^+$ in inequalities (2.3) and (2.4).

Corollary 2.3. *Assume α and f are defined according to Theorem 2.1. If $|f'|$ is a P-function on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{a+}f\Big(\frac{a+b}{2}\Big)+\Im^{\alpha}_{b-}f\Big(\frac{a+b}{2}\Big)\right]\right| \leq \frac{b-a}{4}C_{\alpha}\big[\,|f'(a)|+|f'(b)|\,\big]$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{8}\big[\,|f'(a)|+|f'(b)|\,\big].$$

3. Set $h(\lambda)=\dfrac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$ in Theorem 2.1 and Corollary 2.1.

Corollary 2.4. *Assume α and f are defined according to Theorem 2.1. If $|f'|$ is a n-fractional polynomial convex mapping on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{a+}f\Big(\frac{a+b}{2}\Big)+\Im^{\alpha}_{b-}f\Big(\frac{a+b}{2}\Big)\right]\right| \leq \frac{b-a}{4n}\sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^{\frac{1}{k}}C_{\alpha}\big[\,|f'(a)|+|f'(b)|\,\big]$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{8n}\sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^{\frac{1}{k}}\big[\,|f'(a)|+|f'(b)|\,\big]. \tag{2.5}$$

Putting $n=1$ in inequality (2.5), we get inequality (1.2).

Theorem 2.2. *Let h be a B-function on $(0,1)$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and assume that α, f are defined as in Lemma 2.1. If $|f'|^p$ is a h-convex mapping on $[a,b]$, we get the following Bullen type inequality:*

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{a+}f\Big(\frac{a+b}{2}\Big)+\Im^{\alpha}_{b-}f\Big(\frac{a+b}{2}\Big)\right]\right| \\ \leq\frac{b-a}{4}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}\left(h\Big(\frac{1}{2}\Big)\right)^{\frac{1}{p}}\big[\,|f'(a)|^p+|f'(b)|^p\,\big]^{\frac{1}{p}} \\ \leq\frac{b-a}{4}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}\left(h\Big(\frac{1}{2}\Big)\right)^{\frac{1}{p}}\big[\,|f'(a)|+|f'(b)|\,\big].\end{aligned} \tag{2.6}$$

Proof. Using the absolute value of identity (2.1), we get

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{a+}f\Big(\frac{a+b}{2}\Big)+\Im^{\alpha}_{b-}f\Big(\frac{a+b}{2}\Big)\right]\right| \\ &\leq\frac{b-a}{8}\int_0^1|1-2t^{\alpha}|\left|f'\left(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\right)\right|dt+\frac{b-a}{8}\int_0^1|1-2t^{\alpha}|\left|f'\left(\Big(\frac{1-t}{2}\Big)a+\Big(\frac{1+t}{2}\Big)b\right)\right|dt.\end{aligned}$$

Applying Hölder inequality and $A^{\frac{1}{p}}+B^{\frac{1}{p}}=2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, we conclude that

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^{\alpha}_{b^-}f\Big(\frac{a+b}{2}\Big)\right]\right|\\
&\qquad\le\frac{b-a}{8}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f'\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)\right|^p dt\right)^{\frac{1}{p}}\\
&\qquad\quad+\frac{b-a}{8}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f'\Big(\Big(\frac{1-t}{2}\Big)a+\Big(\frac{1+t}{2}\Big)b\Big)\right|^p dt\right)^{\frac{1}{p}}\\
&\qquad\le\frac{b-a}{8}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}2^{1-\frac{1}{p}}\\
&\qquad\quad\times\left[\int_0^1\left|f'\Big(\Big(\frac{1+t}{2}\Big)a+\Big(\frac{1-t}{2}\Big)b\Big)\right|^p dt+\int_0^1\left|f'\Big(\Big(\frac{1-t}{2}\Big)a+\Big(\frac{1+t}{2}\Big)b\Big)\right|^p dt\right]^{\frac{1}{p}}.\end{aligned}$$

Assuming $|f'|^p$ is an h-convex function, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^{\alpha}_{b^-}f\Big(\frac{a+b}{2}\Big)\right]\right|\\
&\quad\le\frac{b-a}{8}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}2^{\frac{1}{q}}\left[\int_0^1\left(h\Big(\frac{1+t}{2}\Big)|f'(a)|^p+h\Big(\frac{1-t}{2}\Big)|f'(b)|^p\right)dt\right.\\
&\qquad\left.+\int_0^1\left(h\Big(\frac{1-t}{2}\Big)|f'(a)|^p+h\Big(\frac{1+t}{2}\Big)|f'(b)|^p\right)dt\right]^{\frac{1}{p}}\\
&\quad\le\frac{b-a}{8}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}2^{\frac{1}{q}}\left(\int_0^1\left[h\Big(\frac{1+t}{2}\Big)+h\Big(\frac{1-t}{2}\Big)\right]dt\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.\end{aligned}$$

Applying inequality (1.5) for $\lambda=\frac{1+t}{2}$, we obtain

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^{\alpha}_{b^-}f\Big(\frac{a+b}{2}\Big)\right]\right|\\
&\qquad\qquad\le\frac{b-a}{4}\left(\int_0^1|1-2t^{\alpha}|^q\,dt\right)^{\frac{1}{q}}\left(h\Big(\frac{1}{2}\Big)\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.\end{aligned}$$

This completes the proof of the first inequality in (2.6).

For $p>1$ and $A,B\ge 0$, we get $A^p+B^p\le(A+B)^p$, yielding the second inequality in (2.6). □

Putting $\alpha=1$, we have

$$|1-2t|^q=\begin{cases}(1-2t)^q, & t\in\Big(0,\dfrac{1}{2}\Big),\\[2mm](2t-1)^q, & t\in\Big(\dfrac{1}{2},1\Big),\end{cases}$$

thus

$$\int_0^1|1-2t|^q\,dt=\int_0^{\frac{1}{2}}(1-2t)^q\,dt+\int_{\frac{1}{2}}^1(2t-1)^q\,dt=\frac{1}{q+1}$$

and the following Bullen inequalities hold via the Riemann integral for an h-convex function.

Corollary 2.5. *Let h be a B-function on $(0,1)$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and assume that f are defined as in Lemma 2.1. If $|f'|^p$ is a h-convex mapping on $[a,b]$, we get the following Bullen type inequality:*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(h\Big(\frac{1}{2}\Big)\right)^{\frac{1}{p}}\left[\,|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}} \leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(h\Big(\frac{1}{2}\Big)\right)^{\frac{1}{p}}\left[\,|f'(a)|+|f'(b)|\,\right].$$

Now, some special cases on an h-convex function are established.

1. Given $h(\lambda)=\lambda^s$ with $s\in(0,1]$ in Theorem 2.2 and Corollary 2.5, we deduce the following result.

Corollary 2.6. *Assume α and f are defined according to Theorem 2.2. If $|f'|^p$ is an s-convex function on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{a+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^\alpha_{b-}f\Big(\frac{a+b}{2}\Big)\right]\right| \leq \frac{b-a}{4}\left(\int_0^1|1-2t^\alpha|^q\,dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\,|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}} \leq \frac{b-a}{4}\left(\int_0^1|1-2t^\alpha|^q\,dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\,|f'(a)|+|f'(b)|\,\right] \tag{2.7}$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\,|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}} \leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\,|f'(a)|+|f'(b)|\,\right]. \tag{2.8}$$

Remark 2.1. Putting $s=1$ in (2.8) yields the following: for $p,q>1$, where $\frac{1}{p}+\frac{1}{q}=1$, if $|f'|^p$ is a convex function on $[a,b]$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4(q+1)^{\frac{1}{q}}}\left(\frac{|f'(a)|^p+|f'(b)|^p}{2}\right)^{\frac{1}{p}}. \tag{2.9}$$

2. Setting $h(\lambda)=1$ in Theorem 2.2 and Corollary 2.5 gives the following new result for the class of P-functions. Consider $s\to0^+$ in inequalities (2.7) and (2.8).

Corollary 2.7. *Assume α and f are defined according to Theorem 2.2. If $|f'|^p$ is a P-function on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{a+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^\alpha_{b-}f\Big(\frac{a+b}{2}\Big)\right]\right|$$

$$\leq \frac{b-a}{4}\left(\int_0^1 |1-2t^\alpha|^q\,dt\right)^{\frac{1}{q}}\big[\,|f'(a)|^p+|f'(b)|^p\big]^{\frac{1}{p}}$$

$$\leq \frac{b-a}{4}\left(\int_0^1 |1-2t^\alpha|^q\,dt\right)^{\frac{1}{q}}\big[\,|f'(a)|+|f'(b)|\,\big]$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\big[\,|f'(a)|^p+|f'(b)|^p\big]^{\frac{1}{p}}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\big[\,|f'(a)|+|f'(b)|\,\big].$$

3. Setting $h(\lambda)=\frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$ in Theorem 2.2 and Corollary 2.5, we get the following new result for the class of n-fractional polynomial convex functions.

Corollary 2.8. *Assume α and f are defined according to Theorem 2.1. If $|f'|^p$ is an n-fractional polynomial convex mapping on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{a^+}f\Big(\frac{a+b}{2}\Big)+\mathfrak{I}^\alpha_{b^-}f\Big(\frac{a+b}{2}\Big)\right]\right|$$

$$\leq \frac{b-a}{4}\left(\int_0^1 |1-2t^\alpha|^q\,dt\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^{\frac{1}{k}}\right)^{\frac{1}{p}}\big[\,|f'(a)|^p+|f'(b)|^p\big]^{\frac{1}{p}}$$

$$\leq \frac{b-a}{4}\left(\int_0^1 |1-2t^\alpha|^q\,dt\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^{\frac{1}{k}}\right)^{\frac{1}{p}}\big[\,|f'(a)|+|f'(b)|\,\big]$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^{\frac{1}{k}}\right)^{\frac{1}{p}}\big[\,|f'(a)|^p+|f'(b)|^p\big]^{\frac{1}{p}}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{2}\Big)^{\frac{1}{k}}\right)^{\frac{1}{p}}\big[\,|f'(a)|+|f'(b)|\,\big]. \quad (2.10)$$

Setting $n=1$ in (2.10) yields inequality (2.9).

3 Applications

We consider the means for arbitrary positive numbers $b>a>0$ as follows,

- The arithmetic mean:
$$A(a,b)=\frac{a+b}{2}.$$

- The generalized logarithmic mean:

$$L_n(a,b) = \Big(\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}\Big)^{\frac{1}{n}}, \quad n \in \mathbf{R} - \{-1, 0\}.$$

Proposition. Let $b > a > 0$, $n > 1$ and $p > 1$. Then the following inequality holds:

$$\Big|\frac{A(a^n, b^n) + A^n(a,b)}{2} - L_n^n(a,b)\Big| \le \frac{b-a}{4(q+1)^{\frac{1}{q}}}\, A^{\frac{1}{p}}(a^{(n-1)p}, b^{(n-1)p}).$$

Proof. Applying Remark 2.1 and taking $f(t) = t^n$ for $t > 0$, one gets $f'(t) = n\, t^{n-1}$. Since

$$\big(|f'(t)|^p\big)'' = n^p p(n-1)(p(n-1)-1)t^{p(n-1)-2} > 0,$$

the function $|f'(t)|^p$ is convex. □

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Authors' addresses:

Bouharket Benaissa
Faculty of Material Science, University of Tiaret, Algeria
E-mail: bouharket.benaissa@univ-tiaret.dz

Noureddine Azzouz
Faculty of Sciences, University Center Nour Bachir El Bayadh, Algeria
E-mail: n.azzouz@cu-elbayadh.dz