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LIMIT CYCLES OF PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY LINEAR CENTER OR FOCUS AND CUBIC UNIFORM ISOCHRONOUS CENTER

Abstract. Due to the wide interdisciplinary use of discontinuous piecewise differential systems and the main role of the periodic solutions in understanding and explaining many natural phenomena, scientists developed many methods and tools for studying the periodic solutions of such differential systems, like the averaging theory of given order. In this paper, by using the averaging theory up to seven-order for computing the periodic solutions of discontinuous piecewise differential systems, we prove that five is the maximum number of limit cycles that can bifurcate from the discontinuous piecewise differential systems formed by an arbitrary linear focus or center and an arbitrary cubic uniform isochronous center separated by a straight line.

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1 Introduction and statement of the main results

The classification of the center's problem of planar polynomial differential systems and the study of their cyclicity are two well-known and challenging problems in the qualitative theory of planar polynomial differential systems. We recall that the cyclicity problem asks for the maximal number of limit cycles that can bifurcate from either the center that we called in the literature the *Hopf bifurcation* or from some of the periodic orbits around this center (for more details see [15]). The problem of bifurcation of limit cycles is related to the 16th Hilbert's problem, which is still an open problem until now.

We know that one of the fundamental open problems of polynomial differential systems is the identification of their maximum number of limit cycles and their possible positions in the plane. The bifurcation of limit cycles in continuous planar differential systems has recently attracted the attention of numerous publications (see [4,11,12,17] and the references therein). But in the real world many phenomena are described by using discontinuous differential equations (see, e.g., [6,18] and the references therein). Discontinuous piecewise differential systems have been widely used in recent years to describe many natural phenomena in mechanics, electronics, economics, neuroscience, etc., due to their significant importance and extensive application (see, e.g., [6,14,16]). The bifurcation of limit cycles of discontinuous piecewise differential systems becomes an interesting topic of research. There are a lot of papers that focus on the study of such classes of piecewise differential systems using the averaging method at such orders (for more information see [7,9,10,13]).

Recently, many papers studied the bifurcation of limit cycles in planar differential systems that have a uniform isochronous center (see [1,8]). Isochronicity is significant in physical applications as well as stability theory, bifurcation issues, the uniqueness and existence of solutions for many boundary values, and so on. Moreover, the attraction to this issue has also been rekindled due to the expansion of powerful computational methods.

In this paper, our main objective is to study the limit cycles that can bifurcate from the discontinuous piecewise differential systems separated by the straight line $y = 0$ and formed by a linear differential system having a center or focus of the form

$$\dot{x} = Ax + By + C, \quad \dot{y} = -Bx + Ay + D, \tag{1.1}$$

in the half-plane $y \geq 0$, where $A, B, C, D \in \mathbb{R}$, and by an arbitrary uniform cubic isochronous center in the half-plane $y \leq 0$, characterized in the following lemma.

Lemma 1.1. *A cubic polynomial differential system has a uniform isochronous center at the origin if and only if after an affine change of variables and a rescaling of the independent variable it can be written as*

$$\dot{x} = -y + xf(x, y), \quad \dot{y} = x + yf(x, y), \tag{1.2}$$

where $f(x, y) = ax + by + cx^2 + dxy - cy^2$*, and satisfies* $a^2c - b^2c + abd = 0$.

The last result is due to Collins [5], who classified the cubic polynomial uniform isochronous centers. For other proof of Lemma 1.1, see Section 2 of [2].

The averaging theory described in Section 2 allows us to study analytically the existence of limit cycles of a non-autonomous differential system by studying the simple zeros of the averaged function. Here, we use the averaging theory up to the seventh order to study a number of limit cycles that can bifurcate from the discontinuous piecewise differential systems formed by (1.1) for $y \geq 0$, when we perturb it inside the class of all polynomial differential systems of degree 1 as follows:

$$\dot{x} = \sum_{i=1}^{7} P_{1i}(x, y)\epsilon^i, \quad \dot{y} = \sum_{i=1}^{7} Q_{1i}(x, y)\epsilon^i,$$

and by the differential system (1.2) for $y \leq 0$, when we perturb it inside the class of all polynomial differential systems of degree 3 as follows:

$$\dot{x} = \sum_{i=1}^{7} P_{3i}(x, y)\epsilon^i, \quad \dot{y} = \sum_{i=1}^{7} Q_{3i}(x, y)\epsilon^i.$$

Here, $\varepsilon > 0$ is a small parameter, $i = 1, \dots, 7$, P_{1i} and Q_{1i} are real polynomials of degree 1 in the variables x and y, and P_{3i}, Q_{3i} are real polynomials of degree 3 in the variables x and y.

The following theorem states the main result of our paper, which deals with the maximum number of limit cycles using the averaging theory up to the seventh order.

Theorem 1.1. *For $|\epsilon| \neq 0$ sufficiently small, the maximum number of limit cycles of the piecewise differential system separated by the straight line $y = 0$ and formed by systems* (1.1) *and* (1.2) *is at most five.*

Theorem 1.1 is proved in Section 3.

2 Preliminary results

To compute the limit cycles of discontinuous piecewise differential systems that we analyze in this work, we introduce the fundamental results that relate to the averaging theory up to the seventh order and the Descartes Theorem.

The next result is a summary of an improvement of the classical averaging theory for computing limit cycles of planar discontinuous piecewise differential systems which goes back to Itikawa, Llibre, and Novaes in [9].

Let us define the discontinuous differential systems in polar coordinates of the form

$$\dot{r}(\theta) = \begin{cases} F^{+}(r,\theta,\epsilon) = \sum\limits_{i=1}^{7} F_i^{+}(\theta,r,\epsilon)\epsilon^i + R^{+}(\theta,r,\epsilon)\epsilon^8, \\ F^{-}(r,\theta,\epsilon) = \sum\limits_{i=1}^{7} F_i^{-}(\theta,r,\epsilon)\epsilon^i + R^{-}(\theta,r,\epsilon)\epsilon^8, \end{cases} \tag{2.1}$$

where $\epsilon \neq 0$ is a sufficiently small parameter, $\theta \in \mathbb{S}$ which is the unite circle, and $r \in D$, where D is an open subset of $\mathbb{R}^{+}$.

Now, we give the functions $y_i^{\pm}(t,r)$ with $i = 1, \dots, 7$ defined in [9] as follows:

$$\begin{aligned}
y_1^{\pm}(t,r) &= \int_0^t F_1^{\pm}(s,r)\,ds, \\
y_2^{\pm}(t,r) &= \int_0^t \big(2F_2^{\pm}(s,r) + 2\partial F_1^{\pm}(s,r)y_1^{\pm}(s,r)\big)\,ds, \\
y_3^{\pm}(t,r) &= \int_0^t \Big(6F_3^{\pm}(s,r) + 6\partial F_2^{\pm}(s,r)y_1^{\pm}(t,r) + 3\partial^2 F_1^{\pm}(s,r)y_1^{\pm}(s,r)^2 + 3\partial F_1^{\pm}(s,r)\,y_2^{\pm}(s,r)\Big)\,ds, \\
y_4^{\pm}(t,r) &= \int_0^t \Big(24F_4^{\pm}(s,r) + 24\partial F_3^{\pm}(s,r)y_1^{\pm}(s,r) + 12\partial^2 F_2^{\pm}(s,r)y_1^{\pm}(s,r)^2 + 12\partial F_2^{\pm}(s,r)y_2^{\pm}(s,r) \\
&\quad + 12\partial^2 F_1^{\pm}(s,r)y_1^{\pm}(s,r)y_2^{\pm}(s,r) + 4\partial^3 F_1^{\pm}(s,r)y_1^{\pm}(s,r)^3 + 4\partial F_1^{\pm}(s,r)y_3^{\pm}(s,r)\Big)\,ds, \\
y_5^{\pm}(t,r) &= \int_0^t \Big(120F_5^{\pm}(s,r) + 120\partial F_4^{\pm}(s,r)y_1^{\pm}(s,r) + 60\partial^2 F_3^{\pm}(s,r)y_1^{\pm}(s,r)^2 \\
&\quad + 60\partial F_3^{\pm}(s,r)y_2^{\pm}(s,r) + 60\partial^2 F_2^{\pm}(s,r)y_1^{\pm}(s,r)y_2^{\pm}(s,r) + 20\partial^3 F_2^{\pm}(s,r)y_1^{\pm}(s,r)^3 \\
&\quad + 20\partial F_2^{\pm}(s,r)y_3^{\pm}(s,r) + 20\partial^2 F_1^{\pm}(s,r)y_1^{\pm}(s,r)y_3^{\pm}(s,r) + 15\partial^2 F_1^{\pm}(s,r)y_2^{\pm}(s,r)^2 \\
&\quad + 30\partial^3 F_1^{\pm}(s,r)y_1^{\pm}(s,r)^2 y_2^{\pm}(s,r) + 5\partial^4 F_1^{\pm}(s,r)y_1^{\pm}(s,r)^4 + 5\partial F_1^{\pm}(s,r)y_4^{\pm}(s,r)\Big)\,ds,
\end{aligned}$$

$$
\begin{aligned}
y_6^\pm(t,r) = 720\int_0^t \Big(& F_6^\pm(s,r) + \partial F_5^\pm(s,r)y_1^\pm(s,r) + \frac{1}{2}\,\partial^2 F_4^\pm(s,r)y_1^\pm(s,r)^2 \\
&+ \frac{1}{2}\,\partial F_4^\pm(s,r)y_2^\pm(s,r) + \frac{1}{6}\,\partial F_3^\pm(s,r)y_1^\pm(s,r)y_3^\pm(s,r) + \frac{1}{6}\,\partial^3 F_3^\pm(s,r)y_1^\pm(s,r)^3 \\
&+ \frac{1}{2}\,\partial^2 F_3^\pm(s,r)y_1^\pm(s,r)y_2^\pm(s,r) + \frac{1}{24}\,\partial F_2^\pm(s,r)y_4^\pm(s,r) + \frac{1}{8}\,\partial^2 F_2^\pm(s,r)y_2^\pm(s,r)^2 \\
&+ \frac{1}{6}\,\partial^2 F_2^\pm(s,r)y_1^\pm(s,r)y_3^\pm(s,r) + \frac{1}{4}\,\partial^3 F_2^\pm(s,r)y_1^\pm(s,r)^2y_2^\pm(s,r) \\
&+ \frac{1}{24}\,\partial^4 F_2^\pm(s,r)y_1^\pm(s,r)^4 + \frac{1}{8}\,\partial^3 F_1^\pm(s,r)y_1^\pm(s,r)y_2^\pm(s,r)^2 \\
&+ \frac{1}{12}\,\partial^4 F_1^\pm(s,r)y_1^\pm(s,r)^3y_2^\pm(s,r) + \frac{1}{120}\,\partial^4 F_1^\pm(s,r)y_1^\pm(s,r)^5 \\
&+ \frac{1}{12}\,\partial^2 F_1^\pm(s,r)y_2^\pm(s,r)y_3^\pm(s,r) + \partial^3 F_1^\pm(s,r)y_1^\pm(s,r)^2y_3^\pm(s,r)\Big)\,ds,
\end{aligned}
$$

where the i-th partial derivative of the function $F_i^\pm(s,r)$ with respect to the variable r has been denoted by $\partial^i F_i^\pm(s,r)$ with $i=1,\dots,7$. From [9], the (upper/lower) averaged functions $f_i^\pm(r)$ for $i=1,\dots,7$ are

$$
\begin{aligned}
f_1^\pm(r) &= \int_0^T F_1^\pm(t,r)\,dt, \\
f_2^\pm(r) &= \int_0^T \Big(F_2^\pm(t,r) + \partial F_1^\pm(t,r)y_1^\pm(t,r)\Big)\,dt, \\
f_3^\pm(r) &= \int_0^T \Big(F_3^\pm(t,r) + \partial F_2^\pm(t,r)y_1^\pm(t,r) + \frac{1}{2}\,\partial^2 F_1^\pm(t,r)y_1^\pm(t,r)^2 + \frac{1}{2}\,\partial F_1^\pm(t,r)y_2^\pm(t,r)\Big)\,dt, \\
f_4^\pm(r) &= \int_0^T \Big(F_4^\pm(t,r) + \partial F_3^\pm(t,r)y_1^\pm(t,r) + \frac{1}{2}\,\partial^2 F_2^\pm(t,r)y_1^\pm(t,r)^2 + \frac{1}{2}\,\partial F_2^\pm(t,r)y_2^\pm(t,r) \\
&\quad + \frac{1}{2}\,\partial^2 F_1^\pm(t,r)y_1^\pm(t,r)y_2^\pm(t,r) + \frac{1}{6}\,\partial^3 F_1^\pm(t,r)y_1^\pm(t,r)^3 + \frac{1}{6}\,\partial F_1^\pm(t,r)y_3^\pm(t,r)\Big)\,dt, \\
f_5^\pm(r) &= \int_0^T \Big(F_5^\pm(t,r) + \partial F_4^\pm(t,r)y_1^\pm(t,r) + \frac{1}{2}\,\partial^2 F_3^\pm(t,r)y_1^\pm(t,r)^2 + \frac{1}{2}\,\partial F_3^\pm(t,r)y_2^\pm(t,r) \\
&\quad + \frac{1}{2}\,\partial^2 F_2^\pm(t,r)y_1^\pm(t,r)y_2^\pm(t,r) + \frac{1}{6}\,\partial^3 F_2^\pm(t,r)y_1^\pm(t,r)^3 + \frac{1}{6}\,\partial F_2^\pm(t,r)y_3^\pm(t,r) \\
&\quad + \frac{1}{6}\,\partial^2 F_1^\pm(t,r)y_1^\pm(t,r)y_3^\pm(t,r) + \frac{1}{8}\,\partial^2 F_1^\pm(t,r)y_2^\pm(t,r)^2 + \frac{1}{4}\,\partial^3 F_1^\pm(t,r)y_1^\pm(t,r)^2 \\
&\quad y_2^\pm(t,r) + \frac{1}{24}\,\partial^4 F_1^\pm(t,r)y_1^\pm(t,r)^4 + \frac{1}{24}\,\partial F_1^\pm(t,r)y_4^\pm(t,r)\Big)\,dt, \\
f_6^\pm(r) &= \int_0^T \Big(F_6^\pm(t,r) + \partial F_5^\pm(t,r)y_1^\pm(t,r) + \frac{1}{2}\,\partial F_4^\pm(t,r)y_2^\pm(t,r) + \frac{1}{2}\,\partial^2 F_4^\pm(t,r)y_1^\pm(t,r)^2 \\
&\quad + \frac{1}{6}\,\partial F_3^\pm(t,r)y_3^\pm(t,r) + \frac{1}{2}\,\partial^2 F_3^\pm(t,r)y_1^\pm(t,r)y_2^\pm(t,r) + \frac{1}{6}\,\partial^3 F_3^\pm(t,r)y_1^\pm(t,r)^3 \\
&\quad + \frac{1}{24}\,\partial F_2^\pm(t,r)y_4^\pm(t,r) + \frac{1}{6}\,\partial^2 F_2^\pm(t,r)y_1^\pm(t,r)y_3^\pm(t,r) + \frac{1}{4}\,\partial^3 F_2^\pm(t,r)y_1^\pm(t,r)^2y_2^\pm(t,r) \\
&\quad + \frac{1}{8}\,\partial^2 F_2^\pm(t,r)y_2^\pm(t,r)^2 + \frac{1}{24}\,\partial^4 F_2^\pm(t,r)y_1^\pm(t,r)^4 + \frac{1}{120}\,\partial F_1^\pm(t,r)y_5^\pm(t,r)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{24}\,\partial^2F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_4^{\pm}(t,r)+\frac{1}{12}\,\partial^2F_1^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r)\\
&+\frac{1}{12}\,\partial^3F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2y_3^{\pm}(t,r)+\frac{1}{12}\,\partial^4F_2^{\pm}(t,r)y_1^{\pm}(t,r)^3y_2^{\pm}(t,r)\\
&+\frac{1}{8}\,\partial^3F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)^2+\frac{1}{120}\,\partial^5F_1^{\pm}(t,r)y_1^{\pm}(t,r)^5\Big)\,dt,
\end{aligned}
$$

$$
\begin{aligned}
f_7^{\pm}(r)=\int_0^T\Big(&F_7^{\pm}(t,r)+\partial F_6^{\pm}(t,r)y_1^{\pm}(t,r)+\frac{1}{2}\,\partial F_5^{\pm}(t,r)y_2^{\pm}(t,r)+\frac{1}{2}\,\partial^2F_5^{\pm}(t,r)y_1^{\pm}(t,r)^2\\
&+\frac{1}{6}\,\partial F_4^{\pm}(t,r)y_3^{\pm}(t,r)+\frac{1}{2}\,\partial^2F_4^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)+\frac{1}{6}\,\partial^3F_4^{\pm}(t,r)y_1^{\pm}(t,r)^3\\
&+\frac{1}{24}\,\partial F_3^{\pm}(t,r)y_4^{\pm}(t,r)+\frac{1}{6}\,\partial^2F_3^{\pm}(t,r)y_1^{\pm}(t,r)y_3^{\pm}(t,r)\\
&+\frac{1}{4}\,\partial^3F_3^{\pm}(t,r)y_1^{\pm}(t,r)^2y_2^{\pm}(t,r)+\frac{1}{24}\,\partial^4F_3^{\pm}(t,r)y_1^{\pm}(t,r)^4+\frac{1}{8}\,\partial^2F_3^{\pm}(t,r)y_2^{\pm}(t,r)^2\\
&+\frac{1}{120}\,\partial F_2^{\pm}(t,r)y_5^{\pm}(t,r)+\frac{1}{24}\,\partial^2F_2^{\pm}(t,r)y_1^{\pm}(t,r)y_4^{\pm}(t,r)\\
&+\frac{1}{12}\,\partial^3F_2^{\pm}(t,r)y_1^{\pm}(t,r)^2y_3^{\pm}(t,r)+\frac{1}{8}\,\partial^3F_2^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)^2\\
&+\frac{1}{12}\,\partial^4F_2^{\pm}(t,r)y_1^{\pm}(t,r)^3y_2^{\pm}(t,r)+\frac{1}{120}\,\partial^5F_2^{\pm}(t,r)y_1^{\pm}(t,r)^5\\
&+\frac{1}{12}\,\partial^2F_2^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r)+\frac{1}{720}\,\partial F_1^{\pm}(t,r)y_6^{\pm}(t,r)\\
&+\frac{1}{120}\,\partial^2F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_5^{\pm}(t,r)+\frac{1}{48}\,\partial^3F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2y_4^{\pm}(t,r)\\
&+\frac{1}{48}\,\partial^2F_1^{\pm}(t,r)y_2^{\pm}(t,r)^2y_4^{\pm}(t,r)+\frac{1}{36}\,\partial^4F_1^{\pm}(t,r)y_1^{\pm}(t,r)^3y_3^{\pm}(t,r)\\
&+\frac{1}{72}\,\partial^2F_1^{\pm}(t,r)y_3^{\pm}(t,r)^2+\frac{1}{48}\,\partial^5F_1^{\pm}(t,r)y_1^{\pm}(t,r)^4y_2^{\pm}(t,r)\\
&+\frac{1}{16}\,\partial^4F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2y_2^{\pm}(t,r)^2+\frac{1}{48}\,\partial^3F_1^{\pm}(t,r)y_2^{\pm}(t,r)^3\\
&+\frac{1}{12}\,\partial^3F_1^{\pm}(t,r)y_1^{\pm}(t,r)y_2^{\pm}(t,r)y_3^{\pm}(t,r)+\frac{1}{720}\,\partial^6F_1^{\pm}(t,r)\Big)\,dt.
\end{aligned}
$$

Now, we call each function of the form

$$f_j(r)=f_j^+(r)-f_j^-(r)$$

an averaged function of order j. The averaging theory for studying the periodic solutions of discontinuous piecewise differential systems works as follows. Suppose that the average functions $f_j(r)=0$ for $j=1\cdots k$ and $f_k(r)\neq 0$ for some $k\in\mathbb{N}$. If r^* is a simple zero of $f_k(r)$, then there is a limit cycle denoted by $r(\theta,\epsilon)$ of system (2.1) such that $r(0,r^*)=r^*+O(\epsilon)$ (for more details see [9]).

In order to prove our results concerning the number of zeros of a real polynomial, we need to state the Descartes Theorem.

Theorem 2.1 (Descartes theorem)**.** *Consider the real polynomial $r(x)=a_{i_1}x^{i_1}+a_{i_2}x^{i_2}+\cdots+a_{i_r}x^{i_r}$ with $0=i_1<i_2<\cdots<i_r$ and $a_{i_j}\neq 0$ real constant for $j\in\{1,\ldots,r\}$. When $a_{i_j}a_{i_{j+1}<0}$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of the sign. If the number of variations of signs is m, then $r(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $r(x)$ in such a way that $r(x)$ has exactly $r-1$ positive real roots.*

For more details, see [3].

3 Proof of Theorem 1.1

Now, we provide the maximum number of limit cycles that can bifurcate from the discontinuous piecewise differential systems separated by the straight line $y=0$ and formed by two differential systems. In the half-plane $y \geq 0$, the linear differential center has a weak focus of form (1.1) and in the other half-plane $y \leq 0$, the cubic differential system has a uniform isochronous center (1.2).

In order to apply the averaging theory of order seven, we have to perturb two differential systems (1.1) and (1.2), where we developed the parameters of the differential systems until order seven in ϵ. Concerning the development of B, we must add -1 to guarantee that the origin of system (1.1) is a center. Then

$$a = \sum_{i=1}^{7} a_i\epsilon^i, \quad b = \sum_{i=1}^{7} b_i\epsilon^i, \quad c = \sum_{i=1}^{7} c_i\epsilon^i, \quad d = \sum_{i=1}^{7} d_i\epsilon^i,$$

and

$$A = \sum_{i=1}^{7} A_i\epsilon^i, \quad B = -1 + \sum_{i=1}^{7} B_i\epsilon^i, \quad C = \sum_{i=1}^{7} C_i\epsilon^i, \quad D = \sum_{i=1}^{7} D_i\epsilon^i.$$

First of all, to apply the averaging theory, we need to write the discontinuous piecewise differential system, systems (1.1) and (1.2), in the normal form (2.1). Hence we need to perform the polar change of coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ to transform the differential systems $(\dot{x}, \dot{y})$ onto the normal form (2.1). Then we take θ as the new independent variable to move from the differential system $(\dot{r}, \dot{\theta})$ onto the differential equation $dr/d\theta$, and carrying out a Taylor expansion truncated at seventh order in ϵ, we obtain the differential equation (2.1).

Now, we only give the functions $F_i^{\pm}(r,\theta)$ for $i = 1, \ldots, 4$ and omit the ones with $i = 5, 6, 7$ because of their large expressions,

$$\begin{aligned}
F_1^+(r,\theta) &= C_1\cos(\theta) + D_1\sin(\theta) + A_1 r,\\
F_1^-(r,\theta) &= r^2\Big(\cos(\theta)(a_1 + d_1 r\sin(\theta)) + b_1\sin(\theta) + c_1 r\cos(2\theta)\Big),\\
F_2^+(r,\theta) &= \frac{1}{2r}\Big(C_1^2\sin(2\theta) + 2A_2r^2 + 2B_1C_1r\cos(\theta) + 2C_2r\cos(\theta) + 2D_2r\sin(\theta)\Big),\\
F_2^-(r,\theta) &= \frac{1}{2}\Big(2a_2r^2\cos(\theta) + 2b_2r^2\sin(\theta) + 2c_2r^3\cos(2\theta) + d_2r^3\sin(2\theta)\Big),\\
F_3^+(r,\theta) &= C_3\cos(\theta) + D_3\sin(\theta)\\
&\quad + \frac{1}{r^2}\Big(C_1\cos(\theta)\big(C_1^2\sin^2(\theta) + r^2(B_1^2 + B_2) + r\sin(\theta)(2B_1C_1 + C_2)\big)\Big)\\
&\quad + A_3r + \frac{1}{r}\big(C_2\cos(\theta)(C_1\sin(\theta) + B_1r)\big),\\
F_3^-(r,\theta) &= r^2\Big(\cos(\theta)(a_3 + d_3r\sin(\theta)) + b_3\sin(\theta) + c_3r\cos(2\theta)\Big),\\
F_4^+(r,\theta) &= \frac{1}{8r^3}\Big(2C_1^4\sin(2\theta) - C_1^4\sin(4\theta) + 8A_4r^4 + 8B_1^3C_1r^3\cos(\theta) + 8B_1^2C_2r^3\cos(\theta)\\
&\quad + 16B_1B_2C_1r^3\cos(\theta) + 8B_1C_3r^3\cos(\theta) + 8B_2C_2r^3\cos(\theta) + 8B_3C_1r^3\cos(\theta)\\
&\quad + 8C_4r^3\cos(\theta) + 8D_4r^3\sin(\theta) + 12B_1^2C_1^2r^2\sin(2\theta) + 16B_1C_1C_2r^2\sin(2\theta)\\
&\quad + 8B_2C_1^2r^2\sin(2\theta) + 8C_1C_3r^2\sin(2\theta) + 4C_2^2r^2\sin(2\theta) + 6B_1C_1^3r\cos(\theta)\\
&\quad - 6B_1C_1^3r\cos(3\theta) + 6C_1^2C_2r\cos(\theta) - 6C_1^2C_2r\cos(3\theta)\Big),\\
F_4^-(r,\theta) &= \frac{1}{2}\Big(2a_4r^2\cos(\theta) + 2b_4r^2\sin(\theta) + 2c_4r^3\cos(2\theta) + d_4r^3\sin(2\theta)\Big).
\end{aligned}$$

From Section 2, for $i = 1$, the averaging function of order one is

$$f_1(r) = -2b_1r^2 + 2D_1 + \pi A_1 r.$$

Since $f_1(r) = 0$ is a quadratic equation, we find that $f_1(r)$ may have at most two positive real solutions r_1 and r_2 which means that at most two small limit cycles of radii r_1 and r_2, respectively, can bifurcate from the discontinuous piecewise differential system $(1.1), (1.2)$ for ϵ sufficiently small.

Now, in order to apply the averaging theory of second order to the discontinuous piecewise differential system $(1.1), (1.2)$, we have to eliminate the coefficient of $f_1(r)$ by choosing $b_1 = D_1 = A_1 = 0$

to get $f_1(r) \equiv 0$. Calculating the averaging function of the second order, we get

$$f_2(r) = \frac{2}{3}\, a_1 c_1 r^4 - 2b_2 r^2 + \pi A_2 r + 2D_2.$$

The function $f_2(r) = 0$ is a quartic equation, which may have at most three positive real solutions by Descartes Theorem 2.1. Then till now, the maximum number of limit cycles for the discontinuous piecewise differential system (1.1), (1.2) is at most three.

In the same way, to apply the averaging theory of the third order, we take $b_2 = A_2 = D_2 = 0$ to eliminate the coefficients of r^2, r^1, and r^0. So, to get $f_2(r) \equiv 0$, we have two different cases $a_1 = 0$ and $c_1 \neq 0$, or $c_1 = 0$ and $a_1 \neq 0$. We start with the first case.

Case A: $a_1 = 0$ ***and*** $c_1 \neq 0$**.** In this case, the resulting averaging function of the third order is

$$f_3(r) = -\frac{1}{8}\, 3\pi(c_2 d_1 - c_1 d_2) r^5 - \frac{4}{3}\, a_2 c_1 r^4 - 2b_3 r^2 + \pi A_3 r + 2D_3.$$

Then, by Descartes Theorem, $f_3(r) = 0$ has at most four positive real solutions. Consequently, at most four small limit cycles can bifurcate from the discontinuous piecewise differential system (1.1), (1.2). Now, taking $d_2 = \frac{c_2 d_1}{c_1}$, $a_2 = 0$, $b_3 = 0$, $A_3 = 0$ and $D_3 = 0$, we obtain $f_3(r) \equiv 0$. So, we can compute the averaging theory of the fourth order which has the expression

$$f_4(r) = \frac{2}{3}\, a_3 c_1 r^4 - 2b_4 r^2 + \pi A_4 r + 2D_4.$$

The averaging function of the fourth order $f_4(r)$ may have at most 3 positive real roots. By considering $a_3 = b_4 = A_4 = D_4 = 0$ and computing the averaging function of fifth order, we have

$$f_5(r) = (c_3 d_1 - c_1 d_3)\Big(\frac{9}{32}\, \pi d_1 r^7 + \frac{(3\pi c_2) r^5}{8c_1}\Big) + \frac{2}{3}\, a_4 c_1 r^4 - 2b_5 r^2 + \pi A_5 r + 2D_5.$$

The equation $f_5(r) = 0$ is a polynomial of degree 7 which may have at most five positive real solutions according to Descartes Theorem. Then the maximum number of limit cycles for the discontinuous piecewise differential system (1.1), (1.2) is at most five. To apply the averaging theory of order six, we need to choose $a_4 = b_5 = A_5 = D_5 = 0$ to eliminate the coefficients of r^4, r^2, r_1 and the constant coefficient. So, to get $f_5(r) \equiv 0$, we distinguish two different subcases $c_2 = d_1 = 0$ or $d_3 = \frac{c_3 d_1}{c_1}$

Subcase A.1: $c_2 = d_1 = 0$**.** Here, the resulting averaging function of order six is

$$f_6(r) = \frac{9}{128}\, \pi c_1^3 d_3 r^9 + \frac{2}{3}\, a_5 c_1 r^4 - 2b_6 r^2 + \pi A_6 r + 2D_6.$$

Clearly, $f_5(r) = 0$ may have at most four positive real solutions. Setting $b_6 = a_5 = d_3 = A_6 = D_6 = 0$, we get $f_5(r) \equiv 0$. Then we can apply the seventh order averaging theory, which gives

$$f_7(r) = -\frac{1}{640}\, 27\pi c_1^3 d_4 r^9 \frac{2}{3}\, a_6 c_1 r^4 - 2b_7 r^2 + \pi A_7 r + 2D_7.$$

This polynomial may have at most four positive real roots.

Now, we start the computations from $f_3(r)$ just before Subcase A.1 taking the second subcase $a_2 = 0$.

Subcase A.2: $d_3 = \frac{c_3 d_1}{c_1}$ **.** In this subcase, the resulting averaging function of order six is

$$f_6(r) = (c_4 d_1 - c_1 d_4)\Big(\frac{9}{32}\, \pi d_1 r^7 + \frac{3\pi c_2 r^5}{8c_1}\Big) + \frac{2}{3}\, a_5 c_1 r^4 - 2b_6 r^2 + \pi A_6 r + 2D_6.$$

The equation $f_6(r) = 0$ is a polynomial of degree 7, which may have at most five real positive roots by the Descartes Theorem. To apply the averaging theory of order seven, we need to choose $a_5 = b_6 = A_6 = D_6 = 0$ to eliminate the coefficients of r^4, r^2, r and the constant coefficient. So, to get $f_6(r) \equiv 0$, we have two different subcases $c_2 = d_1 = 0$ or $d_4 = \frac{c_4 d_1}{c_1}$. Starting with the first subcase.

Subcase A.2.1: $c_2 = 0$ ***and*** $d_1 = 0$. Under these conditions, the resulting averaging function of order 7 is

$$f_7(r) = -\frac{1}{640}\, 27\pi c_1^3 d_4 r^9 + \frac{2}{3}\, a_6 c_1 r^4 - 2b_7 r^2 + \pi A_7 r + 2D_7.$$

This polynomial may have at most four positive real roots. Now, we go to the second Subcase A.2.2 continuing the computation of the averaging function of order seven.

Subcase A.2.2: $d_4 = \frac{c_4 d_1}{c_1}$ **.** The next averaging function is

$$f_7(r) = \frac{9}{32}\, \pi d_1(c_5 d_1 - c_1 d_5) r^7 + \frac{3\pi c_2(c_5 d_1 - c_1 d_5)}{8c_1}\, r^5 + \frac{2}{3}\, a_6 c_1 r^4 - 2b_7 r^2 + \pi A_7 r + 2D_7.$$

This polynomial may have at most five positive roots. Now, we go to the second case $c_1 = 0$ and $a_1 \neq 0$, but we continue the computations just after $f_2(r)$, i.e., we continue from $f_3(r)$.

Case B: $c_1 = 0$ ***and*** $a_1 \neq 0$. The resulting averaging function of the third order is

$$f_3(r) = -\frac{1}{8}\, 3\pi c_2 d_1 r^5 + 2a_1 c_2 r^4 - 2b_3 r^2 + \pi A_3 r + 2D_3.$$

Then, by Descartes Theorem, $f_3(r) = 0$ has at most four positive real solutions. Taking $c_2 = b_3 = A_3 = D_3 = 0$, we get $f_3(r) = 0$. Consequently, the averaging function of the fourth order is of the form

$$f_4(r) = \frac{2}{3}\, a_1 c_3 r^4 - 2b_4 r^2 + \pi A_4 r + 2D_4.$$

It is clear that since $f_4(r)$ is a polynomial, by Descartes Theorem, $f_4(r) = 0$ may have at most three positive solutions. We consider $c_3 = b_4 = A_4 = D_4 = 0$ and by computing the averaging function of the fifth order, we get

$$f_5(r) = \frac{2}{3}\, a_1 c_4 r^4 - 2b_5 r^2 + \pi A_5 r + 2D_5.$$

The polynomial $f_5(r)$ may have at most three positive real roots. Then by considering $c_4 = b_5 = A_5 = D_5 = 0$ and computing the averaging function of order six, we have

$$f_6(r) = \frac{2}{3}\, a_1 c_5 r^4 - 2b_6 r^2 + \pi A_6 r + 2D_6.$$

The averaging function $f_6(r)$ may have at most three positive real roots. If we take $c_5 = b_6 = A_6 = D_6 = 0$, the averaging function of order six is

$$f_7(r) = \frac{2}{3}\, a_1 c_6 r^4 - 2b_7 r^2 + 2D_7 + \pi A_7 r.$$

In this case, $f_5(r) = 0$ is a quartic polynomial with at most three positive solutions.

In short, in the previous cases we have found that the averaging functions $f_i(r)$ for $i = 1, \ldots, 7$, may have at most 2, 3, 4 and 5 real positive roots. Consequently, using the averaging theory up to the seventh order, the maximum number of limit cycles is at most five limit cycles. This completes the proof of Theorem 1.1.

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