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EXISTENCE OF SOLUTIONS FOR A FOURTH-ORDER DISCRETE PROBLEM WITH $(p_1(k), p_2(k))$-LAPLACIAN OPERATOR

Abstract. The purpose of this paper is to study the existence of solutions for a non-linear fourth-order discrete problem involving the operator $(p_1(k), p_2(k))$-Laplacian under appropriate assumptions on the nonlinearity and the parameter λ, when the approach is based on the variational methods and critical point theory.

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1 Introduction

Let $T \geq 2$ be a positive integer, $[a, b]_{\mathbb{Z}} = \{a; a+1; \dots; b\}$ be the discrete interval, where a and b are integers with $a < b$, and let λ be a positive parameter. The main goal of this paper is to establish the existence of solutions for the following discrete boundary value problem:

$$
(\mathrm{P}): \quad \begin{cases} \Delta^2\Big(\displaystyle\sum_{i=1}^{2} \alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big) = \lambda f(k, u(k)), & k \in [1, T]_{\mathbb{Z}}, \\ u(-1) = u(0) = u(T+1) = u(T+2) = 0, \end{cases}
$$

where $f : [1, T+2]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha_i : [-1, T+2]_{\mathbb{Z}} \to [1, \infty)$, $p_i : [-1, T+2]_{\mathbb{Z}} \to [2, \infty)$ for $i = 1, 2$ are the given functions, $\Delta u(k) = u(k+1) - u(k)$ for all $k \in [-1, T+2]_{\mathbb{Z}}$ is the forward difference operator and $\phi_{p(k)}$ is called the $p(k)$-Laplacian operator defined as $\phi_{p(k)}(s) = |s|^{p(k)-2}s$, $s \in \mathbb{R}$.

From the definition of the forward difference operator, it is clear that

$$
\Delta^2 u(k) = \Delta(\Delta u(k)) = u(k+2) - 2u(k+1) + u(k) \ \text{ for all } \ k \in [1, T]_{\mathbb{Z}},
$$

moreover, $u(-1) = u(0) = u(T+1) = u(T+2) = 0$ implies that $\Delta u(-1) = \Delta u(T+1) = 0$.

We say that a function $u : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R}$ is a solution of problem (P) if it satisfies both equations of (P).

For convenience, denote

$$
\begin{aligned}
p_i^+ &:= \max_{k \in [-1, T+2]_{\mathbb{Z}}} p_i(k), & p_i^- &:= \min_{k \in [-1, T+2]_{\mathbb{Z}}} p_i(k), \\
p^+ &:= \max\{p_1^+, p_2^+\}, & p^- &:= \min\{p_1^-, p_2^-\}, \\
\alpha_i^+ &:= \max_{k \in [-1, T+2]_{\mathbb{Z}}} \alpha_i(k), & \alpha^+ &:= \max\{\alpha_1^+, \alpha_2^+\}.
\end{aligned}
$$

The theory of nonlinear difference equations has been intensively used to study the discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In recent years, a great deal of work has been done in the study of the existence of solutions for discrete boundary value problems. For the background and recent results, we refer the reader to [1–11, 14, 15] and the references therein. It is well known that the critical point theory is a powerful tool to investigate the problems for differential equations.

However, to the best of our knowledge, research concerning the discrete anisotropic problems like (P) involving variable exponents has been initiated by Kone and Ouaro in [14] and by Mihăilescu, Rădulescu and Tersian in [15], where the critical point theory and more known tools are applied to get the existence and multiplicity of solutions. Further tools and ideas to study anisotropic discrete nonlinear problems one can be found in [3] and [6].

We can consider problem (P) as the discrete counterpart of the following functional differential equation:

$$
\begin{cases} \dfrac{d^2}{dt^2}\Big(\alpha_1(t)\Big|\dfrac{d^2u(t)}{dt^2}\Big|^{p_1-2}\Big(\dfrac{d^2u(t)}{dt^2}\Big)\Big) + \dfrac{d^2}{dt^2}\Big(\alpha_2(t)\Big|\dfrac{d^2u(t)}{dt^2}\Big|^{p_2-2}\Big(\dfrac{d^2u(t)}{dt^2}\Big)\Big) = f(t, u(t)), & t \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}
$$

A particular equation of the above equation is

$$
\begin{cases} u^{(4)}(t) = f(t, u(t)), & t \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}
$$

which is used to model deformations of elastic beams and image processing [7, 11]. Also, for the continuous counterpart of the fourth order discrete problems, one can see [16].

In this paper, we are inspired by the results in [10] where the authors study the existence and multiplicity of a Dirichlet boundary value problem by means of the critical point theorems with variable exponent, in fact, we are trying to prove some of the results with different boundary conditions, of course, with necessary modifications.

The rest of this paper is structured as follows. In Section 2, we introduce some basic properties and provide several auxiliary inequalities useful for our approach. After variational framework, in Section 3, we state and prove our main results.

2 Preliminaries

In this section, we recall some notations, definitions and properties.

Let E be a real finite-dimensional space and $\Phi, G : E \to \mathbb{R}$ be two continuously Gâteaux differentiable mappings with derivatives $\varphi, g : E \to E^*$. We consider the following equation:

$$\varphi(u) = g(u), \quad u \in E. \tag{2.1}$$

We give the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \Phi(u) - G(u).$$

Proposition 2.1 (see [9])**.** *Assume that G and Φ are convex on E and $W \subset E$ contains at least two points. Let there exist $u_0 \in E$ and $v \in W$ satisfying $\varphi(v) = g(u_0)$ such that $J(u_0) \leq \inf_{x \in W} J(x)$. Then u_0 is a critical point to J, so u_0 is a solution of (2.1). Moreover, if J is anti-coercive, then (2.1) has another solution, different from u_0.*

In the present paper, solutions to (P) will be investigated in the Banach space

$$E := \Big\{ u : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R} : \quad u(0) = u(-1) = u(T+1) = u(T+2) = 0 \Big\},$$

and we define the norm

$$\|u\|_- := \Big(\sum_{i=1}^{2} \sum_{k=1}^{T+2} \alpha_i(k-2) \big|\Delta^2 u(k-2)\big|^{p^-} \Big)^{1/p^-}.$$

Let us also introduce other equivalent norms, namely,

$$\begin{aligned}
\|u\|_+ &= \Big(\sum_{i=1}^{2} \sum_{k=1}^{T+2} \alpha(k-2) \big|\Delta^2 u(k-2)\big|^{p^+} \Big)^{1/p^+}, \\
\|u\|_\infty &= \max_{k \in [1,T]_{\mathbb{Z}}} |u(k)|, \\
|u|_p &= \Big(\sum_{k=1}^{T+2} |u(k)|^p \Big)^{1/p} \quad \text{for all } \; p > 1
\end{aligned}$$

and the Luxemburg norm defined by

$$\|u\|_{p(\,\cdot\,)} = \inf \Big\{ \mu > 0 : \quad \sum_{i=1}^{2} \sum_{k=1}^{T+2} \alpha_i(k-2) \Big| \frac{\Delta^2 u(k-2)}{\mu} \Big|^{p_i(k-2)} \leq 1 \Big\}.$$

Moreover, we note that there exists a constant $M > 0$ satisfying

$$\|u\|_- \leq M \|u\|_{p(\,\cdot\,)}. \tag{2.2}$$

In the next lemma we present some auxiliary inequalities that we will use later.

Lemma 2.1. *For all $u \in E$, we have*

$$\text{(A.1)} \qquad \sum_{k=1}^{T+2} \left|\Delta^2 u(k-2)\right|^p \leq 3^{p-1}(2^p+2)\sum_{k=1}^{T+2}|u(k)|^p \ \ \textit{for any} \ \ p>1,$$

$$\text{(A.2)} \qquad \|u\|_- \leq (T+2)^{\frac{1}{p^-}-\frac{1}{p^+}}(2\alpha^+ C_p)^{\frac{1}{p^-}}|u|_{p^+} \ \ \textit{with} \ \ C_p = 3^{p^- -1}(2^p+2),$$

$$\text{(A.3)} \qquad \|u\|_+ \leq \big((T+2)\alpha_+\big)^{\frac{p^+-p^-}{p^+p^-}}\|u\|_-,$$

$$\text{(A.4)} \qquad \|u\|_\infty \leq \frac{1}{8}C^{\frac{2p^- -1}{p^-}}\|u\|_- \ \ \textit{with} \ \ C = (T+2)\alpha_+.$$

Proof. To obtain relation $(A.1)$, we use a similar argument as in [12, 13].

Note that the function $x \mapsto |x|^p$ is convex on $\mathbb{R}$ for any $p>1$, then, by Jensen's inequality, we have

$$\left|a_1+a_2+a_3\right|^p \leq 3^{p-1}\big(|a_1|^p+|a_2|^p+|a_3|^p\big) \ \text{ for any } \ p>1,$$

where a_1, a_2 and a_3 are the real numbers. Thus

$$\begin{aligned}\sum_{i=1}^{T+2}\left|\Delta^2 u(i-2)\right|^p &= \sum_{i=1}^{T+2}\left|u(i)-2u(i-1)+u(i-2)\right|^p \\ &\leq 3^{p-1}\Big(\sum_{i=1}^{T+2}|u(i)|^p + 2^p\sum_{i=1}^{T+2}|u(i-1)|^p + \sum_{i=1}^{T+2}|u(i-2)|^p\Big) \leq 3^{p-1}(2^p+2)\sum_{i=1}^{T+2}|u(i)|^p.\end{aligned}$$

So, (A.1) holds.

By (A.1), we get (A.2) as follows:

$$\|u\|_-^{p^-} \leq 2\alpha^+\sum_{i=1}^{T+2}\left(|\Delta^2 u(i-2)|^{p^-}\right) \leq \left(2\alpha^+ 3^{p^- -1}(2^{p^-}+2)\right)\sum_{i=1}^{T+2}|u(i)|^{p^-}. \tag{2.3}$$

By the Hölder inequality, we get

$$\sum_{i=1}^{T+2}|u(i)|^{p^-} \leq (T+2)^{1-\frac{p^-}{p^+}}\Big(\sum_{i=1}^{T+2}|u(i)|^{p^+}\Big)^{\frac{p^-}{p^+}}. \tag{2.4}$$

We combine (2.3) and (2.4) and obtain (A.2).

By similar arguments as in [11], we obtain relation (A.3). In fact, by the Hölder inequality, we observe that

$$\|u\|_+^{p^+} \leq C^{\frac{p^- - p^+}{p^-}}\Big(\sum_{i=1}^{2}\sum_{k=1}^{T+2}\alpha_i(k-2)\left|\Delta^2 u(k-2)\right|^{p^-}\Big)^{\frac{p^+}{p^-}} \leq C^{\frac{p_- - p^+}{p_- p^+}}\|u\|_-^{p^+}.$$

To prove (A.4), for all $u \in E$ and $k \in [-1, T+2]_{\mathbb{Z}}$, we have

$$|u(k)| = \Big|\sum_{i=1}^{k}\Delta u(i-1)\Big| \leq \sum_{i=1}^{k}|\Delta u(i-1)|$$

and

$$|u(k)| = \Big|\sum_{i=k+1}^{T+2}\Delta u(i-1)\Big| \leq \sum_{i=k+1}^{T+2}|\Delta u(i-1)|.$$

Combining the above inequalities and adding the left- and right-hand sides, we can see that

$$2|u(k)| \leq \sum_{i=1}^{T+2} |\Delta u(i-1)|.$$

Since $u(-1)=0$, for any $k \in [-1, T+2]$, we get

$$|u(k)| \leq \frac{1}{2} \sum_{i=1}^{T+2} |\Delta u(i-1)|. \tag{2.5}$$

Arguing similarly, for any $k \in [1, T+2]$, we obtain

$$|\Delta u(k-1)| \leq \frac{1}{2} \sum_{i=1}^{T+2} \left|\Delta^2 u(i-2)\right|. \tag{2.6}$$

Note that for $i = 1, 2$ we have $\alpha_i(k) \geq 1$ for any $k \in [-1, T+2]_{\mathbb{Z}}$, then, by (2.5) and (2.6), we get

$$|u(k)| \leq \frac{1}{4} \sum_{i=1}^{2} \sum_{j=1}^{T+2} \alpha_i(j-2)|\Delta u(j-1)| \text{ for any } k \in [-1, T+2]_{\mathbb{Z}}$$

and

$$|\Delta u(k-1)| \leq \frac{1}{2} \sum_{i=1}^{T+2} \alpha(i-2)\left|\Delta^2 u(i-2)\right| \text{ for any } k \in [1, T+2]_{\mathbb{Z}}.$$

So, for any $k \in [-1, T+2]_{\mathbb{Z}}$, by the Hölder inequality, we get

$$\begin{aligned}
|u(k)| &\leq \frac{1}{2} \max_{k\in[1,T+2]} |\Delta u(k-1)| \sum_{j=1}^{T+2} \alpha_i(j-2) \\
&\leq \frac{1}{4}(T+2)\alpha_+ \sum_{j=1}^{T+2} \alpha_i(j-2)\left|\Delta^2 u(j-2)\right| \leq \frac{1}{8}(T+2)\alpha_+ \sum_{i=1}^{2}\sum_{j=1}^{T+2} \alpha_i(j-2)\left|\Delta^2 u(j-2)\right| \\
&\leq \frac{1}{8}\left((T+2)\alpha_+\right)^{\frac{2p^- -1}{p^-}} \Big(\sum_{i=1}^{2}\sum_{j=1}^{T+2} \alpha_i(j-2)\left|\Delta^2 u(j-2)\right|^{p^-}\Big)^{\frac{1}{p^-}} = \frac{1}{8} C^{\frac{2p^- -1}{p^-}} \|u\|_-.
\end{aligned}$$

Thus the proof of Lemma 2.1 is complete. □

Let $\psi : E \to \mathbb{R}$ be the functional given by the formula

$$\psi(u) := \sum_{i=1}^{2} \sum_{k=1}^{T+2} \alpha_i(k-2)\left|\Delta^2 u(k-2)\right|^{p_i(k-2)}.$$

Then we have the following inequalities (see [4]):

$$\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|_{p(\cdot)}^{p^-} \leq \psi(u) \leq \|u\|_{p(\cdot)}^{p^+}. \tag{2.7}$$

Lemma 2.2. *For all $u \in E$, we have*

$$\psi(u) \leq C^{\frac{p^+ - p^-}{p^-}} \|u\|_-^{p^+} + 2\alpha^+(T+2). \tag{2.8}$$

Proof. By a similar argument as in [13], for $i = 1, 2$ and for any $u \in E$, we have

$$\begin{aligned}
&\sum_{k=1}^{T+2} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p_i(k-2)} \\
&\quad \leq \sum_{k\in[1,T+2]_{\mathbb{Z}}:\ \{|\Delta^2 u(k-2)|<1\}} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p^-} \\
&\qquad + \sum_{k\in[1,T+2]_{\mathbb{Z}}:\ \{|\Delta^2 u(k-2)|\geq 1\}} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p^+} \\
&\quad = \sum_{k=1}^{T+2} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p^+} \\
&\qquad + \sum_{k\in[1,T+2]_{\mathbb{Z}}:\ |\Delta^2 u(k-2)|<1\}} \alpha_i(k-2)\Big(\big|\Delta^2 u(k-2)\big|^{p^-} - \big|\Delta^2 u(k-2)\big|^{p^+}\Big) \\
&\quad \leq \sum_{k=1}^{T+2} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p^+} + \alpha^+(T+2).
\end{aligned}$$

So,

$$\sum_{i=1}^{2}\sum_{k=1}^{T+2} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p_i(k-2)} \leq \sum_{i=1}^{2}\sum_{k=1}^{T+2} \alpha_i(k-2)\big|\Delta^2 u(k-2)\big|^{p^+} + 2\alpha^+(T+2).$$

Therefore, in view of (A.4), we deduce inequality (2.8). □

3 Variational framework

Let $u \in E$. We put

$$\Phi(u) := \sum_{i=1}^{2}\sum_{k=1}^{T+2} \Big(\frac{\alpha_i(k-2)}{p_i(k-2)}\,\big|\Delta^2 u(k-2)\big|^{p_i(k-2)}\Big) \tag{3.1}$$

and

$$G(u) := \sum_{k=1}^{T+2} F(k, u(k)), \tag{3.2}$$

where

$$F(k,r) := \int_0^r f(k,s)\,ds \ \text{ for all } \ (k,r) \in [1, T+2]_{\mathbb{Z}} \times \mathbb{R}.$$

Let $\lambda \in (0, +\infty)$ be a positive parameter and $J_\lambda : E \to \mathbb{R}$ be the functional defined by

$$J_\lambda(u) = \Phi(u) - \lambda G(u) = \sum_{i=1}^{2}\sum_{k=1}^{T+2} \Big(\frac{\alpha_i(k-2)}{p_i(k-2)}\,\big|\Delta^2 u(k-2)\big|^{p_i(k-2)}\Big) - \lambda \sum_{k=1}^{T+2} F(k, u(k)).$$

The derivative of G reads as

$$G'(u)(v) = \sum_{k=1}^{T+2} f(k, u(k))v(k)$$

for all $u, v \in E$. For the functional Φ, by considering the boundary values and summing by parts twice, we can observe that

$$\Phi'(u)(v) = \sum_{i=1}^{2}\sum_{k=1}^{T+2} \Delta^2\Big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big)v(k)$$

for all $u, v \in E$. In fact, it is easy to see that

$$\Phi'(u)(v) = \sum_{i=1}^{2} \sum_{k=1}^{T+2} \alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Delta^2 v(k-2).$$

Let $u, v \in E$, we take into account the boundary conditions, then for $i = 1, 2$ we have

$$\begin{aligned}
&\sum_{k=1}^{T+2} \alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Delta^2 v(k-2)\\
&\quad = \alpha_i(T)\phi_{p_i(T)}(\Delta^2 u(T))\Delta^2 v(T) + \sum_{k=1}^{T+1} \alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Delta^2 v(k-2)\\
&\quad = \alpha_i(T)\phi_{p_i(T)}(\Delta^2 u(T))\Delta^2 v(T) + \Big[\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Delta v(k-2)\Big]_1^{T+2}\\
&\qquad\qquad - \sum_{k=1}^{T+1} \Delta\big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\big)\Delta v(k-1)\\
&\quad = \sum_{k=1}^{T+1} \Delta\Big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big)\Delta v(k-1)\\
&\quad = -\Big[\Delta\Big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big)v(k-1)\Big]_1^{T+2}\\
&\qquad\qquad + \sum_{k=1}^{T+1} \Delta^2\Big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big)v(k)\\
&\quad = \sum_{k=1}^{T+2} \Delta^2\big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\big)v(k).
\end{aligned}$$

Hence, for all $u, v \in E$, we have

$$\Phi'(u)(v) = \sum_{i=1}^{2} \sum_{k=1}^{T+2} \Delta^2\Big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big)v(k).$$

It is obvious that Φ and G are of the class C^1 on E. Then J_λ is also of the class C^1 on E.

Lemma 3.1. *The function $\widetilde{u} \in E$ is a solution of problem* (P) *if and only if $\widetilde{u}$ is a critical point of J_λ in E.*

Proof. Let $\widetilde{u}$ be a critical point of J_λ in E. Then for all $v \in E$, $J'_\lambda(\widetilde{u})(v) = 0$ and $\Delta\widetilde{u}(-1) = \Delta\widetilde{u}(T+1) = \widetilde{u}(-1) = \widetilde{u}(T+2) = 0$. Thus, for every $v \in E$, taking twice summation by parts formula and also $v(-1) = v(0) = v(T+1) = v(T+2) = 0$, we have

$$0 = J'_\lambda(\widetilde{u})(v) = \sum_{i=1}^{2} \sum_{k=1}^{T+2} \Big(\Delta^2\big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2\widetilde{u}(k-2)\big)\big)\Big)v(k) - \lambda \sum_{k=1}^{T+2} f(k, \widetilde{u}(k))v(k).$$

Since $v \in E$ is arbitrary, we get

$$\sum_{i=1}^{2} \Delta^2\Big(\alpha_i(k-2)\big|\Delta^2\widetilde{u}(k-2)\big|^{p_i(k-2)-2}\Delta^2\widetilde{u}(k-2)\Big) = \lambda f(k, \widetilde{u}(k)) \tag{3.3}$$

for all $k \in [1, T]_{\mathbb{Z}}$. Therefore, $\widetilde{u}$ is a solution of (P). We deduce that any critical point of J_λ in E is a solution of problem (P). $\square$

Remark 3.1. From Lemma 3.1, we conclude that finding the solutions to problem (P) is equivalent to finding the critical points of the functional J_λ.

Lemma 3.2. *The functional Φ is coercive, i.e., $\Phi_\lambda(u) \to +\infty$ as $\|u\|_- \to +\infty$.*

Proof. Note that $\|u\|_- \to +\infty$ implies $\|u\|_{p(\,\cdot\,)} \to +\infty$, so, for $\|u\|_-$ large enough, by (2.2) and (2.7), we get

$$\Phi(u) = \sum_{i=1}^{2}\sum_{k=1}^{T+2}\left(\frac{\alpha_i(k-2)}{p_i(k-2)}\left|\Delta^2 u(k-2)\right|^{p_i(k-2)}\right) \geq \frac{\psi(u)}{p^+} \geq \frac{\|u\|_-^{p^-}}{p^+ M^{p^-}}.$$

We conclude that $\Phi(u) \to +\infty$ as $\|u\|_- \to +\infty$, so, the functional Φ is coercive. $\square$

Now, we state the following assumptions.

(H.1) $x \longmapsto F(k,x)$ is convex on $\mathbb{R}$ for all $k \in [1, T+2]_{\mathbb{Z}}$, and

$$F^\infty := \min_{k\in[1,T+2]_{\mathbb{Z}}} \limsup_{x\to+\infty} \frac{F(k,x)}{|x|^{p^+}} > 0.$$

(H.2)

$$F_0 := \max_{k\in[1,T+2]_{\mathbb{Z}}} \liminf_{x\to 0} \frac{F(k,x)}{|x|^{p^-+1}} < \infty.$$

We put the notations

$$\lambda^* := \frac{C^{\frac{p^- - p^+}{p^-}}}{p^- F^\infty (2\alpha^+ + C_p)^{-\frac{p^+}{p^-}} (T+2)^{1-\frac{p^+}{p^-}}}$$

and

$$\lambda^{**} = \frac{4C^{\frac{1-2p^-}{p^-}}}{(p^-+1)F_0 M^{p^-+2}}.$$

Lemma 3.3. *Suppose that* (H.1) *holds. Then for any $\lambda > \lambda^*$, the functional J_λ is anti-coercive, i.e., $J_\lambda(u) \to -\infty$ as $\|u\|_- \to +\infty$.*

Proof. Take $\lambda > \lambda^*$. Since $\limsup\limits_{x\to+\infty} \frac{F(k,x)}{|x|^{p^+}} \geq F^\infty$, there exists $\epsilon > 0$ such that

$$F(k,x) \geq F^\infty |x|^{p^+}$$

for all $k \in [1, T+2]_{\mathbb{Z}}$ and $x \in \mathbb{R}$ with $|x| > \epsilon$.

For $\|u\|_-$ large enough, by (A.3), we get

$$-\lambda G(u) \leq -\lambda F^\infty \sum_{k=1}^{T+2} |u(k)|^{p^+} \leq -\lambda F^\infty (2\alpha^+ C_p)^{-\frac{p^+}{p^-}} (T+2)^{1-\frac{p^+}{p^-}} \|u\|_-^{p^+} \tag{3.4}$$

and by (2.8), we have

$$\Phi(u) \leq \frac{\psi(u)}{p^-} \leq \frac{1}{p^-}\Big(C^{\frac{p^- - p^+}{p^-}} \|u\|_-^{p^+} + 2(T+2)\alpha^+\Big). \tag{3.5}$$

So, combining (3.4) and (3.5), it follows that

$$\begin{aligned}
J_\lambda(u) &= \Phi(u) - \lambda G(u) \\
&= \sum_{i=1}^{2}\sum_{k=1}^{T+2}\left(\frac{\alpha_i(k-2)}{p_i(k-2)}\left|\Delta^2 u(k-2)\right|^{p_i(k-2)}\right) - \lambda \sum_{k=1}^{T+2} F(k,u) \\
&\leq \frac{1}{p^-}\Big(C^{\frac{p^- - p^+}{p^-}} \|u\|_-^{p^+} + 2(T+2)\alpha^+\Big) - \lambda F^\infty (2\alpha^+ C_p)^{-\frac{p^+}{p^-}} (T+2)^{1-\frac{p^+}{p^-}} \|u\|_-^{p^+} \\
&= \frac{2(T+2)\alpha^+}{p^-} + \left(\frac{1}{p^-} C^{\frac{p^- - p^+}{p^-}} \|u\|_-^{p^+} - \lambda F^\infty (2\alpha^+ C_p)^{-\frac{p^+}{p^-}} (T+2)^{1-\frac{p^+}{p^-}}\right) \|u\|^{p^+} \\
&= \frac{2(T+2)\alpha^+}{p^-} + F^\infty (2\alpha^+ C_p)^{-\frac{p^+}{p^-}} (T+2)^{1-\frac{p^+}{p^-}} (\lambda^* - \lambda) \|u\|_-^{p^+}.
\end{aligned}$$

We conclude that $J_\lambda(u) \to -\infty$ as $\|u\|_- \to +\infty$ because $F^\infty > 0$ and $\lambda > \lambda^*$. $\square$

4 Main results

Our main results are the following.

Theorem 4.1. *Assume that assumption* (H.2) *holds. Then, for any* $\lambda \in (0, \lambda^{**})$ *problem* (P) *has at least one nontrivial solution.*

Proof. From (H.2), one can conclude that for all $|x| \leq M$ and all $k \in [1, T+2]_{\mathbb{Z}}$,

$$f(k,x) \leq (p^- + 1)F_0|x|^{p^-}, \tag{4.1}$$

where $M > 0$ is defined in (2.2). Let us take the set $W \subset E$ defined by

$$W = \{x \in E : \ \|x\|_{p(\cdot)} \leq M\}.$$

In order to apply Proposition 2.1, we start by recalling that for any $\lambda \in (0, \lambda^{**})$ is fixed. The functional J_λ is continuous and the subset W is closed and bounded, therefore there exists a minimum of J_λ over W, which we denote by u_0, moreover,

$$\|u_0\|_{p(\cdot)} < M. \tag{4.2}$$

Now, on the space E, we consider the following boundary value problem connected to (P):

$$\begin{cases} \Delta^2\Big(\displaystyle\sum_{i=1}^{2} \alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 u(k-2)\big)\Big) = \lambda f(k, u_0(k)), \quad k \in [1,T]_{\mathbb{Z}}, \\ u(-1) = u(0) = u(T+1) = u(T+2) = 0. \end{cases} \tag{4.3}$$

Note that the functionals Φ and G are convex and of the class C^1 on E. Then, from Lemma 3.2, the functional $J : E \to \mathbb{R}$ corresponding to (4.3) defined by

$$J(x) = \Phi(x) - \lambda G(u_0)$$

is C^1 coercive and strictly convex on E, so there exists $v \in E$ that solves problem (4.3).

Next, we prove that $v \in W$. Consider the following cases.

Case 1. Suppose that $\|v\|_- \geq 1$. Multiplying

$$\Delta^2\Big(\sum_{i=1}^{2} \alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 v(k-2)\big)\Big) = \lambda f(k, u_0(k))$$

by v and summing from 1 to $T+2$, we have

$$\sum_{i=1}^{2}\sum_{k=1}^{T+2} \Delta^2\Big(\alpha_i(k-2)\phi_{p_i(k-2)}\big(\Delta^2 v(k-2)\big)\Big)v(k) = \lambda \sum_{k=1}^{T+2} f(k, u_0(k))v(k).$$

By taking twice the summation by parts and taking into account that $v(-1) = v(0) = v(T+1) = v(T+2) = 0$, we get

$$\psi(v) = \lambda \sum_{k=1}^{T+2} f(k, u_0(k))v(k).$$

Moreover, in view of (2.7), we have

$$\|v\|_{p(\cdot)}^{p^-} \leq \psi(v).$$

Then, from (2.2), (4.1) and (4.2), we obtain

$$\begin{aligned} \lambda \sum_{k=1}^{T+2} f(k, u_0(k))v(k) &\leq \lambda \sum_{k=1}^{T+2} (p^- + 1)F_0|u_0(k)|^{p^-} v(k) \leq (p^- + 1)\lambda F_0 \|v\|_\infty \sum_{k=1}^{T+2} |u_0|^{p^-} \\ &\leq (p^- + 1)\lambda F_0 M^{p^-} \|u_0\|_{p(\cdot)}^{p^-} \|v\|_\infty \leq (p^- + 1)\lambda F_0 M^{2p^-} \|v\|_\infty \\ &\leq \frac{1}{4}\,(p^- + 1)\lambda F_0 M^{2p^-} C^{\frac{2p^- - 1}{p^-}} \|v\|_- \leq \frac{1}{4}\,(p^- + 1)\lambda F_0 M^{2p^- + 1} C^{\frac{2p^- - 1}{p^-}} \|v\|_{p(\cdot)}. \end{aligned}$$

So,

$$\|v\|_{p(\,\cdot\,)}^{p^-} \leq \frac{1}{4}\,(p^- + 1)\lambda F_0 M^{2p^- + 1} C^{\frac{2p^- - 1}{p^-}} \|v\|_{p(\,\cdot\,)}$$

and thus

$$\|v\|_{p(\,\cdot\,)}^{p^- - 1} \leq \frac{1}{4}\,(p^- + 1)\lambda F_0 M^{2p^- + 1} C^{\frac{2p^- - 1}{p^-}}.$$

Hence, for any $0 < \lambda < \lambda^{**}$, we obtain

$$\|v\|_{p(\,\cdot\,)}^{p^- - 1} \leq \frac{1}{4}\,(p^- + 1)\lambda F_0 M^{2p^- + 1} C^{\frac{2p^- - 1}{p^-}} < M^{p^- - 1}.$$

Therefore, $v \in W$.

Case 2. If $\|v\|_- < 1$, the conclusion is immediate.

Finally, applying Proposition 2.1, we prove that problem (P) has at least one nontrivial solution. □

Theorem 4.2. *Assume that assumption* (H.1) *is satisfied. Then, for any* $\lambda > \lambda^*$, *problem* (P) *has at least one nontrivial solution.*

Proof. The functional J_λ is of the class C^1 on E, moreover, in view of Lemma 3.3, for any $\lambda \in (\lambda^*, +\infty)$, J_λ is anti-coercive in a finite-dimensional space, so. it has obviously at least one maximizer, therefore it has a critical point.

Thus, by Lemma 3.1, problem (P) has at least one nontrivial solution. □

Remark 4.1. If $\lambda^* < \lambda^{**}$, combining the results of Theorems 4.1 and 4.2, we conclude that for any parameter λ such that $\lambda^* < \lambda < \lambda^{**}$, problem (P) has at least two nontrivial solutions.

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