Completeness by Modal Definitions

Levan Uridia[1(✉)] and Dirk Walther[2]

[1] Razmadze Institute of Mathematics, Tbilisi, Georgia
l.uridia@freeuni.edu.ge
[2] Fraunhofer IVI, Dresden, Germany

Abstract. In this paper, we develop a framework for obtaining completeness results for extensions of modal logics. A modal language is extended by fresh modalities, which are then specified using definitions formulated in the original logic. When adding the modal definitions to the axiom system, completeness of the extended logic is guaranteed by the main result of the paper. We demonstrate the technique by applying it to extensions of the modal logic $\mathsf{S5}$.

1 Introduction

We show how to obtain Kripke completeness for certain extensions of modal logics. We consider extensions of a modal logic L with *modal definitions* of the form

$$\boxplus p \leftrightarrow \varphi(p),$$

where '$\boxplus$' is a fresh box-modality, and p is a proposition occurring in φ. That is, the modality $\boxplus$ is defined in terms of φ in which $\boxplus$ does not occur. We state the conditions on φ under which we obtain Kripke completeness of the extended logic. We pose as an interesting open problem to find a syntactic characterisation of modal definitions that give rise to what we call relational semantics. The related problem of characterising elementary formulas (i.e., modal formulas that define a first-order frame property) has been studied extensively; see, e.g., [5,8,14,18]. However, elementarity is neither a necessary nor sufficient criterion for a modal formula to be used in relational modal definitions.

The idea to add modal definitions to existing normal modal logics is quite common, e.g., for (dynamic) epistemic logics. The following formulas are examples of modal definitions: $E_A p \leftrightarrow \bigwedge_{a\in A} \Box_a p$ is the axiom for 'everyone knows' in epistemic logic, i.e., every agent in the group A knows p [13]; $[!\varphi]p \leftrightarrow (\varphi \rightarrow p)$ is the reduction axiom for the announcement operator $[!\varphi]$ in Public Announcement Logic [4,17]; $\Box_{\mathsf{S4}} p \leftrightarrow \Box_{\mathsf{K4}} p \wedge p$ is a definition of an $\mathsf{S4}$-box modality in terms of a $\mathsf{K4}$-box modality [11]; $[\varphi]_K p \leftrightarrow [\top]_K p \vee (\varphi \wedge [\top]_K(\varphi \rightarrow p))$ was used as the definition of the modal operator 'Modest Enrichment (Type B)' in [12]; and $[\varphi]p \leftrightarrow \Box p \vee (\varphi \wedge \Box(\varphi \rightarrow p))$ is the reduction axiom used for the epistemic logic $\mathsf{S5}^r$ for reasoning about knowledge under hypotheses in [19].

We show that we can obtain a finite axiomatisation of normal modal logics extended with relational modal definitions in a straightforward way. We illustrate

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this technique with one extensions of the modal logic $\mathsf{S5}$. In Sect. 4, we recall the logic $\mathsf{S5}^r$, which extends $\mathsf{S5}$ with a modal operator '$[\cdot]$' that can be parameterised with a hypothesis. The modality $[\varphi]$ represents the knowledge state under the hypothesis φ. The formula $[\varphi]\psi$ states that 'under the hypothesis φ, the agent knows ψ'. If φ happens to be true at the current world and the agent knows that φ implies ψ, then the agent knows ψ; otherwise, i.e., if φ is false, the agent knows only what it would know anyway, i.e. without any assumptions. We give a new completeness proof for the logic $\mathsf{S5}^r$ based on techniques developed in Sect. 3.

The paper is organised as follows. In the following section, we review standard definitions of modal logic and modal definability. In Sect. 3 we introduce the notion of relational modal definitions and pose the problem of finding a syntactic characterisation for it. Additionally, we show how to obtain completeness for modal logics extended with a relational modal definition as new axiom schema. We illustrate this technique with extensions of the modal logic S5 in Sect. 4. Finally, we conclude the paper in Sect. 5.

2 Preliminaries

In this section, we briefly review some standard definitions for modal logic and modal definability, cf. [7]. First, we fix a signature $\langle \Pi, M\rangle$ consisting of countable sets Π and M of symbols for propositions and modalities, respectively. The *propositional modal language* $\mathcal{L}$ for this signature consists of formulas φ that are built up inductively according to the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_m \varphi,$$

where p ranges over proposition symbols in Π and m over modality symbols in M. The logical symbols '$\top$' and '$\bot$', and the additional connectives such as '$\vee$', '$\rightarrow$' and '$\leftrightarrow$' and the dual modalities '$\Diamond_m$' with $m \in M$ are defined as usual, i.e.: $\top := p \vee \neg p$ for some atomic proposition p; $\bot := \neg\top$; $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$; $\varphi \rightarrow \psi := \neg\varphi \vee \psi$; $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$; and $\Diamond_m \varphi := \neg\Box_m\neg\varphi$.

A subset L of the propositional modal language $\mathcal{L}$ is a *modal logic* iff it contains all propositional tautologies, is closed under substitution, modus ponens and modal replacement (MREP) $\frac{p \leftrightarrow q}{\Box_m p \leftrightarrow \Box_m q}$, for $m \in M$. The modal logic L is called *monotonic* iff it contains the formulas (C) $\Box_m(p \wedge q) \rightarrow \Box_m q$, for $m \in M$, and L is *normal* iff it additionally contains the formulas (S) $\Box_m p \wedge \Box_m q \rightarrow \Box_m(p \wedge q)$ and ($\top$) $\Box_m \top$. Alternatively, it is also sufficient to state normal modal logics contain the formulas (K) $\Box_m(p \rightarrow q) \rightarrow (\Box_m p \rightarrow \Box_m q)$ and are closed under (NEC) $\frac{p}{\Box_m p}$ (i.e., instead of stating (C), (S), ($\top$) and (MREP)). The smallest normal modal logic is commonly denoted with K.

The relational semantics for the propositional modal language $\mathcal{L}$ is based on labelled graphs (Kripke structures) for the signature of $\mathcal{L}$. That is, the points are labelled by propositions from Π and the edges are binary relations, one for every modality in M. Formally, an *M-frame* is a tuple $\mathfrak{F} = (W, \{R_m\}_{m\in M})$, where W is a non-empty set of worlds and each $R_m \subseteq W^2$ is a binary relation over W labeled with a symbol m, for every $m \in M$. Formally R_m is a shorthand for (R_m, m). A *Kripke structure for* $\langle \Pi, M\rangle$ is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ consisting of an M-frame $\mathfrak{F} = (W, \{R_m\}_{m\in M})$ together with a valuation function $V : \Pi \to 2^W$ assigning to every proposition p in Π a set $V(p)$ of worlds. A Kripke structure $\mathfrak{M} = (\mathfrak{F}, V)$ is said to be *based on the frame* $\mathfrak{F}$. We also refer to a Kripke structure as a 'model'. We denote the class of all Kripke structures for $\langle \Pi, M\rangle$ as $\mathcal{K}_{\langle \Pi,M\rangle}$, or simply $\mathcal{K}$ if the signature is understood. Later we will use $\mathcal{C}$ to denote a class of models.

An interpretation of formulas from $\mathcal{L}$ is given by means of a *satisfaction relation* '$\models$', which is a binary relation between pointed models and formulas. A pointed model is a pair $\langle \mathfrak{M}, w\rangle$, where $\mathfrak{M} = (W, \{R_m\}_{m\in M}, V)$ is a model from the class $\mathcal{C}$ of all models and w a world from W. The satisfaction relation is defined inductively on the structure of formulas φ as:

- $\langle \mathfrak{M}, w\rangle \models p$ iff $w \in V(p)$;
- $\langle \mathfrak{M}, w\rangle \models \neg\psi$ iff $\langle \mathfrak{M}, w\rangle \not\models \psi$;
- $\langle \mathfrak{M}, w\rangle \models \psi \wedge \chi$ iff $\langle \mathfrak{M}, w\rangle \models \psi$ and $\langle \mathfrak{M}, w\rangle \models \chi$;
- $\langle \mathfrak{M}, w\rangle \models \Box_m\psi$ iff for all $v \in W$ with $(w, v) \in R_m$, $\langle \mathfrak{M}, v\rangle \models \psi$.

A formula φ is said to be *true* at w in $\mathfrak{M}$ iff $\langle \mathfrak{M}, w\rangle \models \varphi$; φ is *satisfiable* iff there is a pointed model $\langle \mathfrak{M}, w\rangle$ at which it is true; φ is *valid in* $\mathfrak{M}$ (written '$\mathfrak{M} \models \varphi$') iff $\langle \mathfrak{M}, w\rangle \models \varphi$ for all w in $\mathfrak{M}$; φ is *valid* on $\mathfrak{F}$ (written '$\mathfrak{F} \models \varphi$') iff φ is valid in all models based on $\mathfrak{F}$; and φ is *valid* in the class $\mathcal{C}$ of models (written '$\models_{\mathcal{C}} \varphi$') iff it is valid in every model from $\mathcal{C}$.

The set of $\mathcal{L}$-formulas that are valid in all models from a class $\mathcal{C}$ of models is called the *$\mathcal{L}$-theory* $\mathsf{Th}_{\mathcal{L}}(\mathcal{C})$ of $\mathcal{C}$, i.e.:

$$\mathsf{Th}_{\mathcal{L}}(\mathcal{C}) := \{\varphi \in \mathcal{L} \mid \text{for every } \mathfrak{M} \text{ from } \mathcal{C}, \varphi \text{ is valid in } \mathfrak{M}\}.$$

A modal logic L is said to be *Kripke complete w.r.t.* $\mathcal{C}$ iff $L = \mathsf{Th}_{\mathcal{L}}(\mathcal{C})$. In what follows, we will also just say 'complete'. For instance, K is complete w.r.t. the class of all models, and $\mathsf{S4}$ is complete w.r.t. the class of models which are based on frames that are pre-orders (i.e., frames with reflexive and transitive relations). A modal logic L is *complete w.r.t. a class $\mathcal{F}$ of frames* iff L is complete w.r.t. the class of models that are each based on a frame from $\mathcal{F}$. Not all normal modal logics are complete w.r.t. a class of frames.

The relationship to first-order logic is made precise by the so-called *standard translation* $\textsc{st}(\cdot)$, which assigns to a modal formula φ a corresponding first-order formula $\textsc{st}_x(\varphi)$ with one free variable x. The signature of the first-order language contains unary predicate symbols P and binary predicate symbols R_m, one P for every $p \in \Pi$ and one R_m for every $m \in M$. The translation function $\textsc{st}(\cdot)$ is inductively defined as follows:

$$\begin{aligned} \textsc{st}_x(p) &:= P(x) \\ \textsc{st}_x(\neg\varphi) &:= \neg\textsc{st}_x(\varphi) \\ \textsc{st}_x(\varphi \wedge \psi) &:= \textsc{st}_x(\varphi) \wedge \textsc{st}_x(\psi) \\ \textsc{st}_x(\Box_m\varphi) &:= \forall y(R_m(x, y) \to \textsc{st}_y(\varphi)) \end{aligned}$$

where y is a fresh variable for every occurrence of a box-modality.

A Kripke structure $\mathfrak{M} = (W, \{R_m\}_{m\in M}, V)$ for $\langle \Pi, M\rangle$ can be seen as a first-order structure interpreting the formulas $\textsc{st}_x(\varphi)$. While a predicate symbol R_m

is interpreted as the same called binary relation over W that is interpreting the modality m in M, a predicate symbol P is interpreted as the subset $V(p)$ of W, where p is the proposition symbol from Π that corresponds to P. Neither constants nor function symbols are introduced by the standard translation. In the first-order structure $\mathfrak{M}$, however, we introduce a dedicated constant c_w for every world $w \in W$ and we interpret c_w as w. At the level of pointed models $\langle \mathfrak{M}, w \rangle$, the relationship between φ and $\textsc{st}_x(\varphi)$ is such that:

$$\langle \mathfrak{M}, w \rangle \models \varphi \text{ iff } \mathfrak{M} \models \textsc{st}_x(\varphi)[x \mapsto c_w],$$

where $[x \mapsto c_w]$ substitutes every occurrence of the free variable x in $\textsc{st}_x(\varphi)$ with the constant c_w. Note that $\textsc{st}_x(\varphi)[x \mapsto c_w]$ is a sentence, i.e. a first-order formula without free variables.

When considering the notion of validity on frames $\mathfrak{F}$, we have that φ corresponds to the monadic second-order formula $\forall \boldsymbol{P}\, \forall x\, \textsc{st}_x(\varphi)$ as follows:

$$\mathfrak{F} \models \varphi(\boldsymbol{p}) \text{ iff } \mathfrak{F} \models \forall \boldsymbol{P}\, \forall x\, \textsc{st}_x(\varphi),$$

where $\boldsymbol{p}$ are the propositions from Π that occur in φ and $\boldsymbol{P}$ the corresponding unary predicates.

For modal formulas φ that are commonly considered as axioms, such as the formulas of the axioms (K), (T), (4), etc., there exists a first-order equivalent of the second-order formula $\forall \boldsymbol{P}\, \forall x\, \textsc{st}_x(\varphi)$. A modal formula that defines a first-order frame property is also said to be *elementary*. For instance, (4) is elementary as it is valid on all frames with transitive relations and the class of transitive frames can be defined with first-order formulas $\forall xyz(R(x,y) \wedge R(y,z) \rightarrow R(x,z))$, one for every relation R in the frame. However, there are modal formulas that are non-elementary, among them are the Löb formula $\Box(\Box p \rightarrow p) \rightarrow \Box p$ and the McKinsey formula $\Box\Diamond p \rightarrow \Diamond\Box p$. The Sahlqvist formulas define a set of elementary modal formulas [18], but it does not cover all elementary formulas. The problem of determining whether or not a modal formula is elementary is undecidable [8]. Conversely, there are elementary frame classes that are not modally definable, e.g. the class of irreflexive and the class of antisymmetric frames.

3 Modal Definitions

In this section, we show for certain extensions of modal logics how to obtain Kripke completeness w.r.t. a specific class of models. Later, in the next section, we apply this technique to extensions of the modal logic S5.

By *extending a modal logic L with a formula φ* we mean obtaining a modal logic L' as a set of formulas that is minimal w.r.t. $\subseteq$, that contains all tautologies over the symbols for propositions occuring in $L \cup \{\varphi\}$, that contains all formulas from $L \cup \{\varphi\}$ and that is closed under substitution, modus ponens and modal replacement. It can readily be seen that $L \cup \{\varphi\}$ is not necessarily a modal logic. Moreover, an extension of a modal logic that is Kripke complete w.r.t. a class $\mathcal{C}$ of models is not necessarily complete w.r.t. $\mathcal{C}$ itself nor any other class of models. We are interested in studying formulas of a specific form (modal definitions) that, when used to extend a modal logic, yield a modal logic that is complete w.r.t. a specific class of models.

Before formulating the completeness result, we introduce the notion of modal definitions.

Definition 1. *Let $\mathcal{L}$ be a propositional modal language over the signature $\langle \Pi, M \rangle$. Let $\varphi(\boldsymbol{p})$ be a formula in $\mathcal{L}$, where $\boldsymbol{p}$ are the propositions occurring in φ. Let '+' be a fresh symbol for a unary modality not in M, and $\boxplus$ the box-version of this modality. A* modal definition *in $\mathcal{L}$ is a formula of the form*

$$\boxplus p \leftrightarrow \varphi(\boldsymbol{p}),$$

where $\boldsymbol{p}$ contains p.

The box-modality $\boxplus$ is defined in terms of a modal formula in which $\boxplus$ does not occur. Notice that the modal definition $\boxplus p \leftrightarrow \varphi(\boldsymbol{p})$ itself is a formula in the propositional modal language over the extended signature $\langle \Pi, M \cup \{+\} \rangle$. For the sake of simplicity, we consider $+$ to be a unary modality symbol. We leave generalising Definition 1 and the results below to polyadic modality symbols for future work. Moreover, we will only consider the modal definitions for the box-version of $+$. The results for the dual modality can be obtained in a similar way.

In this paper, we only consider modal definitions $\boxplus p \leftrightarrow \varphi(\boldsymbol{p})$, where the box-modality $\boxplus$ does not occur in $\varphi(\boldsymbol{p})$. It is interesting, however, to also consider the more general setting, where this restriction may be weakened. For instance, the axiom for common knowledge and the axiom for the star-programme of PDL are not covered by Definition 1. We leave this for future work as well.

A modal definition is interpreted in models $\mathfrak{M} = (\mathfrak{F}, V)$ that are based on $M \cup \{+\}$-frames $\mathfrak{F} = (W, \{R_m\}_{m \in M} \cup \{R_+\})$, i.e., frames that are extended with a binary relation R_+ to interpret the new box-modality $\boxplus$. The semantics of $\boxplus$ can be defined in the usual way as for any other box-modality:

- $\langle \mathfrak{M}, w \rangle \models \boxplus \psi$ iff for all $v \in W$ with $(w, v) \in R_+$, it holds that $\langle \mathfrak{M}, v \rangle \models \psi$.

We want to interpret $\boxplus$ as specified in the modal logic L' obtained from the modal logic L extended with a modal definition of $\boxplus$. To this end, we have to confine outselves to the models from $\mathcal{C}(L')$, i.e., all models from $\mathcal{K}_{\langle \Pi, M \cup \{+\} \rangle}$ in which all formulas of L' are valid. It is now interesting to investigate the relationship between the modal definition of $\boxplus$ and the properties of the relation R_+ in the models from $\mathcal{C}(L')$.

Example 1. Let $\mathcal{L}$ be a propositional modal language over $\langle \Pi, M \rangle$. Additionally, let '+' be a fresh symbol for a modality not in M. Finally, let $L \subseteq \mathcal{L}$ be a modal logic.

The modal definition $\alpha_1 = \boxplus p \leftrightarrow p$ yields that R_+ is the identity relation.

Another simple example of a modal definition is $\boxplus p \leftrightarrow \Box_m p$, for some $m \in M$. Here we have that R_+ equals R_m in every model. Consider two more examples: $\boxplus p \leftrightarrow p \vee \neg p$ and $\boxplus p \leftrightarrow p \wedge \neg p$. In the former case, R_+ is the empty relation, whereas in the latter the modal definition does not yield any relation.

As the examples show, not all modal definitions yield a relational semantics for the logic extended with the newly defined modality. Taking the standard translation of a formula φ that is used in a definition $\boxplus p \leftrightarrow \varphi(\boldsymbol{p})$ results in the second-order formula $\forall \boldsymbol{P} \forall x\, \text{ST}_x(\varphi)$, where the predicates in $\boldsymbol{P}$ correspond to the propositional variables in $\boldsymbol{p}$. We are interested in elementary formulas, i.e., those formulas φ for which there exists a first-order formula that is equivalent to the second-order formula $\forall \boldsymbol{P} \forall x\, \text{ST}_x(\varphi)$, that additionally yield a relational semantics for the new modality $+$. It is a non-trivial problem to give a syntactic characterisation of such formulas φ that are suitable for defining fresh modalities.

In this paper, we will not solve this problem, but we will show how such modal definitions can be used to obtain an axiomatisation of the extended logic. To this end, we introduce the notion of 'relational modal definition'.

Definition 2. *Let $\mathcal{L}$ be a propositional modal language over the signature $\langle \Pi, M \rangle$. Let $\varphi(p, p_1, \ldots, p_n)$ with $n \geq 0$ be a formula in $\mathcal{L}$, where $p, p_1, \ldots, p_n$ are the propositions occurring in φ. Let '$+$' be a fresh symbol for a unary modality not in M, and $\boxplus$ the box-version of this modality.*

A modal definition $\boxplus p \leftrightarrow \varphi(p, p_1, \ldots, p_n)$ is called a relational modal definition *if there exists a first-order formula $\Psi_+(x, y)$ with two free variables x and y using only predicates that occur in $\text{ST}_x(\varphi(p, p_1, \ldots, p_n))$ such that for every $\psi \in \mathcal{L}$, it holds that for all pointed models $\langle \mathfrak{M}, w \rangle$,*

$$\langle \mathfrak{M}, w \rangle \models (\forall y)(\Psi_+(x, y) \Rightarrow \text{ST}_y(\psi)) \text{ iff } \mathfrak{M} \models \text{ST}_x(\varphi(\psi, p_1, \ldots, p_n))[x \mapsto c_w].$$

We note that elementarity is not a sufficient condition for modal formulas being suitable for a relational modal definition. For instance, the modal formula $\Diamond_m \top$ is elementary as it is valid on all frames in which the relation R_m is serial and the class of serial frames can be defined with first-order formulas $\forall x \exists y (R(x, y))$, one for every relation R in the frame. However, it can readily be seen that there is no first-order formula corresponding to $\boxplus p \leftrightarrow \Diamond_m \top$ in the sense of Definition 2. Another example is the formula $\Diamond_m \Box_m \bot$ which together with Axiom (4) states the reachability of a world without successors from any world. Furthermore, elementarity is not a necessary condition either; see, e.g., the reduction axiom for $\mathsf{S5}^r$ in the following section which yields a relational modal definition despite it being non-elementary.

Let $\Psi_+(x, y)$ be the first-order formula with two free variables x and y corresponding to a relational modal definition. Given a model $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{F} = (W, \{R_m\}_{m \in M})$, we uniquely construct the model $\mathfrak{M}_+ = (\mathfrak{F}_+, V)$, where the underlying frame $\mathfrak{F}_+$ is obtained from $\mathfrak{F}$ by adding the binary relation $R_+ \subseteq W \times W$ defined as:

$$(v, w) \in R_+ \text{ iff } \mathfrak{M} \models \Psi_+(x, y)[x \mapsto c_v, y \mapsto c_w].$$

For a class $\mathcal{C}$ of models, we denote with $\mathcal{C}_+$ the class consisting of the models $\mathfrak{M}_+$, where $\mathfrak{M}$ ranges over the models in $\mathcal{C}$.

Formulas from the extended language $\mathcal{L}_+$ can be translated to formulas in $\mathcal{L}$ in a straightforward way.

Definition 3. *Let $\mathcal{L}$ and $\mathcal{L}_+$ be propositional modal languages over the signatures $\langle \Pi, M\rangle$ and $\langle \Pi, M \cup \{+\}\rangle$, respectively, where $+$ is a fresh unary modality not in M. The translation function $^*: \mathcal{L}^+ \to \mathcal{L}$ for the relational modal definition $\boxplus p \leftrightarrow \varphi_+(p, p_1, \ldots, p_n)$ is inductively defined as follows:*

- $p^* = p$;
- $(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$;
- $(\neg\varphi)^* = \neg\varphi^*$;
- $(\Box_m\varphi)^* = \Box_m\varphi^*$, *for* $m \in M$;
- $(\boxplus\psi)^* = \varphi_+(\psi^*, p_1, \ldots, p_n)$.

Lemma 1. *Let $\mathcal{L}$ and $\mathcal{L}_+$ be propositional modal languages over the signatures $\langle \Pi, M\rangle$ and $\langle \Pi, M \cup \{+\}\rangle$, respectively, where $+$ is a fresh unary modality not in M. Let $L \subseteq \mathcal{L}$ be a normal modal logic, and obtain $L_+ \subseteq \mathcal{L}_+$ from L by adding a relational modal definition $\boxplus p \leftrightarrow \varphi(p, p_1, \ldots, p_n)$ as an only axiom schema for $\boxplus$.*

Then for every $\psi, \chi \in \mathcal{L}_+$, it holds that:

(i) if $\psi \leftrightarrow \chi \in L_+$, then $\boxplus\psi \leftrightarrow \boxplus\chi \in L_+$; and
(ii) $\psi \in L_+$ iff $\psi^* \in L$.

Proof. We first show Item *(i)*. Due to the reduction axiom it suffices to show that if $\psi \leftrightarrow \chi \in L_+$, then $\varphi(\psi, p_1, \ldots, p_n) \leftrightarrow \varphi(\chi, p_1, \ldots, p_n) \in L_+$. We show this by induction on the structure of φ. Recall that $\varphi(p, p_1, .., p_n)$ is a formula of the language $\mathcal{L}$, i.e., not containing $\boxplus$. We use the following as induction hypothesis. For every $\varphi(p, p_1, .., p_n) \in \mathcal{L}$ and every two formulas $\psi, \chi \in \mathcal{L}_+$ with $\psi \leftrightarrow \chi \in L_+$, it holds that $\varphi(\psi, p_1, \ldots, p_n) \leftrightarrow \varphi(\chi, p_1, \ldots, p_n) \in L_+$. For the base case, we distinguish two cases. *Case 1* $\varphi(p, p_1, ..p_n) = q$ where q is a propositional letter distinct from p. For this case $\varphi(\psi, p_1, \ldots, p_n) = q = \varphi(\chi, p_1, \ldots, p_n)$ and indeed $q \leftrightarrow q \in L_+$. *Case 2* $\varphi(p, p_1, ..p_n) = p$. For this case after substitution we get $\varphi(\psi, p_1, \ldots, p_n) = \psi$ and $\varphi(\chi, p_1, \ldots, p_n) = \chi$ and by assumption $\varphi \leftrightarrow \chi \in L_+$. Now assume for every formula before some constructive step k the inductive claim holds. Let $\varphi(p, p_1, .., p_n)$ be the formula constructed on step k. Then either $\varphi(p, p_1, .., p_n) = \varphi_1(p, p_1, .., p_n) \wedge \varphi_2(p, p_1, .., p_n)$ or $\varphi(p, p_1, .., p_n) = \neg\varphi_1(p, p_1, .., p_n)$ or $\varphi(p, p_1, .., p_n) = \Box_m\ \varphi_1(p,\ p_1, .., p_n)$ for some formulas $\varphi_1(p, p_1, .., p_n), \varphi_2(p, p_1, .., p_n)$ constructed on previous steps. For each case by inductive assumption we have that substitution keeps the equivalence. Let us check this only for the last case other cases are similar. So assume that $\varphi(p, p_1, .., p_n) = \Box_m\varphi_1(p, p_1, .., p_n)$. By inductive assumption we know that $\varphi_1(\psi, p_1, .., p_n) \leftrightarrow \varphi_1(\chi, p_1, .., p_n) \in L_+$. Hence $\Box_m(\varphi_1(\psi, p_1, .., p_n) \leftrightarrow \varphi_1(\chi, p_1,\ ..,\ p_n)) \in L_+$ since $L_+ \supseteq L$. By properties of box modality we obtain $\Box_m\varphi_1(\psi, p_1, .., p_n) \leftrightarrow \Box_m\varphi_1(\chi, p_1, .., p_n) \in L_+$.

Consider Item *(ii)*. We show by induction on the structure of $\varphi \in \mathcal{L}$ that $\vdash_{L_+} \varphi \leftrightarrow \varphi^*$. The only non-trivial case is when $\varphi = \boxplus\beta$. We omit the other cases. Suppose that $\varphi = \boxplus\beta$. Then by the induction hypothesis it holds that $\vdash_{L_+} \beta \leftrightarrow \beta^*$. By Item *(i)* we obtain that $\vdash_{L_+} \boxplus\beta \leftrightarrow \boxplus\beta^*$. Due to the reduction axiom, we have that $\vdash_{L_+} \boxplus\beta^* \leftrightarrow \varphi(\beta^*, p_1, \ldots, p_n)$. Hence $\vdash_{L_+} \boxplus\beta \leftrightarrow \varphi(\beta^*, p_1, \ldots, p_n)$. By Definition 3 we obtain $\boxplus\beta \leftrightarrow (\boxplus\beta)^*$.

As a result we obtain that $\varphi \in L_+$ iff $\varphi^* \in L_+$, and since $\varphi^* \in \mathcal{L}$ and the logic L_+ is defined without further axioms or rules involving the symbol $+$, it follows that $\varphi^* \in L$. The other direction of *(ii)* is immediate since the logic L_+ extends L.

Lemma 2. *Let $\mathcal{L}_+$ be a propositional modal language over the signature $\langle \Pi, M \cup \{+\}\rangle$, where $+$ is a fresh unary modality not in M. Let L_+ be the logic in the language $\mathcal{L}_+$ obtained from L by adding a modal definition $\vdash \boxplus p \leftrightarrow \varphi(p, p_1, .., p_n)$ as an only axiom schemata involving $\boxplus$.*

Then for every $\psi \in \mathcal{L}_+$, it holds that $\langle \mathfrak{M}_+, w\rangle \models \psi$ iff $\langle \mathfrak{M}, w\rangle \models \psi^*$.

Proof. The proof proceeds by induction on the structure of the formula ψ. For ψ being a proposition in Π, the lemma is immediate since both models have the same valuation function. The Boolean cases and the case for the box-modalities $\Box_m$ with $m \in M$ are standard. Let $\psi = \boxplus\alpha$. Assume that $\langle \mathfrak{M}_+, w\rangle \models \boxplus\alpha$. This is equivalent to the implication $(\forall v)((w, v) \in R_+ \Rightarrow \langle \mathfrak{M}_+, v\rangle \models \alpha)$. By the induction hypothesis this is equivalent to $(\forall v)(\mathfrak{M} \models \Psi(w, v) \Rightarrow \langle \mathfrak{M}, v\rangle \models \alpha^*)$. By Definition 2, this is equivalent to $\mathfrak{M} \models \textsc{st}_x(\varphi(\alpha^*, p_1, \ldots, p_n))[x \mapsto c_w]$, and by Definition 3 to $\langle \mathfrak{M}, w\rangle \models (\boxplus\alpha)^*$.

Theorem 1. *Let $\mathcal{L}$ and $\mathcal{L}_+$ be propositional modal languages over the signatures $\langle \Pi, M\rangle$ and $\langle \Pi, M \cup \{+\}\rangle$, respectively, where $+$ is a fresh unary modality not in M. Let $L \subseteq \mathcal{L}$ be a normal modal logic that is sound and complete w.r.t. a class $\mathcal{F}$ of Kripke frames. Obtain $L_+ \subseteq \mathcal{L}_+$ from L by adding a relational modal definition $\boxplus p \leftrightarrow \varphi(p, p_1, \ldots, p_n)$ as an only axiom schema for $\boxplus$.*

Then the logic L_+ is sound and complete w.r.t. the class $\mathcal{F}_+$.

Proof. Completeness. Assume $\nvdash \varphi$ in the logic L_+. By Lemma 1, we have that $\nvdash \varphi^*$ in the logic L. As L is complete w.r.t. $\mathcal{F}$, there is a model $\mathfrak{M}$ based on a frame in $\mathcal{F}$ and a world w in $\mathfrak{M}$ such that $\langle \mathfrak{M}, w\rangle \not\models \varphi^*$. By Lemma 2, it follows that $\langle \mathfrak{M}_+, w\rangle \not\models \varphi$. Hence, $\mathcal{C}_+ \not\models \varphi$.

4 The Modal Logic $\mathsf{S5}^r$

In this section, we recall the multi-modal logic $\mathsf{S5}^r$ from [19] together with the completeness result w.r.t. a particular class of models called *basic structures*. The language of $\mathsf{S5}^r$ is the language of propositional logic extended with modal operators parameterised with $\mathsf{S5}^r$-formulas. Formally, this is done as follows.

Definition 4 (Syntax of $\mathsf{S5}^r$). *Let Π be a countable set of propositions. Formulas φ of the language $\mathcal{L}$ are defined inductively over Π by the following grammar:*

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid [\varphi]_K \psi,$$

where p ranges over propositions in Π and $_K$ is a part of modality symbol indicating that we deal with knowledge modality.

The logical symbols '$\top$' and '$\bot$', and additional operators such as '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', and the dual modalities '$\langle\varphi\rangle_K$' are defined as usual.

Modal formulas are commonly evaluated in models containing a binary relation over the domain, one for each modality in the modal language. In this case, however, every binary relation is determined by the valuation of the atomic propositions in the domain. Therefore, it is sufficient to consider models without relations, which we call *basic structures*. Formally, a basic structure $\mathfrak{M}$ is a tuple $\mathfrak{M} = (W, V)$, where W is a non-empty set of *worlds* and $V : \Pi \rightarrow 2^W$ a *valuation function* mapping every atomic proposition p to a set of worlds $V(p)$ at which it is true. The relations that are required to evaluate the modalities are defined alongside the satisfaction relation. But first we introduce an auxiliary notion, a binary operation '$\otimes$' on sets yielding a binary relation. Let X and Y be two sets. Let $X \otimes Y$ be a binary relation over $X \cup Y$ such that

$$X \otimes Y = X^2 \cup (X \times Y) \cup Y^2. \tag{1}$$

We illustrate this notion with an example.

Example 2. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ be two sets. Then, according to (1), $X \otimes Y$ is a binary relation over $X \cup Y$ that is composed of the relations X^2, $X \times Y$ and Y^2 by taking their union. It holds that $X^2 = \{(x_1, x_2), (x_2, x_1)\} \cup \mathsf{id}(X)$, $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$ and $Y^2 = \{(y_1, y_2), (y_2, y_1), (y_1, y_3), (y_3, y_1), (y_2, y_3), (y_3, y_2)\} \cup \mathsf{id}(Y)$. Then the relation $X \otimes Y = X^2 \cup (X \times Y) \cup Y^2$ contains two fully connected clusters X^2 and Y^2, and directed edges between every point in X to every point in Y. Figure 1 below gives a graphical representation of $X \otimes Y$ (leaving out the reflexive and symmetric edges).

We are now ready to introduce the semantics of $\mathsf{S5}^r$. It differs from the semantics of Public Announcement Logic [10,17] in that the model does not change during the evaluation of formulas.

Definition 5 (Semantics of $\mathsf{S5}^r$). *Let $\mathfrak{M} = (W, V)$ be a basic structure. The logical satisfaction relation '$\models$' is defined by induction on the structure of $\mathsf{S5}^r$-formulas as follows: For all $p \in \Pi$ and all $\varphi, \psi \in \mathcal{L}$,*

- *$\langle\mathfrak{M}, w\rangle \models p$ iff $w \in V(p)$;*
- *$\langle\mathfrak{M}, w\rangle \models \varphi \vee \psi$ iff $\langle\mathfrak{M}, w\rangle \models \varphi$* or *$\langle\mathfrak{M}, w\rangle \models \psi$;*
- *$\langle\mathfrak{M}, w\rangle \models [\varphi]_K\psi$ iff for all $v \in W$ with $(w, v) \in R_\varphi$, it holds that $\langle\mathfrak{M}, v\rangle \models \psi$;*

where $R_\varphi = (W \setminus [\![\varphi]\!]_{\mathfrak{M}}) \otimes [\![\varphi]\!]_{\mathfrak{M}}$ as defined in Eq. (1) and $[\![\varphi]\!]_{\mathfrak{M}} = \{\, x \in W \mid \langle\mathfrak{M}, w\rangle \models \varphi \,\}$ is the extension *of φ in $\mathfrak{M}$.*

We say that a $\mathsf{S5}^r$-formula φ is *satisfiable* if there is a model $\mathfrak{M}$ and a world w in $\mathfrak{M}$ such that $\langle\mathfrak{M}, w\rangle \models \varphi$; φ is *valid in* $\mathfrak{M}$ if $\langle\mathfrak{M}, w\rangle \models \varphi$ for all w in $\mathfrak{M}$; and φ is *valid* if φ is valid in all models. We will refer to the relation R_φ as being *determined* by φ and a model.

According to the semantics, a formula determines a binary relation in a model. The following proposition states the properties of such relations.

Proposition 1. *Let φ be an $S5^r$-formula and let $\mathfrak{M} = (W, V)$ be a basic structure. Then, the relation R_φ determined by φ and $\mathfrak{M}$ (cf. Definition 5) is a* one-step *total preorder, i.e., R_φ satisfies the following conditions:*

– *R_φ is transitive: $\forall xyz(R_\varphi(x, y) \wedge R_\varphi(y, z) \to R_\varphi(x, z))$;*
– *R_φ is total: $\forall xy(R_\varphi(x, y) \vee R_\varphi(y, x))$;* and
– *R_φ is one-step: $\forall xyz(R_\varphi(x, y) \wedge \neg R_\varphi(y, x) \wedge R_\varphi(x, z) \to (zR_\varphi y))$.*

Instead of 'preorder' also the term 'quasiorder' is often used in the literature. Note that totality implies reflexivity and that a symmetric total preorder is an equivalence relation. The proposition is readily checked as any relation R_φ in a model determined by φ is defined using the operation '$\otimes$', which always yields a so-called 'one-step total preorder'. As the domain of a model is non-empty, it contains at least one point and, thus, the smallest relation R_φ is the edge of a single reflexive point.

Proposition 2. *The relation R_φ for every formula $\varphi \in S5^r$ is characterised by the following condition: $R_\varphi(w, v)$ iff $w \in [\![\varphi]\!]$ implies that $v \in [\![\varphi]\!]$.*

Figure 1 illustrates the relation R_φ in a model $\mathfrak{M}$. The domain of $\mathfrak{M}$ is partitioned into two clusters, the worlds in each of which are fully connected (reflexive and symmetric edges within the clusters are not shown). Between the clusters there are outgoing directed edges from worlds in the cluster on the left- to worlds in the cluster on the right-hand side, but not vice versa. Revisit Example 2 to see in detail how R_φ is computed (where $X = W \setminus [\![\varphi]\!]_\mathfrak{M}$ and $Y = [\![\varphi]\!]_\mathfrak{M}$).

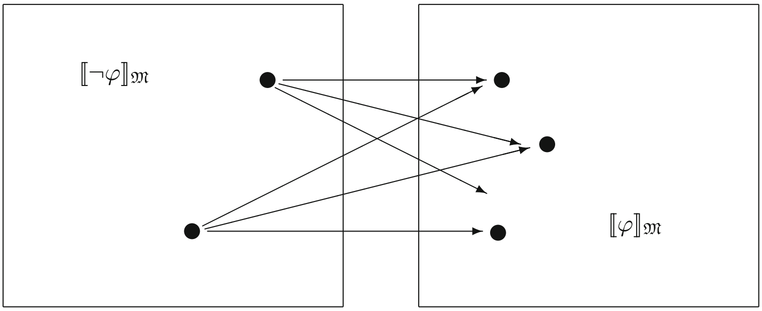

Fig. 1. Model $\mathfrak{M}$ with relation R_φ

Consider the following example, which illustrates the effect that hypotheses can have on an agent's knowledge.

Example 3. Let $\mathfrak{M} = (W, V)$ be a basic structure with $W = \{x, y\}$, $V(p_h) = V(p_c) = \{x\}$ and $V(p_u) = \{x, y\}$. Intuitively, the three propositions p_h, p_c and p_u stand for hypothesis, conclusion and universal or already established knowledge, respectively. Then, $[p_h]_K p_u$ is true at x and y in $\mathfrak{M}$. In fact, we have that $\langle \mathfrak{M}, x\rangle \models [\varphi]_K p_u$ for every $\mathsf{S5}^r$-formula φ, because p_u holds everywhere in $\mathfrak{M}$. But $[p_h]_K p_c$ holds only at x and not at y, because $\langle \mathfrak{M}, x\rangle \models p_h$ and p_h implies p_c everywhere in $\mathfrak{M}$.

We conclude this section with a discussion on how $\mathsf{S5}^r$ could possibly be used to reason about the knowledge of multiple agents; see, e.g., [13,16] for standard references. Syntactically, $\mathsf{S5}^r$ is a single-agent logic. That is, it does not provide us with syntactic markers to distinguish agents such as a different modality for each agent as in the modal epistemic logic $\mathsf{S5}_n$. Consequently, there is no way to distinguish different agents other than by what they know. In $\mathsf{S5}^r$ we can represent the individuality of agents in the hypothesis itself. For instance, in order to represent what the agents a and b know, we can use different hypotheses p_a and p_b, which are atomic propositions labelling the states which the agents a and b, respectively, consider possible. Thus $[p_a]_K\varphi$ states 'a knows φ' and $[p_b]_K\psi$ states that 'b knows ψ'.

4.1 Axiomatisation

We now present a sound and complete axiomatisation of $\mathsf{S5}^r$ from [19]. The axiom system consists of all propositional tautologies and the following axioms:

(K) $[\varphi]_K(p \to q) \to ([\varphi]_K p \to [\varphi]_K q)$
(T) $[\top]_K p \to p$
(4) $[\top]_K p \to [\top]_K[\top]_K p$
(B) $p \to [\top]_K\neg[\top]_K\neg p$
(R) $[\varphi]_K p \leftrightarrow [\top]_K p \vee (\varphi \wedge [\top]_K(\varphi \to p))$.

The first four axioms are similar to the axioms known from the modal epistemic logic $\mathsf{S5}$ characterising any modality $[\varphi]_K$ in our logic $\mathsf{S5}^r$ as epistemic operator that can be used to represent what is known under the hypothesis φ.

The axioms (T), (4), and (B) are for the modality $[\top]_K$ only, whereas we need additional instances of the axioms (K) and (R), namely the ones for each modal parameter φ (cf. Definition 4). The reduction axiom (R) states that every modality $[\varphi]_K$ is definable in terms of the basic modal operator $[\top]_K$, which corresponds to the $\mathsf{S5}$-box or the universal modality. As it was already mentioned in the introduction, Axiom (R) corresponds to the definition of the modal operator 'Modest Enrichment (Type B)' in [12].

Theorem 2 ([19]). *The system $\mathsf{S5}^r$ is sound and complete w.r.t. the class of basic structures.*

We note that the completeness proof that we present here is different from the canonical model proof envisioned in [19].

Proof. We first show soundness. The axioms (K), (T), (4), and (B) are sound w.r.t. basic structures. We show that the reduction axiom is also valid. Let $\mathfrak{M} = (W, V)$ be a basic structure and let w be a world in it. Suppose that $w \models [\varphi]_K\psi$. For every $v \in W$, it holds that if $R_\varphi(w, v)$, then $\langle\mathfrak{M}, v\rangle \models \psi$. By Proposition 2 we obtain that for every $v \in W$, if $w \in [\![\varphi]\!] \Rightarrow v \in [\![\varphi]\!]$, then $\langle\mathfrak{M}, v\rangle \models \psi$. We now show that $\langle\mathfrak{M}, w\rangle \models [\top]_K\psi \vee (\varphi \wedge [\top]_K(\varphi \to \psi))$. We distinguish two cases. In the first case, it holds that $w \notin [\![\varphi]\!]$. The implication $w \in [\![\varphi]\!] \Rightarrow v \in [\![\varphi]\!]$ holds for every $v \in W$. Hence, for every $v \in W$, we have that $\langle\mathfrak{M}, v\rangle \models \psi$. This implies that $\langle\mathfrak{M}, w\rangle \models [\top]_K\psi$. In the second case, it holds that $w \in [\![\varphi]\!]$. Then $\langle \mathfrak{M}, w\rangle \models \varphi$ and also $\langle \mathfrak{M}, w\rangle \models [\top]_K(\varphi \to \psi))$. This is because only R_φ-successors of w satisfy φ and every R_φ-successor of w satisfies ψ. Therefore, every world where φ is true also satisfies ψ. The converse direction can be shown similarly.

For showing completeness, it suffices to show that the reduction axiom $[\varphi]_K\psi \leftrightarrow [\top]_K\psi \vee (\varphi \wedge [\top]_K(\varphi \to \psi))$ is a relational modal definition defining the relation R_φ (cf. Theorem 1). Let $\Psi_\varphi(x, y) = \mathrm{ST}_x(\varphi) \Rightarrow \mathrm{ST}_y(\varphi)$ be a formula with the two free variables x and y. We want to show that $(\forall y)(\Psi_\varphi(x, y) \Rightarrow \mathrm{ST}_y(\psi))$ is equivalent to $\mathrm{ST}_x([\top]_K\psi \vee (\varphi \wedge [\top]_K(\varphi \to \psi)))$. The standard translation $\mathrm{ST}_x([\top]_K\psi \vee (\varphi \wedge [\top]_K(\varphi \to \psi)))$ is a disjunction of the formulas $(\forall y)(\mathrm{ST}_y(\psi))$ and $(\mathrm{ST}_x(\varphi) \wedge (\forall y)(\mathrm{ST}_y(\varphi) \Rightarrow \mathrm{ST}_y(\psi))$. We show that for a model $\mathfrak{M}$ and a world w, it holds that $(\forall y)(\mathrm{ST}_x(\varphi) \Rightarrow \mathrm{ST}_y(\varphi)) \Rightarrow \mathrm{ST}_y(\psi)[x \leftarrow c_w]$ iff $(\forall y)(\mathrm{ST}_y(\psi)) \vee (\mathrm{ST}_x(\varphi) \wedge (\forall y)(\mathrm{ST}_y(\varphi) \Rightarrow \mathrm{ST}_y(\psi))[x \leftarrow c_w]$.

5 Conclusions

In this paper we present a method for obtaining Kripke completeness of Kripke complete modal logics extended with a special kind of axioms which we call relational modal definitions. The notion of relational modal definition ensures that the newly defined modality has a relational semantics. The method applies to several existing modal logics, e.g., variants of dynamic epistemic logic. As an illustration we show completeness of the multi-modal logic $\mathsf{S5}^r$. The logic $\mathsf{S5}^r$ was introduced as a logic of hypotheses [19]. We think that it is an interesting non-trivial problem to give an explicit syntactic characterisation of the class of all relational modal definitions. Similar questions have been addressed in [20] and [9], although to the best of our knowledge no such characterisation has been given yet. Here we have considered some instances of relational modal definitions, e.g., classes of formulas constructed in a manner similar to Sahlqvist formulas. The first author was partially supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) grant number YS17-71.

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