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# Filtered Hirsch algebras

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**Abstract**

Motivated by the cohomology theory of loop spaces, we consider a special class of higher order homotopy commutative differential graded algebras and construct the filtered Hirsch model for such an algebra $A$. When $x \in H(A)$ with $\mathbb{Z}$ coefficients and $x^2 = 0$, the symmetric Massey products $\langle x\rangle^n$ with $n \geq 3$ have a finite order (whenever defined). However, if $\Bbbk$ is a field of characteristic zero, $\langle x\rangle^n$ is defined and vanishes in $H(A \otimes \Bbbk)$ for all $n$. If $p$ is an odd prime, the Kraines formula $\langle x\rangle^p = -\beta\mathcal{P}_1(x)$ lifts to $H^*(A \otimes \mathbb{Z}_p)$. Applications of the existence of polynomial generators in the loop homology and the Hochschild cohomology with a $G$-algebra structure are given.



*Keywords:* Hirsch algebra; Filtered model; Multiplicative resolution; Symmetric Massey product; Steenrod operation; Hochschild cohomology

## 1. Introduction

In this paper we investigate a special class of homotopy commutative algebras called *Hirsch algebras* [20]. When the structural operations of a Hirsch algebra $A$ agree component-wise with those of a homotopy $G$-algebra (HGA), the pre-Jacobi axiom can fail [7,8,19,37] and the induced product on the bar construction $BA$ is not necessarily associative. Indeed, the theory of loop space cohomology suggests that it is impossible in general, to construct a small model for $H^*(\Omega X)$ in the category of HGAs. The investigation here applies a perturbation theory that extends the well-developed perturbation theories for differential graded modules and differential graded algebras (dgas) [3,9,13,11,27,28].

One difficulty encountered when constructing a theory of homological algebra for Hirsch algebras is that the Steenrod cochain product $a \smile_1 b$ fails to be a cocycle even for cocycles $a$ and $b$. Consequently $a \smile_1 b$ does not necessarily lift to cohomology. We control such difficulties by introducing the notion of a *filtered* Hirsch algebra, which can be thought of as a specialization of a distinguished resolution in the sense of [10] (see also [14]). On the other hand, the filtered Hirsch model $(RH, d + h)$ of a Hirsch algebra $A$ is itself a Hirsch algebra whose structural operations $E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \longrightarrow RH$ are completely determined by the commutative graded algebra (cga) structure of $H = H(A, d_A)$; furthermore, the perturbation $h : RH \to RH$ of the resolution differential $d$ is

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determined by the Hirsch algebra structure on $A$ (Theorem 1). Thus by ignoring the operations $E_{p,q}$ we obtain a multiplicative resolution $(RH, d) \to (H, 0)$ of the cga $H$ thought of as a non-commutative version of its Tate–Jozefiak resolution [35,16] and the filtered model of the dga $A$ is the perturbation $(RH, d + h) \to (A, d_A)$ in [27] (such a filtered model in the category of cdgas over a field of characteristic zero was constructed by Halperin and Stasheff in [11]).

A Hirsch resolution always admits a binary operation $\cup_2$, which can be viewed as *divided* Steenrod $\smile_2$-operation. This leads to the notion of a *quasi-homotopy commutative* Hirsch algebra (QHHA) introduced here. We note that in general, the construction of a Hirsch map $(RH, d + h) \to A$ compatible with a QHHA structure on $A$ is obstructed by the non-free action of $Sq_1$ on its cohomology $H(A)$.

Every cdga $H$ can be thought of as a trivial Hirsch algebra in which the operations $E_{p,q} \equiv 0$ for all $p, q \geq 1$. However, we exhibit an example of a cohomology algebra $H = H(A)$ with a non-trivial Hirsch algebra structure determined by $Sq_1$.

For a Hirsch algebra $A$ over the integers, we establish some formulas relating the structural operations $E_{p,q}$ with syzygies in $(RH, d)$ that arise from a single element $x \in H(A)$ with $x^2 = 0$. Whereas the $n$-fold symmetric Massey product $\langle x\rangle^n$ with $n \geq 3$ is defined in $H(A)$ [23,22], our formulas imply that $\langle x\rangle^n$ has finite order. Note that when $A$ is an algebra over a field $\Bbbk$ of characteristic zero, $\langle x\rangle^n$ is defined and vanishes for all $n \geq 3$ (Theorem 2). As a consequence we have (compare [4]):

**Theorem A.** *Let $X$ be a simply connected space, let $\Bbbk$ be a field of characteristic zero and let $\sigma_* : H_*(\Omega X; \Bbbk) \to H_{*+1}(X; \Bbbk)$ be the suspension map. If $y \notin \operatorname{Ker} \sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ for all $n \geq 2$.*

Given an odd prime $p$, consider the Hirsch algebra $A \otimes \mathbb{Z}_p$, let $x \in H^{2m+1}(A \otimes \mathbb{Z}_p)$, and let $\beta$ be the Bockstein operator. We obtain the formula

$$\langle x\rangle^p = -\beta\mathcal{P}_1(x), \tag{1.1}$$

which has the same form as Kraines's formula in [23], however, the cohomology operation $\mathcal{P}_1 : H^{2m+1}(A \otimes \mathbb{Z}_p) \to H^{2mp+1}(A\otimes\mathbb{Z}_p)$ in (1.1) is canonically determined by the iteration of the $\smile_1$-product on $A\otimes\mathbb{Z}_p$ (Theorem 3). Dually, if $A$ is the singular chains on the triple loop space $\Omega^3 X$, we can identify $\mathcal{P}_1$ with the Dyer–Lashof operation (see [22]). In fact the validity of (1.1) in a general algebraic framework is conjectured by May [25, Section 6]. Furthermore, when $X = BF_4$, the classifying space of the exceptional group $F_4$, we exhibit explicit perturbations in the filtered model of $X$ and recover formula (1.1) in $H^*(X; \mathbb{Z}_3)$.

Although Theorem 1 provides a theoretical model of a Hirsch algebra $A$ endowed with higher order operations $E_{p,q}$, in practice one can construct a small *multiplicative* model for recognizing $H^*(BA)$ as an algebra in which the product is determined only by the binary operation $E_{1,1} = \smile_1$. Thus, a (minimal) multiplicative resolution of $H^*(A)$ endowed with a $\smile_1$-product provides an economical way to calculate the algebra $H^*(BA)$. We apply this technique to the Hochschild cochain complex $A = C^\bullet(P; P)$ of an associative algebra $P$ over a field $\Bbbk$ of characteristic zero to establish the following.

**Theorem B.** *If the Hochschild cohomology $H^* = H(C^\bullet(P; P))$ is a free algebra, then the Lie algebra structure on $Tor^A_*(\Bbbk, \Bbbk)$ is completely determined by that of the $G$-algebra $H^*$. Consequently, the product $\mu^*$ on $Tor^A_*(\Bbbk, \Bbbk)$ is commutative if and only if the $G$-product on $H^*$ is trivial.*

Some applications of filtered Hirsch algebras considered in an earlier version of this paper are also considered in [31,32] (see also [29,33]).

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## 2. The category of Hirsch algebras

This section defines the generalized notion of a Hirsch algebra applied here, the morphisms between them, and the notion of a Hirsch resolution.

Let $\Bbbk$ be a commutative ring with unity 1 and characteristic $\nu$; in the applications, $\Bbbk$ will be the integers $\mathbb{Z}$, a finite field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ with $p$ prime, or a field of characteristic zero. Graded $\Bbbk$-modules $A^*$ are assumed to be graded over $\mathbb{Z}$. A module $A^*$ is connected if $A^0 = \Bbbk$, and a non-negatively graded, connected module $A^*$ is 1-*reduced* if $A^1 = 0$.

For a module $A$, let $T(A)=\bigoplus_{i=0}^{\infty}A^{\otimes i}$, where $A^0=\Bbbk$, be the tensor module of $A$. An element $a_1\otimes\cdots\otimes a_n\in A^{\otimes n}$ is denoted by $[a_1|\cdots|a_n]$ when $T(A)$ is viewed as the tensor coalgebra or by $a_1\cdots a_n$ when $T(A)$ is viewed as the tensor algebra. We denote by $s^{-1}A$ the desuspension of $A$, i.e., $(s^{-1}A)^i=A^{i+1}$.

A dga $(A,d_A)$ is assumed to be supplemented; in particular, it has the form $A=\tilde{A}\oplus\Bbbk$. The (reduced) bar construction $BA$ on $A$ is the tensor coalgebra $T(\bar{A})$, $\bar{A}=s^{-1}\tilde{A}$, with differential $d=d_1+d_2$ given for $[\bar{a}_1|\cdots|\bar{a}_n]\in T^n(\bar{A})$ by

$$d_1[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\le i\le n}(-1)^{\epsilon^a_{i-1}}[\bar{a}_1|\cdots|\overline{d_A(a_i)}|\cdots|\bar{a}_n]$$

and

$$d_2[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\le i<n}(-1)^{\epsilon^a_i}[\bar{a}_1|\cdots|\overline{a_ia_{i+1}}|\cdots|\bar{a}_n],$$

where $\epsilon^x_i=|x_1|+\cdots+|x_i|+i$.

Let us generalize (slightly) the definition of a Hirsch algebra [20]. Let $A$ be a dga and consider the dg module $(Hom(BA\otimes BA,A),\nabla)$, where $\nabla$ is the canonical $Hom$ differential. Since the tensor product $BA\otimes BA$ is a dgc with the standard coalgebra structure, the $\smile$-product induces a dga structure on $(Hom(BA\otimes BA,A),\nabla,\smile)$.

**Definition 1.** A **Hirsch algebra** is an associative dga $A$ equipped with multilinear maps

$$E_{p,q}:A^{\otimes p}\otimes A^{\otimes q}\to A,\quad p,q\ge 0,\ p+q>0,$$

satisfying the following conditions:

(i) $\deg E_{p,q}=1-p-q$;
(ii) $E_{1,0}=Id=E_{0,1}$ and $E_{p>1,0}=0=E_{0,q>1}$;
(iii) The homomorphism $E:BA\otimes BA\to A$ defined by

$$E([\bar{a}_1|\cdots|\bar{a}_p]\otimes[\bar{b}_1|\cdots|\bar{b}_q])=E_{p,q}(a_1,\ldots,a_p;b_1,\ldots,b_q)\tag{2.1}$$

is a twisting cochain in the dga $(Hom(BA\otimes BA,A),\nabla,\smile)$, i.e., $\nabla E=-E\smile E$.

A *morphism* $f:A\to B$ between two Hirsch algebras is a dga map $f$ that commutes with $E_{p,q}$ for all $p,q$.

Condition (iii) implies that $\mu_E:BA\otimes BA\to BA$ is a chain map; thus $BA$ is a dg bialgebra whose multiplication $\mu_E$ is not necessarily associative (compare [8,37,5,21,26]); in particular, $\mu_{E_{10}+E_{01}}$ is the shuffle product on $BA$, and a Hirsch algebra with $E_{p,q}\equiv 0$ for all $p,q\ge 1$ is just a cdga (cf. (2.3)). It is useful to express Eq. (2.1) component-wise:

$$\begin{aligned}
&dE_{p,q}(a_1,\ldots,a_p;b_1,\ldots,b_q)\\
&\quad=\sum_{1\le i\le p}(-1)^{\epsilon^a_{i-1}}E_{p,q}(a_1,\ldots,da_i,\ldots,a_p;b_1,\ldots,b_q)\\
&\qquad+\sum_{1\le j\le q}(-1)^{\epsilon^a_p+\epsilon^b_{j-1}}E_{p,q}(a_1,\ldots,a_p;b_1,\ldots,db_j,\ldots,b_q)\\
&\qquad+\sum_{1\le i<p}(-1)^{\epsilon^a_i}E_{p-1,q}(a_1,\ldots,a_ia_{i+1},\ldots,a_p;b_1,\ldots,b_q)\\
&\qquad+\sum_{1\le j<q}(-1)^{\epsilon^a_p+\epsilon^b_j}E_{p,q-1}(a_1,\ldots,a_p;b_1,\ldots,b_jb_{j+1},\ldots,b_q)\\
&\qquad+\sum_{\substack{0\le i\le p\\ 0\le j\le q\\ (i,j)\ne(0,0)}}(-1)^{\epsilon_{i,j}}E_{i,j}(a_1,\ldots,a_i;b_1,\ldots,b_j)\cdot E_{p-i,q-j}(a_{i+1},\ldots,a_p;b_{j+1},\ldots,b_q),
\end{aligned}\tag{2.2}$$

$$\epsilon_{i,j}=\epsilon^a_i+\epsilon^b_j+(\epsilon^a_i+\epsilon^a_p)\epsilon^b_j+1.$$

In particular, the operation $E_{1,1}$ satisfies conditions similar to Steenrod's cochain $\smile_1$-product:

$$dE_{1,1}(a;b) - E_{1,1}(da;b) + (-1)^{|a|}E_{1,1}(a;db) = (-1)^{|a|}ab - (-1)^{|a|(|b|+1)}ba; \tag{2.3}$$

consequently, $E_{1,1}$ measures the non-commutativity of the product $\cdot$ on $A$. We shall use the notation $a \smile_1 b = E_{1,1}(a;b)$ interchangeably. The following special cases will also be important for us, so we write them explicitly:

The Hirsch formulas up to homotopy

$$\begin{aligned} dE_{2,1}(a,b;c) = {} & E_{2,1}(da,b;c) - (-1)^{|a|}E_{2,1}(a,db;c) + (-1)^{|a|+|b|}E_{2,1}(a,b;dc) \\ & - (-1)^{|a|}(ab)\smile_1 c + (-1)^{|a|+|b|+|b||c|}(a\smile_1 c)b + (-1)^{|a|}a(b\smile_1 c) \end{aligned}$$

and

$$\begin{aligned} dE_{1,2}(a;b,c) = {} & E_{1,2}(da;b,c) - (-1)^{|a|}E_{1,2}(a;db,c) + (-1)^{|a|+|b|}E_{1,2}(a;b,dc) \\ & + (-1)^{|a|+|b|}a\smile_1(bc) - (-1)^{|a|+|b|}(a\smile_1 b)c - (-1)^{|a|(|b|-1)}b(a\smile_1 c) \end{aligned}$$

tell us that the deviations of the binary operation $\smile_1$ from left and right derivation of the $\cdot$ product are measured by the respective boundaries of the operations $E_{1,2}$ and $E_{2,1}$ on three variables.

The following definition describes a class of Hirsch algebras in which the $\smile_1$-product itself is homotopy commutative (cf. (2.5)).

**Definition 2.** A **quasi-homotopy commutative Hirsch algebra** (QHHA) is a Hirsch algebra $A$ equipped with a binary product $\cup_2 : A\otimes A \to A$ such that

$$d(a\cup_2 b) = da\cup_2 b + (-1)^{|a|}a\cup_2 db + (-1)^{|a|}a\smile_1 b + (-1)^{(|a|+1)|b|}b\smile_1 a - q(a;b), \tag{2.4}$$

where $q(a;b)$ satisfies:

$(2.4)_1$ Leibniz rule: $dq(a;b) = -q(da;b) - (-1)^{|a|}q(a;db)$;
$(2.4)_2$ Acyclicity: $[q(a,b)] = 0 \in H(A,d)$ for $da = db = 0$.

Note that $(2.4)_1$ follows from the equalities (2.2) and $d^2 = 0$. Obviously, discarding the parameter $q(a;b)$, the above formula just becomes the Steenrod formula for the $\smile_2$-cochain product:

$$d(a\smile_2 b) = da\smile_2 b + (-1)^{|a|}a\smile_2 db + (-1)^{|a|}a\smile_1 b + (-1)^{(|a|+1)|b|}b\smile_1 a. \tag{2.5}$$

However, $q(-;-)$ may be non-zero when passing to models constructed via cohomology as below. In the following four examples, the first is a naturally occurring example of a *cochain* Hirsch algebra (compare Example 5); in the second example QHHA structures are considered for certain Hirsch algebras; in the third and fourth examples a Hirsch algebra structure is lifted to the cohomology level. In fact, the fourth example was the original motivation for this paper.

**Example 1.** The primary examples of Hirsch algebras for topological spaces $X$ are their cubical or simplicial cochain complexes [20,19,21]. In the simplicial case one can choose $E_{p,q} = 0$ for $q \geq 2$ and obtain an HGA structure on the simplicial cochains $C^*(X;\Bbbk)$ [2] (see also [19]). Furthermore, the product $\mu_E$ on $BC^*(X;\Bbbk)$ gives the multiplicative structure of the loop space cohomology $H^*(\Omega X;\Bbbk)$.

Here the cochain complex $C^*(X;\Bbbk)$ of a space $X$ is 1-reduced, since by definition $C^*(X;\Bbbk) = C^*(\mathrm{Sing}^1 X;\Bbbk)/C^{>0}(\mathrm{Sing}\, x;\Bbbk)$ where $\mathrm{Sing}^1 X \subset \mathrm{Sing}\, X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard $n$-simplex $\Delta^n$ to the base point $x$ of $X$. Unlike the cubical cochains, the Hirsch algebra structure of the simplicial cochains is *associative*, i.e., the above product $\mu_E$ is associative.

**Example 2.** First, note that the Hirsch algebras from the previous example are also QHHA's by setting $\cup_2 = \smile_2$ and $q(-;-) = 0$. Let $A$ be a *special Hirsch algebra*, i.e., $A$ is an associative Hirsch algebra and $BA$ also admits a Hirsch algebra structure. Then $A$ is a QHHA since it admits a $\cup_2$-product satisfying (2.5) (cf. [18]). An important example of a special Hirsch algebra is $A = C^*(X;\Bbbk)$ from the previous example (cf. [20,34]). Finally, for a QHHA $A$ with $\nu$

to be zero or odd and $\smile_2$-product satisfying (2.5), define the *divided* $\smile_2$-operation $\cup_2$ as

$$a \cup_2 b = \begin{cases} \frac{1}{2}\, a \smile_2 a, & a = b \\ a \smile_2 b, & \text{otherwise.} \end{cases}$$

Then $A$ with this $\cup_2$-operation is again a QHHA.

**Example 3.** Let $(H, d = 0)$ be a free cga $H = S\langle\mathcal{H}^*\rangle$ generated by a graded set $\mathcal{H}^*$. Then any map of sets $\tilde{E}_{p,q} : \mathcal{H}^{\times p} \times \mathcal{H}^{\times q} \to H$ of degree $1 - p - q$ extends to a Hirsch algebra structure $E_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \to H$ on $H$. Indeed, using formula (2.2) the construction goes by induction on the sum $p + q$. In particular, if only $\tilde{E}_{1,1}$ is non-zero then the image of $E_{p,q}$ for $p + q \geq 3$ is into the submodule of $H$ spanned by the monomials of the form $\tilde{E}_{1,1}(a_1; b_1) \cdots \tilde{E}_{1,1}(a_k; b_k) \cdot x$ for $a_i, b_i \in \mathcal{H}$, $x \in H$, and $k \geq 1$.

**Example 4.** The argument in Example 3 suggests how to lift a Hirsch $\mathbb{Z}_2$-algebra structure from the cochain level to cohomology. Given a Hirsch algebra $A$, let $H = H^*(A)$. For a cocycle $a \in A^m$, one has $d_A E_{1,1}(a, a) = 0$ and $Sq_1 : H^m \to H^{2m-1}$ is defined by

$$[a] \to [E_{1,1}(a, a)].$$

The trick here is to convert the Hirsch formulas up to homotopy on $A$ to the Cartan formula $Sq_1(ab) = Sq_1 a \cdot Sq_0 b + Sq_0 a \cdot Sq_1 b$ on $H$ by fixing a set of multiplicative generators $\mathcal{H} \subset H$. Define the map $\tilde{Sq}_{1,1} : \mathcal{H} \times \mathcal{H} \to H$ for $a, b \in \mathcal{H}$ by

$$\tilde{Sq}_{1,1}(a; b) = \begin{cases} Sq_1 a, & a = b, \\ 0, & \text{otherwise} \end{cases}$$

and extend to the operation $Sq_{1,1} : H \otimes H \to H$ as a (two-sided) derivation with respect to the $\cdot$ product; then in particular, $Sq_{1,1}(u; u) = Sq_1 u$ for all $u \in H$. Define $Sq_{p,q} = E_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \to H$ for $p + q \geq 3$ by means of (2.2). Note that if the multiplicative structure on $H$ is not free, such an extension might not exist. This procedure gives a Hirsch algebra structure $\{Sq_{p,q}\}$ on the cohomology algebra $H$ in the following situations:

(i) $H$ has trivial multiplication (e.g. the cohomology of a suspension).
(ii) $H$ is a polynomial algebra.
(iii) $H$ has the following property: If $a \cdot b = 0$, then $Sq_1 a \cdot b = 0 = Sq_1 a \cdot Sq_1 b$ for all $a, b \in H$.

Obviously we have the following proposition:

**Proposition 1.** *A morphism $f : A \to A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions*

$$Bf : BA \to BA'.$$

*If the modules $A, A'$ are $\Bbbk$-free and $f$ is a homology isomorphism, so is $Bf$.*

This proposition is useful when applying special models for a Hirsch algebra $A$ to calculate the cohomology algebra $H^*(BA) = Tor^A(\Bbbk, \Bbbk)$ (see Section 3.4), and consequently, the loop space cohomology $H^*(\Omega X; \Bbbk)$ when $A = C^*(X; \Bbbk)$ (see, for example, [31]).

Given a Hirsch algebra $A$ with cohomology $H = H(A)$, let us construct a Hirsch algebra model of $A$. The commutative algebra $H$ admits a special *multiplicative* resolution $(RH, d)$, which is endowed with the Hirsch algebra structure $\{E_{p,q}\}$. The perturbed differential $d_h$ on $RH$ gives the desired Hirsch algebra model $(RH, d_h)$ of $A$.

### *2.1. Hirsch resolution*

Let $H^*$ be a graded algebra and recall that a multiplicative resolution $(R^*H^*, d)$ of $H^*$ is the bigraded tensor algebra $T(V)$ generated by the bigraded free $\Bbbk$-module

$$V = \bigoplus_{j,m \geq 0} V^{-j,m},$$

where $V^{-j,m} \subset R^{-j}H^m$. The total degree of $R^{-j}H^m$ is the sum $-j+m$, $d$ is of bidegree $(1,0)$ and $\rho : (RH, d) \to H$ is a map of bigraded algebras inducing an isomorphism $\rho^* : H^*(RH, d) \xrightarrow{\approx} H^*$ where $H^*$ is bigraded via $H^{0,*} = H^*$ and $H^{<0,*} = 0$ ([27]; compare [11,13]). In other words,

$$\left((R^*H^m, d) \xrightarrow{\rho} H^m\right) = (\cdots \xrightarrow{d} R^{-2}H^m \xrightarrow{d} R^{-1}H^m \xrightarrow{d} R^0H^m \xrightarrow{\rho} H^m)$$

is a usual free (additive) resolution of the $\Bbbk$-module $H^m$ for each $m$, and there is a multiplication on the family $\{R^*H^m\}_{m\in\mathbb{Z}}$, which is compatible with both $d$ and the bidegree. When each $H^m$ is $\Bbbk$-free, $\Omega BH$ (the cobar–bar construction of $H$) is an example of $RH$ with $V = BH$. In general, the multiplicative structure of $H^*$ gives rise to (additively) *non-minimal* submodules $(R^*H^m, d)$ even for $H^m$ to be $\Bbbk$-free or $H^m = 0$. The reason for this is that a (multiplicative) relation in $H$ involving elements of degree $<m$ can produce an element $a \in R^{-1}H^k$ with $k < m$, say $m = kn$, some $n \geq 2$, and since the multiplication on $R^*H^*$ respects the bidegree, the non-zero element $a^n$, the $n$th power of $a$, ultimately belongs to $R^{-n}H^m$, the $n$th component of a $\Bbbk$-module resolution of $H^m$ (see the proof of Proposition 3). Furthermore, even for $H$ to be a free cga over a field $\Bbbk$, the non-commutative nature of $RH$ fails to imply $R^*H^m$ to be a minimal $\Bbbk$-module resolution of $H^m$, i.e.,

$$R^0H^m = H^m \quad \text{and} \quad R^{-i}H^m = 0, \quad i > 0;$$

this is quite different from the situation in [11].

For example, consider the polynomial algebra $H = \mathbb{Z}_2[x, y]$ with $x, y \in H^2$ and $x_0, y_0 \in R^0H^2$ satisfying $\rho x_0 = x$ and $\rho y_0 = y$. Then $R^{-1}H^4 \neq 0$ since there is an element $a \in R^{-1}H^4$ such that $da = x_0y_0 + y_0x_0$. In particular, if $H$ is the cohomology of a dga $A$ with a *non-commutative* $\smile_1$-product (and perhaps higher order operations $E_{p,q}$; cf. Examples 1 and 5), then the construction of a *Hirsch algebra* model of $A$ using $RH$ requires to add another element $b$ in $R^{-1}H^4$ with $db = x_0y_0 + y_0x_0$. Then denote $a = x_0 \smile_1 y_0$ and $b = y_0 \smile_1 x_0$ respectively (see Theorem 1). Furthermore, if $H^*$ is 1-reduced and we wish to have a 1-reduced multiplicative resolution $RH$, we must restrict the resolution length of $R^*H^m$ so that $R^{-i}H^m = 0$ for $i \geq m-1$ (e.g. $H^m$ is $\Bbbk$-free for all $m$ or $H^2$ is $\Bbbk$-free and $\Bbbk$ is a principal ideal domain). This motivates the following definition:

**Definition 3.** Let $H^*$ be a cga. An absolute Hirsch resolution of $H$ is a multiplicative resolution

$$\rho : R^*H^* \to H^*, \qquad RH = T(V), \quad V = \langle \mathcal{V} \rangle,$$

endowed with the Hirsch algebra structural operations

$$E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \to V \subset RH$$

such that $V$ is decomposed as $V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$ in which $\mathcal{E}^{0,*} = 0$, $U^{0,*} = V^{0,*}$ and $\mathcal{E}^{*,*} = \bigoplus_{p,q\geq 1} \mathcal{E}^{<0,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q} : \bigoplus_{\substack{i_{(p)}+j_{(q)}=s \\ k_{(p)}+\ell_{(q)}=t}} \left( \underset{1\leq r\leq p}{\otimes} R^{i_r}H^{k_r} \bigotimes \underset{1\leq n\leq q}{\otimes} R^{j_n}H^{\ell_n} \right) \xrightarrow{\approx} \mathcal{E}^{s-p-q+1\,,\,t}_{p,q} \subset V^{*,*}$$

where $x_{(r)} = x_1 + \cdots + x_r$.

Given a Hirsch algebra $(A, \{E_{p,q}\}, d)$, a submodule $J \subset A$ is a *Hirsch ideal* of $A$ if it is an ideal with $E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}) \in J$ whenever $a_i \in J$ for some $i$.

**Definition 4.** Let $\rho_a : (R_aH, d) \to H$ be an absolute Hirsch resolution and $J \subset R_aH$ be a Hirsch ideal such that $d : J \to J$ and the quotient map $g : R_aH \to R_aH/J$ is a homology isomorphism. A Hirsch resolution of $H$ is the Hirsch algebra $RH = R_aH/J$ with a map $\rho : RH \to H$ such that $\rho_a = \rho \circ g$.

Thus an absolute Hirsch resolution is a Hirsch resolution by taking $J = 0$.

**Proposition 2.** *Every cga $H^*$ has an (absolute) Hirsch resolution $\rho : R^*H^* \to H^*$.*

**Proof.** We build a Hirsch resolution of $H^*$ by induction on the resolution degree. Let $\mathcal{H}^* \subset H^*$ be a set of multiplicative generators. Denote $\mathcal{V}^{0,*} = \mathcal{H}^*$; let $V^{0,*} = \langle \mathcal{V}^{0,*} \rangle$ be the free $\Bbbk$-module span of $\mathcal{V}^{0,*}$ and form the free (tensor) graded algebra $R^0H^* = T(V^{0,*})$. Obviously, there is a dga epimorphism $\rho^0 : (R^0H^*, 0) \to H^*$. Inductively, given $n \geq 0$, assume we have constructed a $\Bbbk$-module $R^{(-n)}H^* = \oplus_{0\le r\le n} R^{-r}H^*$ with a map $\rho^{(n)} : (R^{(-n)}H^*, d) \to H^*$ with $\rho^r(R^{-r}H^*) = 0$ for $1 \le r \le n$, where $d : R^{-r}H^* \to R^{-r+1}H^*$ is a differential of bidegree $(1, 0)$ defined for $1 \le r \le n$ and acyclic in resolution degrees $-r$ for $1 \le r < n$; $R^{-r}H^*$ is a component of bidegree $(-r, *)$ of $T(V^{(-r),*})$ for $V^{(-r),*} = V^{0,*} \oplus \cdots \oplus V^{-r,*}$, so that

$$R^{-r}H^* = V^{-r,*} \oplus \mathcal{D}^{-r,*} = \mathcal{E}^{-r,*} \oplus U^{-r,*} \oplus \mathcal{D}^{-r,*}$$

where $\mathcal{E}^{-r,*} = \bigoplus_{p,q\ge 1} \mathcal{E}^{-r,*}_{p,q}$ and $\mathcal{E}^{-r,*}_{p,q}$ spans the set of (formal) expressions $E_{p,q}(a_1,\dots,a_p; b_1,\dots,b_q)$, $a_j \in R^{-i_k}H^*$, $b_\ell \in R^{-j_\ell}H^*$, $r = i_{(p)} + j_{(q)} + p + q - 1$, while $\mathcal{D}^{-r,*}$ is the module of decomposables of bidegree $(-r, *)$ in $T(V^{(-r),*})$; $d$ is given by formula (2.2) on $\mathcal{E}^{-r,*}$, while acts as a derivation on $\mathcal{D}^{-r,*}$.

Let $\mathcal{E}^{-n-1,*} = \bigoplus_{p,q\ge 1} \mathcal{E}^{-n-1,*}_{p,q}$ where $\mathcal{E}^{-n-1,*}_{p,q}$ spans the set of expressions $E_{p,q}(a_1,\dots,a_p; b_1,\dots,b_q)$, $a_k \in R^{-i_k}H^*$, $b_\ell \in R^{-j_\ell}H^*$, $n+1 = i_{(p)} + j_{(q)} + p + q - 1$, and let $\mathcal{D}^{-n-1,*}$ be the module of decomposables of bidegree $(-n-1, *)$ in $T\left(V^{(-n),*} \oplus \mathcal{E}^{-n-1,*}\right)$; define $d$ by formula (2.2) on $\mathcal{E}^{-n-1,*}$ and as a derivation on $\mathcal{D}^{-n-1,*}$ so that

$$\mathcal{E}^{-n-1,*} \oplus \mathcal{D}^{-n-1,*} \xrightarrow{d} R^{-n}H^* \xrightarrow{d} R^{-n+1}H^*.$$

Define a free $\Bbbk$-module $U^{-n-1,*}$ and $d$ on it to achieve acyclicity in resolution degree $-n$, i.e, denoting $V^{-n-1,*} = \mathcal{E}^{-n-1,*} \oplus U^{-n-1,*}$, we obtain a partial resolution for each $m \in \mathbb{Z}$

$$V^{-n-1,m} \oplus \mathcal{D}^{-n-1,m} \xrightarrow{d} R^{-n}H^m \xrightarrow{d} R^{-n+1}H^m \xrightarrow{d} \cdots \xrightarrow{d} R^{-1}H^m \xrightarrow{d} R^0H^m \xrightarrow{\rho} H^m.$$

Define $R^{-n-1}H^* = V^{-n-1,*} \oplus \mathcal{D}^{-n-1,*}$ and $\rho^{n+1} : R^{-n-1}H^* \to H^*$ to be trivial. This completes the inductive step.

Finally, set $R^*H^* = \oplus_n R^{(-n)}H^*$ with $V^{*,*} = \langle \mathcal{V}^{*,*} \rangle$, $\mathcal{E}^{*,*} = \oplus_n \mathcal{E}^{-n,*}$, $U^{*,*} = \oplus_n U^{-n,*}$, $\rho|_{R^0H^*} = \rho^0$ and $\rho|_{R^{-n}H^*} = 0$ for $n > 0$ to obtain the desired resolution map $\rho : RH \to H$. □

Note that in a Hirsch resolution $(RH, \{E_{p,q}\}, d)$, we may have relations among $E_{p,q}$'s (e.g. $E_{p,q} = 0$ for some $p, q \geq 1$; cf. Section 2.6). For example, the Hirsch structure of $RH$ is *associative* if the product $\mu_E$ on the bar construction $B(RH)$ is associative and is equivalent to the equalities among $E_{p,q}$'s as follows.

Given a Hirsch algebra $A$ and an arbitrary triple

$$(\mathbf{a}; \mathbf{b}; \mathbf{c}) = (a_1,\dots,a_k\,; b_1,\dots,b_\ell\,; c_1,\dots,c_r), \quad a_i, b_j, c_s \in A,$$

denote

$$\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}) = \sum_{\substack{k_{(p)}=k;\ell_{(p)}=\ell \\ 1\le p\le k+\ell}} (-1)^{\varepsilon} E_{p,r}(E_{k_1,\ell_1}(a_1,\dots,a_{k_1}; b_1,\dots,b_{\ell_1}), \dots, E_{k_p,\ell_p}(a_{k-k_p+1},\dots,a_k; b_{\ell-\ell_p+1},\dots,b_p)\,; c_1,\dots,c_r)$$

and

$$\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})) = \sum_{\substack{\ell_{(q)}=\ell;r_{(q)}=r \\ 1\le q\le \ell+r}} (-1)^{\delta} E_{k,q}(a_1,\dots,a_k; E_{\ell_1,r_1}(b_1,\dots,b_{\ell_1}; c_1,\dots,c_{r_1}), \dots, E_{\ell_q,r_q}(b_{\ell-\ell_q+1},\dots,b_\ell; c_{r-r_q+1},\dots,c_q)),$$

where we use the convention that $E_{0,1}(-;a) = E_{1,0}(a;-) = a$, $E_{0,m}(-;a_1,\dots,a_m) = E_{m,0}(a_1,\dots,a_m;-) = 0$, $m \geq 2$, and $x_{(n)} = x_1 + \cdots + x_n$, while the signs $\varepsilon$ and $\delta$ are induced by permutations of symbols $a_i, b_j, c_s$ (cf. [37]). Then the associativity of $A$ is equivalent to the equalities

$$\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}) = \mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})), \quad k,\ell,r \geq 1.$$

Now consider the expression

$$\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c}))-\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c})\in\mathcal{E}^{1-k-\ell-r,*}$$

in an absolute Hirsch resolution $RH$. We have that this expression belongs to $\mathcal{E}^{-2,*}$ and is a cocycle for $(\mathbf{a};\mathbf{b};\mathbf{c})=(a;b;c)$, $a,b,c\in R^0H$ (see (2.6) and Fig. 1 in which the boundaries of both hexagons are labeled by the 6 components of $d\mathcal{R}_{1,1,1}(a;(b;c))=d\mathcal{R}_{1,1,1}((a;b);c)$). So there is an element, denoted by $s(\mathcal{R}_{1,1,1}(a;(b;c)))\in V^{-3,*}$ such that $ds(\mathcal{R}_{1,1,1}(a;(b;c)))=\mathcal{R}_{1,1,1}(a;(b;c))-\mathcal{R}_{1,1,1}((a;b);c)$. In general, define elements $s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})))\in V$ such that

$$\begin{aligned}&ds(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})))+s(\mathcal{R}_{k,\ell,r}(d\mathbf{a};(\mathbf{b};\mathbf{c})))+(-1)^{\varepsilon_1}s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(d\mathbf{b};\mathbf{c})))\\&\quad+(-1)^{\varepsilon_2}s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};d\mathbf{c})))=\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c}))-\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c})\\&\varepsilon_1=|\mathbf{a}|+k,\ \varepsilon_2=|\mathbf{a}|+|\mathbf{b}|+k+\ell.\end{aligned}$$

Consequently, $RH=R_aH/J_{ass}$ is an associative Hirsch resolution, where $J_{ass}\subset R_aH$ is a Hirsch ideal generated by

$$\{\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c}))-\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}),\ s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})))\}.$$

In particular, for $(\mathbf{a};\mathbf{b};\mathbf{c})=(a;b;c)$ the associativity of a Hirsch resolution implies the following.

**Proposition 3.** *For $a,b,c\in RH$, there is the equality*

$$\begin{aligned}&(a\smile_1 b)\smile_1 c+E_{2,1}(a,b;c)+(-1)^{(|a|+1)(|b|+1)}E_{2,1}(b,a;c)\\&\quad=a\smile_1(b\smile_1 c)+E_{1,2}(a;b,c)+(-1)^{(|b|+1)(|c|+1)}E_{1,2}(a;c,b).\end{aligned}\tag{2.6}$$

A Hirsch resolution $(RH,d)$ is *minimal* if

$$d(u)\in\mathcal{E}+\mathcal{D}+\kappa_u\cdot V\quad\text{for all }u\in U,$$

where $\mathcal{D}^{*,*}\subset R^*H^*$ denotes the submodule of decomposables $RH^+\cdot RH^+$ ($RH^+$ denotes $RH$ modulo the unital component) and $\kappa_u\in\Bbbk$ is non-invertible. For example, when $\Bbbk=\mathbb{Z}$ we have $\kappa_u\in\mathbb{Z}\setminus\{-1,1\}$; when $\Bbbk$ is a field we have $\kappa_u=0$ for all $u$. Note that a minimal Hirsch resolution is *not* minimal in the category of dgas since the resolution differential does not send multiplicative generators into $\mathcal{D}$ even when $\Bbbk$ is a field. Furthermore, the notion of minimality of $RH$ does not depend upon whether some operation $E_{p,q}$ is zero (cf. Section 2.6). On the other hand, in order to define a $\smile_2$-operation in a simple way on $RH$ we have to consider a non-minimal Hirsch resolution in the next subsection.

Such a flexibility of choice of $RH$ is due to the trivial Hirsch structure of $H$, and, in practice, the choice is suggested by a Hirsch algebra $A$ that realizes $H$ as the cohomology algebra.

### *2.2. QHHA structures on Hirsch algebras*

First, note that one can introduce a $\smile_2$-product on a Hirsch resolution that satisfies (2.5). However, such a QHHA structure on $RH$ in not always satisfactory, and we shall consider a $\cup_2$-operation simultaneously for the reasons explained below. For an even dimensional $a$, or for any $a$ whenever $\nu=2$, we have that $a\smile_1 a$ is cocycle for $da=0$; hence, there is an element $x\in RH$ with $dx=a\smile_1 a$. But we cannot identify $x$ with $a\smile_2 a$ because $d(a\smile_2 a)=0$ according to (2.5). On the other hand, it is helpful to denote $x:=a\cup_2 a$ since certain formulas are conveniently expressed in terms of the binary operation $\cup_2$ (see, for example, Proposition 5 or Remark 7). Furthermore, we can identify $a\cup_2 a$ with $\frac{1}{2}a\smile_2 a$ for $|a|$ even and 2 invertible in $\Bbbk$.

By construction of a Hirsch resolution in Proposition 2, the definition of $\smile_2$ mimics that of $\smile_1$. We start with the consideration of the expression

$$(-1)^a a\smile_1 b+(-1)^{(|a|+1)|b|}b\smile_1 a\ \in\mathcal{E}^{-1,*}\quad\text{for }a,b\in\mathcal{V}^{0,*}.$$

It is a cocycle in $(RH, d)$, and hence, must be killed by a multiplicative generator; denote this generator by $a \smile_2 b \in U^{-2,*}$. Inductively, assume that the right-hand side of (2.5) has been defined as an element of $U^{-n+1,*}$. Then it is bounded by a multiplicative generator $a \smile_2 b \in U^{-n,*}$. Thus, $a \smile_2 b \in U$ for all $a, b \in RH$. In particular, if $dx = 0$, then $d(x \smile_2 x) = 0$ or $d(\frac{\nu}{2} x \smile_2 x) = 0$ for $|x|$ to be odd or for both $|x|$ and $\nu$ to be even respectively in which case a multiplicative generator $y \in U$ with $dy = x \smile_2 x$ is denoted by $x \cup_3 x$.

Now define a $\cup_2$-operation by

$$a \cup_2 b = \begin{cases} a \smile_2 b, & a \neq b,\ a, b \text{ are in a basis of } RH \\ 0, & a = b,\ |a| \text{ and } \nu \text{ are odd,} \end{cases} \tag{2.7}$$

while, otherwise, define $a \cup_2 a \in U$ by

$$d(a \cup_2 a) = \begin{cases} a \smile_1 a + a \smile_2 da + da \cup_3 da, & |a| \text{ is even} \\ \dfrac{\nu}{2}(a \smile_1 a + a \smile_2 da) + da \cup_3 da, & |a| \text{ is odd, } \nu \text{ is even.} \end{cases}$$

Hence, $a \cup_2 b \in U$ for any $a, b \in RH$, and let

$$\mathcal{T} = \{a \cup_2 b \in U \mid a, b \in RH\}.$$

Thus, we obtain the decomposition $U = \mathcal{T} \oplus \mathcal{M}$, some $\mathcal{M}$, and, hence, the decomposition

$$V = \mathcal{E} \oplus U = \mathcal{E} \oplus \mathcal{T} \oplus \mathcal{M}.$$

In particular, $\mathcal{T}$ contains elements of the form $a_1 \cup_2 \cdots \cup_2 a_n$, $a_i \in RH$, obtained by the iteration of the $\cup_2$-product for $n \geq 2$. In particular, for $a_i \in V^{0,2r}$ we have the following equality

$$d(a_1 \cup_2 \cdots \cup_2 a_n) = \sum_{(\mathbf{i};\mathbf{j})} sgn(\mathbf{i};\mathbf{j})(a_{i_1} \cup_2 \cdots \cup_2 a_{i_k}) \smile_1 (a_{j_1} \cup_2 \cdots \cup_2 a_{j_\ell}),$$

where the summation is over unshuffles $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k; j_1 < \cdots < j_\ell)$ of $\underline{n}$ with $(a_{i_1}, \ldots, a_{i_k}) = (a_{i'_1}, \ldots, a_{i'_k})$ if and only if $\mathbf{i} = \mathbf{i}'$ and $sgn(\mathbf{i};\mathbf{j})$ is induced by the permutation sign $a_i \cup_2 a_j = (-1)^{|a_i||a_j|} a_j \cup_2 a_i$ (see also Fig. 1 for $n = 3$); consequently, for $a_1 = \cdots = a_n = a$ and $a^{\cup_2 n} := a \cup_2 \cdots \cup_2 a$, we get

$$da^{\cup_2 n} = \sum_{k+\ell=n} a^{\cup_2 k} \smile_1 a^{\cup_2 \ell}, \quad k, \ell \geq 1. \tag{2.8}$$

Note that the above equalities do not depend on the parity of $a_i$'s when $\nu = 2$.

**Remark 1.** 1. The definition of $\mathcal{T}$ does not depend on the (Hirsch) associativity of $RH$.

2. In a minimal Hirsch resolution one can also minimize the module $\mathcal{T}$ as

$$\mathcal{T} = \{a \cup_2 b \in U \mid a, b \in \mathcal{M}\},$$

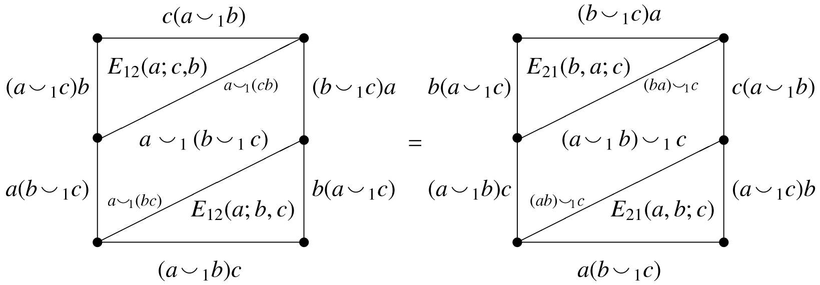
while $a \cup_2 b$ for $a, b \in RH$ is extended by certain *derivation* formulas. These formulas are rather complicated, but they could be written down if necessary.

3. The module $\mathcal{M}$ reflects the complexity of the multiplicative relations of the commutative algebra $H$.

For example, if $H$ is a polynomial algebra and $RH$ is a minimal Hirsch resolution, then $\mathcal{M} = \mathcal{M}^{0,*} = V^{0,*}$ and, consequently, $RH$ is completely determined by the $\smile_1$- and $\cup_2$-operations [31] (see also Theorem 4).

### 2.3. *Some canonical syzygies in the Hirsch resolution*

Below we give topological interpretation of some canonical syzygies in the Hirsch resolution $RH$. In particular these syzygies reflect the non-associativity of the $\smile_1$-product. Remark that higher order canonical syzygies should be also related with the combinatorics of permutahedra. In practice, such relations are helpful to construct small Hirsch resolutions $RH$ (cf. [31], see also Remark 1).



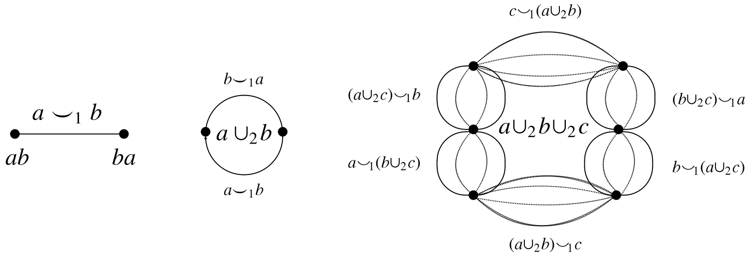




Fig. 1. Topological interpretation of some canonical syzygies in the Hirsch resolution $RH$.

The symbol "=" in the figure above assumes equality (2.6); the picture for $a\cup_2 b\cup_2 c$ is in fact 4-dimensional and must be understood as follows: Whence $a\cup_2 b$ corresponds to the 2-ball, the boundary of $a\cup_2 b\cup_2 c$ consists of the six 3-balls each of which is subdivided into four 3-cells by fixing two equators (these cells just correspond to the four summand components of the differential evaluated on the compositions of the $\smile_1$- and $\cup_2$-products). Then given a 3-ball, two cells from these four cells are glued to the ones of the boundary of the (diagonally) opposite 3-ball, and the other cells are glued to the ones of the boundaries of the neighboring 3-balls according to the relation

$$x\smile_1(y\smile_1 z)+(x\smile_1 y)\smile_1 z=y\smile_1(x\smile_1 z)+(y\smile_1 x)\smile_1 z.$$

## 2.4. *Filtered Hirsch model*

Recall that a dga $(A^*,d)$ is *multialgebra* if it is bigraded $A^n=\underset{n=i+j}{\oplus}A^{i,j}$, $i\leq 0$, $j\geq 0$, and $d=d^0+d^1+\cdots+d^n+\cdots$ with $d^n:A^{p,q}\to A^{p+n,q-n+1}$ [12]. A dga $A$ is bigraded via $A^{0,*}=A^*$ and $A^{i,*}=0$ for $i\neq 0$; consequently, $A$ is a multialgebra. A multialgebra $A$ is *homological* if $d^0=0$ (hence $d^1d^1=0$) and

$$H^i(\cdots\overset{d^1}{\to}A^{i,*}\overset{d^1}{\to}A^{i+1,*}\overset{d^1}{\to}\cdots\overset{d^1}{\to}A^{0,*})=0,\quad i<0.$$

For a homological multialgebra the sum $d^2+d^3+\cdots+d^n+\cdots$ is called a *perturbation* of $d^1$. In the sequel we always consider homological multialgebras, $d^1$ is denoted by $d$, $d^r$ is denoted by $h^r$, and the sum $h^2+h^3+\cdots+h^n+\cdots$ is denoted by $h$. We sometimes denote $d+h$ by $d_h$.

A *multialgebra morphism* $\zeta:A\to B$ between two multialgebras $A$ and $B$ is a dga map of total degree zero that preserves the resolution (column) filtration, so that $\zeta$ has the components $\zeta=\zeta^0+\cdots+\zeta^i+\cdots$, $\zeta^i:A^{s,t}\to B^{s+i,t-i}$. A chain homotopy $s:A\to B$ between two multiplicative maps $f,g:A\to B$ is an $(f,g)$-*derivation* homotopy if $s(ab)=s(a)g(b)+(-1)^{|a|}f(a)s(b)$. A *homotopy* between two morphisms $f,g:A\to B$ of multialgebras is an $(f,g)$-derivation homotopy $s:A\to B$ of total degree $-1$ that lowers the column filtration by 1.

A multialgebra is *quasi-free* if it is a tensor algebra over a bigraded $\Bbbk$-module. Given $m\geq 2$, the map $h^m|_{A^{-m,*}}:A^{-m,*}\to A^{0,*}$ is referred to as the *transgressive* component of $h$ and is denoted by $h^{tr}$. A multialgebra $A$ with a Hirsch algebra structure

$$E_{p,q}:\otimes_{r=1}^{p}A^{i_r,k_r}\bigotimes\otimes_{n=1}^{q}A^{j_k,\ell_n}\longrightarrow A^{s-p-q+1,t}$$

with $(s,t) = \left(i_{(p)} + j_{(q)}\,,\, k_{(p)} + \ell_{(q)}\right),\ p,q \geq 1$, is called *Hirsch multialgebra*. A *homotopy* between two morphisms $f,g : A \to A'$ of Hirsch (multi)algebras is a homotopy $s : A \to A'$ of underlying (multi)algebras and

$$\begin{aligned}
& s(E_{p,q}(a_1,\dots,a_p\,;b_1,\dots,b_q)) \\
&\quad = \sum_{1\le \ell\le q} (-1)^{\epsilon^a_p+\epsilon^b_{\ell-1}} E_{p,q}(fa_1,\dots,fa_p\,;fb_1,\dots,fb_{\ell-1},sb_\ell,gb_{\ell+1},\dots,gb_q) \\
&\qquad + \sum_{1\le k\le p} (-1)^{\epsilon^a_{k-1}} E_{p,q}(fa_1,\dots,fa_{k-1},sa_k,ga_{k+1},\dots,ga_p\,;gb_1,\dots,gb_q) \\
&\qquad - \sum_{\substack{1\le i\le p\\ 1<\ell\le j\le q}} (-1)^{\epsilon_{i,j,\ell}} E_{i,j}(fa_1,\dots,fa_i\,;fb_1,\dots,fb_{\ell-1},sb_\ell,gb_{\ell+1},\dots,gb_j) \\
&\qquad \times E_{p-i,q-j}(fa_{i+1},\dots,fa_{p-1},sa_p\,;gb_{j+1},\dots,gb_q) \\
&\qquad - \sum_{\substack{0\le i<k\le p\\ 1\le j\le q}} (-1)^{\epsilon_{i,j,k}} E_{i,j}(fa_1,\dots,fa_i\,;sb_1,gb_2,\dots,gb_j) \\
&\qquad \times E_{p-i,q-j}(fa_{i+1},\dots,fa_{k-1},sa_k,ga_{k+1},\dots,ga_p\,;gb_{j+1},\dots,gb_q), \qquad (2.9)\\
& \epsilon_{i,j,m} = \epsilon^a_{p-1} + \epsilon^b_{m-1} + (\epsilon^a_p + \epsilon^a_i)\epsilon^b_j, \quad p,q\ge 1,
\end{aligned}$$

in which the first equality is

$$s(a \smile_1 b) = (-1)^{|a|+1} fa \smile_1 sb + sa \smile_1 gb - (-1)^{(|a|+1)(|b|+1)} sb\cdot sa.$$

Denote the homotopy classes of morphisms between two Hirsch (multi)algebras by $[-,-]$.

**Definition 5.** A quasi-free Hirsch homological multialgebra $(A, \{E_{p,q}\}, d+h)$ is a **filtered Hirsch algebra** if it has the following additional properties:

(i) In $A = T(V)$ a decomposition

$$V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$$

is fixed where $\mathcal{E}^{*,*} = \bigoplus_{p,q\ge 1} \mathcal{E}^{<0,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \xrightarrow{\approx} \mathcal{E}_{p,q} \subset V, \quad p,q \ge 1;$$

(ii) The restriction of the perturbation $h$ to $\mathcal{E}$ has no transgressive components $h^{tr}$, i.e., $h^{tr}|_{\mathcal{E}} = 0$.

Given a Hirsch algebra $B$, a *filtered Hirsch model* for $B$ is a filtered Hirsch algebra $A$ together with a Hirsch algebra map $A \to B$ that induces an isomorphism on cohomology. Our next proposition, which is a Adams–Hilton type of statement, exhibits a basic property of filtered Hirsch algebras:

**Proposition 4.** *Let $\zeta : B \to C$ be a map of (filtered) Hirsch algebras that induces an isomorphism on cohomology. If $A$ is a filtered Hirsch algebra, there is a bijection of sets of homotopy classes of (filtered) Hirsch algebra maps*

$$\zeta_\# : [A,B] \xrightarrow{\approx} [A,C].$$

**Proof.** Discarding Hirsch algebra structures, the proof goes by induction on the resolution grading and is similar to that of Theorem 2.5 in [12] (see also [28]). The Hirsch algebra structure serves to specify a choice of homotopy $s$ on the multiplicative generators $\mathcal{E} \subset V$. When constructing a chain homotopy $s : A \to C$ between two multiplicative maps $f,g : A \to C$, we can choose an $s$ on $\mathcal{E}^{i,*}$ that satisfies formula (2.9) in each step of the induction. □

The basic examples of a filtered Hirsch algebra are provided by the following theorem, which states our main result on Hirsch algebras:

**Theorem 1.** *Let $H$ be a cga and let $\rho : (RH, d) \to H$ be an absolute Hirsch resolution. Given a Hirsch algebra $A$, assume there exists an isomorphism $i_A : H \approx H(A, d)$. Then*

(i) *Existence. There is a pair $(h, f)$ where $h : RH \to RH$ is a perturbation of the resolution differential $d$ on $RH$ and*

$$f : (RH, d + h) \to A$$

*is a filtered Hirsch model of $A$ such that $(f|_{R^0H})^* = i_A \rho|_{R^0H} : R^0H \to H(A)$.*

(ii) *Uniqueness. If $(\bar{h}, \bar{f})$ and $\bar{f} : (RH, d + \bar{h}) \to A$ satisfy the conditions of* (i)*, there is an isomorphism of filtered Hirsch models*

$$\zeta : (RH, d + h) \xrightarrow{\approx} (RH, d + \bar{h})$$

*of the form $\zeta = Id + \zeta^1 + \cdots + \zeta^r + \cdots$ with $\zeta^r : R^{-s}H^t \to R^{-s+r}H^{t-r}$ such that $f$ is homotopic to $\bar{f} \circ \zeta$.*

Note that the proof of the theorem uses an induction on resolution grading as it is used by the construction of filtered model due to Halperin–Stasheff [11] (compare also [27,28]); although in the rational case for the existence and the uniqueness of a pair $(h, f)$ the zero characteristic of $\Bbbk$ is essentially involved, the proof below shows that such a restriction can be simply avoided. Here a technical subtlety is that we have certain canonically chosen multiplicative generators on which $(h, f)$ must act by a canonical rule.

**Proof.** *Existence.* Let $RH = T(V)$ with $V = \mathcal{E} \oplus U$. We define a perturbation $h$ and a Hirsch algebra map $f : (RH, d+h) \to (A, d)$ by induction on resolution (column) grading. First consider $R^0H = T(V^{0,*})\,(= T(U^{0,*}))$. Define a chain map $\mathfrak{f}^0 : (V^{0,*}, 0) \to (A, d)$ by $(\mathfrak{f}^0)^* = i_A\rho|_{V^{0,*}} : V^{0,*} \to H(A)$. Extend $\mathfrak{f}^0$ multiplicatively to obtain a dga map $f^0 : R^0H \to A$. There is a map $\mathfrak{f}^1 : V^{-1,*} \to A^{*-1}$ with $f^0d|_{V^{-1,*}} = d\mathfrak{f}^1$; in particular, choose $\mathfrak{f}^1$ on $\mathcal{E}^{-1,*}\,(=\mathcal{E}^{-1,*}_{1,1})$ defined by the formula $\mathfrak{f}^1(a \smile_1 b) = f^0a \smile_1 f^0b$ for $a, b \in R^0H$. Then extend $\mathfrak{f}^0 + \mathfrak{f}^1$ multiplicatively to obtain a dga map $\mathfrak{f}^{(1)}_{\#} : T(V^{(-1),*}) \to (A, d)$; then denote the restriction of $\mathfrak{f}^{(1)}_{\#}$ to $R^{(-1)}H$ by $f^{(1)} : (R^{(-1)}H, d) \to (A, d)$.

Inductively, assume that a pair $(h^{(n)}, f^{(n)})$ has been constructed that satisfies the following conditions:

(1) $h^{(n)} = h^2 + \cdots + h^n$ is a derivation on $RH$,

(2) Equality (2.2) holds on $R^{(-n)}H$ for $d + h^{(n)}$ in which

$$\begin{aligned} h^rE_{p,q}(a_1, \ldots, a_p\,; b_1, \ldots, b_q) = {} & \sum_{i=1}^{p} (-1)^{\epsilon^a_{i-1}} E_{p,q}(a_1, \ldots, h^ra_i, \ldots, a_q\,; b_1, \ldots, b_q) \\ & + \sum_{j=1}^{q} (-1)^{\epsilon^a_p + \epsilon^b_{j-1}} E_{p,q}(a_1, \ldots, a_p\,; b_1, \ldots, h^rb_j, \ldots, b_q), \\ & 2 \le r \le n, \end{aligned}$$

(3) $dh^n + h^nd + \sum_{i+j=n+1} h^ih^j = 0$,

(4) $f^{(n)} : R^{(-n)}H \to A$ is the restriction of a dga map $\mathfrak{f}^{(n)}_{\#} : T\big(V^{(-n),*}\big) \to A$ to $R^{(-n)}H$ for $f^{(n)} = f^0 + \cdots + f^n$;

(5) $f^{(n)}(d + h^{(n)}) = df^{(n)}$ on $R^{(-n)}H$, and

(6) $f^{(n)}$ is compatible with the maps $E_{p,q}$ on $\mathcal{E}^{(-n),*}$.

Consider

$$f^{(n)}(d + h^{(n)})|_{V^{-n-1,*}} : V^{-n-1,*} \to A^{*-n-1};$$

clearly $df^{(n)}(d + h^{(n)}) = 0$. Define a linear map $h^{n+1} : U^{-n-1,*} \to R^0H^{*-n}$ with $\rho h^{n+1} = i_A^{-1}[f^{(n)}(d + h^{(n)})]$ and extend $h^{n+1}$ on $RH$ as a derivation (denoting by the same symbol) with

$$dh^{n+1} + h^{n+1}d + \sum_{i+j=n+2} h^ih^j = 0$$

and

$$h^{n+1}E_{p,q}(a_1,\ldots,a_p\,;b_1,\ldots,b_q) = \sum_{i=1}^{p}(-1)^{\epsilon^a_{i-1}}E_{p,q}(a_1,\ldots,h^{n+1}a_i,\ldots,a_q\,;b_1,\ldots,b_q)$$
$$+\sum_{j=1}^{q}(-1)^{\epsilon^a_p+\epsilon^b_{j-1}}E_{p,q}(a_1,\ldots,a_p\,;b_1,\ldots,h^{n+1}b_j,\ldots,b_q).$$

Then there is a map $\mathfrak{f}^{n+1}: V^{-n-1,*}\to A^{*-n-1}$ such that it is compatible with $E_{p,q}$ on $\mathcal{E}^{-n-1,*}$ and

$$f^{(n)}(d+h^{(n+1)})|_{V^{-n-1,*}} = d\mathfrak{f}^{n+1}.$$

Extend $\mathfrak{f}^{(n+1)} := \mathfrak{f}^0+\cdots+\mathfrak{f}^{n+1}$ multiplicatively to obtain a dga map $\mathfrak{f}^{(n+1)}_{\#}: T(V^{(-n-1),*})\to A$; the restriction of $\mathfrak{f}^{(n+1)}_{\#}$ to $R^{(-n-1)}H$ is denoted by

$$f^{(n+1)}: R^{(-n-1)}H\to A.$$

Thus the construction of the pair $\left(h^{(n+1)}, f^{(n+1)}\right)$ completes the inductive step. Finally, a perturbation $h = h^2+\cdots+h^n+\cdots$ and a Hirsch algebra map $f$ such that $f = f^0+\cdots+f^n+\cdots$ are obtained as desired.

*Uniqueness.* Using Proposition 4 we construct a multialgebra morphism

$$\zeta : (RH, d+h)\to (RH, d+\bar{h}),$$

$\zeta = \zeta^0+\zeta^1+\cdots$, with $\bar{f}\circ\zeta\simeq f$; in addition, it is easy to choose $\zeta$ with $\zeta^0 = Id$. □

### 2.5. *Filtered model for a QHHA*

Referring to Section 2.2, this section considers the compatibility of the perturbation $h$ and the Hirsch map $f$ with the $\cup_2$-product of $RH$ in Theorem 1. Even if $A$ is a QHHA in the theorem, it is impossible to obtain a *QHHA map* $f$ which commutes with $\cup_2$-products because the compatibility of parameters $q(-;-)$ under $f$ is obstructed. When $A$ is a $\mathbb{Z}_2$-algebra, for example, the obstruction is caused by the non-free action of $Sq_1$ on $H$. However, when $q(-;-)=0$ for the $\cup_2$-operation in $A$ (cf. Example 2), one can refine the perturbation $h$ in Theorem 1 as it is stated in Proposition 5 (in particular, item (i) of this proposition is an essential detail of the proof of the main result in [33]).

Let $\mathtt{T}\subset\mathcal{T}$ be a submodule defined by

$$\mathtt{T} = \langle a\cup_2 b\in\mathcal{T}\mid a\neq b \text{ in a basis of } \mathcal{M}\rangle.$$

For $\nu = 2$, let $Sq_1: H^m(A)\to H^{2m-1}(A)$ be the map from Example 4.

**Proposition 5.** *Let $A$ be a QHHA with $\cup_2$-operation satisfying (2.5) (e.g. $A$ is a special Hirsch algebra from Example 2). Then in the filtered Hirsch model $f:(RH,d_h)\to A$ given by Theorem 1, the perturbation $h$ can be chosen such that*

(i) $h^{tr}|_{\mathtt{T}} = 0$;

(ii) *Let $\nu = 2$. Then for $z_i = h^{tr}(a^{\cup_2 2^i})$ with $a\in R^0H$,*

$$\rho z_1 = Sq_1(\rho a) \quad\text{and}\quad h(a^{\cup_2 2^n}) = \sum_{1\le i<n} z_i\cup_2 a^{\cup_2(2^n-2i)} + z_n.$$

**Proof.** (i) First, remark that any element of $\mathtt{T}$ satisfies (2.5) (cf. (2.7)). Following the construction of a pair $(h, f)$ in the proof of Theorem 1, define $f$ for $a\cup_2 b\in\mathtt{T}^{-2,*}$ with $a, b\in\mathcal{V}^{0,*}$ by the formula

$$f(a\cup_2 b) = fa\cup_2 fb. \tag{2.10}$$

Since (2.5), $f$ is chain with respect to the resolution differential $d$ of $RH$, so we can set $h^2(a\cup_2 b) = 0$. Inductively, assume that for $a\cup_2 b\in\mathtt{T}^{-r,*}$, $2\le r<n$, the map $f$ is defined by (2.10), while $h$ is defined by

$$h(a\cup_2 b) = ha\cup_2 b + (-1)^{|a|}a\cup_2 hb. \tag{2.11}$$

Then for $a \cup_2 b \in \mathrm{T}^{-n,*}$ define $h$ again by (2.11). Clearly, $f d_h(a \cup_2 b)$ is a cocycle in $A$ and is bounded by $fa \cup_2 fb$. Therefore, we can define $f$ on $a \cup_2 b$ by (2.10). Consequently, we set $h^{tr}(a \cup_2 b) = 0$ as required.

(ii) Since $f$ is a Hirsch map, it commutes with $\smile_1$-products and the first equality follows from the definition of $Sq_1$. The verification of the second equality follows immediately from (2.8). □

**Remark 2.** Whereas $Sq_1$ induces the product on $H(BA)$, the transgressive values $z_i$ in item (ii) of Proposition 5 are closely related with the existence of the symmetric Massey products of the element $\sigma^*(\rho a) \in H(BA)$ for the suspension map $\sigma^* : H^*(A) \to H^{*-1}(BA)$ (compare Theorem 3 and Remark 7): When $\sigma^*(\rho z_k) = 0$ for $k < i$ (e.g. $z_k \in \mathcal{D}^{0,*}$), the cohomology class $\sigma^*(\rho z_i)$ is automatically identified with the symmetric Massey product $\langle \sigma^*(\rho a)\rangle^{2^i}$.

Unlike Example 1, the Hirsch algebra $A$ provided by the following example does not have a $\smile_2$-product. This fact allows us to lift a combination $a \smile_1 b \pm b \smile_1 a$ for cocycles $a, b \in A$ to the cohomology level as a non-trivial (binary) product (see also Section 3.4).

**Example 5.** It is known that the Hochschild cochain complex $C^\bullet(P;P)$ of an associative algebra $P$ admits an HGA structure [17,8], which is a particular Hirsch algebra. Furthermore, whereas the Hochschild cohomology $H = H(C^\bullet(P;P))$ is a cga, $H$ is also endowed with the binary operation $x * y$ defined for $x = [a]$ and $y = [b]$ by $x * y = [a \circ b - (-1)^{(|a|+1)(|b|+1)} b \circ a]$, where $\circ (= \smile_1)$ is Gerstenhaber's operation on the Hochschild cochain complex. The $*$ product on the Hochschild cohomology is referred to as the G-algebra structure. Since $H$ is a cga, we can apply Theorem 1 for $A = C^\bullet(P;P)$ and obtain the filtered Hirsch model $f : (RH, d+h) \to C^\bullet(P;P)$. Given $a, b \in V^{0,*}$, obviously we have $\rho h^2(a \cup_2 b) = \rho a * \rho b$ (since $f^1(a \smile_1 b) = f^0 a \circ f^0 b$). In other words, the non-triviality of the G-algebra structure on $H$ implies the non-triviality of perturbation $h^2$ restricted to the submodule $\mathcal{T} \subset V$. Consequently, the operation $a \cup_2 b$ with $q(a,b)$ satisfying item $(2.4)_2$ does not exist on the filtered Hirsch model of $C^\bullet(P;P)$ in general.

### 2.6. A small Hirsch resolution $R_\varsigma H$

Let $A$ be a Hirsch algebra over $\Bbbk$. Whereas $(RH, d_h) = (T(V), d_h)$ in a filtered Hirsch model $f : (RH, d_h) \to A$, the calculation of $H(BA)$ can be carried out in terms of $V$ as follows. Denote $\bar V = s^{-1}(V^{>0}) \oplus \Bbbk$ and define the differential $\bar d_h$ on $\bar V$ by the restriction of $d+h$ to $V$ to obtain the cochain complex $(\bar V, \bar d_h)$. There are isomorphisms

$$H^*(\bar V, \bar d_h) \approx H^*(B(RH), d_{B(RH)}) \overset{Bf^*}{\approx} H^*(BA, d_{BA}) \approx Tor^A(\Bbbk; \Bbbk). \tag{2.12}$$

In particular, for $A = C^*(X;\Bbbk)$ with $X$ simply connected (cf. Example 1),

$$H^*(\bar V, \bar d_h) \approx H^*(BC^*(X;\Bbbk), d_{BC}) \approx H^*(\Omega X;\Bbbk).$$

**Remark 3.** Note that the first isomorphism of (2.12) is a consequence of a general fact about tensor algebras [6], while the second follows from Proposition 1.

Furthermore, to conveniently involve the multiplicative structure of (2.12), one can reduce $V$ at the cost of $\mathcal{E} \subset V$ in the manner we shall describe. Let $J_\varsigma \subset R_a H$ be the Hirsch ideal of an absolute Hirsch resolution $R_a H$ generated by

$$\{E_{p,q}(a_1,\dots,a_p;a_{p+1},\dots,a_{p+q}),\ dE_{p,q}(a_1,\dots,a_p;a_{p+1},\dots,a_{p+q}) \mid p+q \ge 3\}$$

with

$$\begin{array}{ll} a_1,\dots,a_p \in R_aH,\ a_{p+1} \in V, & p \ge 1,\ q = 1 \\ a_1,\dots,a_{p+q} \in R_aH, & p \ge 1,\ q > 1. \end{array}$$

Then

$$R_\varsigma H = R_a H / J_\varsigma$$

is a Hirsch resolution of $H$. Indeed, using (2.2) we see that $d : J_\varsigma \to J_\varsigma$ and $H(J_\varsigma, d) = 0$. Thus $g_\varsigma : (R_aH, d) \to (R_\varsigma H, d)$ is a homology isomorphism. We have an obvious projection $\rho_\varsigma : (R_\varsigma H, d) \to H$ such that $\rho = \rho_\varsigma \circ g_\varsigma$.

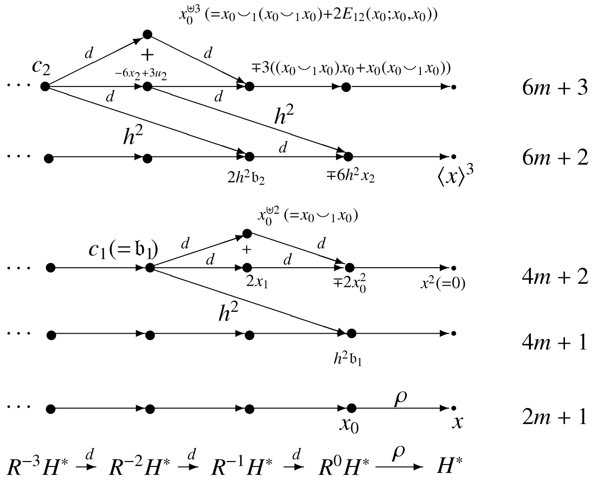



Fig. 2. A fragment of the filtered Hirsch $\mathbb{Z}$-algebra obtained as a perturbed resolution $(RH, d+h)$ of a cga $H$.

Consequently, $\rho_\varsigma$ is also a resolution map. Furthermore, we have $h : J_\varsigma \to J_\varsigma$ so that $(R_\varsigma H, d_h)$ is a Hirsch algebra (in fact an HGA) and $g_\varsigma$ extends to a quasi-isomorphism of filtered Hirsch algebras

$$g_\varsigma : (R_a H, d_h) \to (R_\varsigma H, d_h). \tag{2.13}$$

Thus, the Hirsch (HGA) structure of $R_\varsigma H = T(V_\varsigma)$ is generated by the $\smile_1$-product and (2.2) is equivalent to the following two equalities:

1. *The (left) Hirsch formula.* For $a, b, c \in R_\varsigma H$:

$$c \smile_1 ab = (c \smile_1 a)b + (-1)^{(|c|+1)|a|} a(c \smile_1 b).$$

2. *The (right) generalized Hirsch formula.* For $a, b \in R_\varsigma H$ and $c \in V_\varsigma$ with $d_h(c) = \sum c_1 \cdots c_q,\ c_i \in V_\varsigma$:

$$ab \smile_1 c = \begin{cases} a(b \smile_1 c) + (-1)^{|b|(|c|+1)}(a \smile_1 c)\, b, & q = 1,\\ a(b \smile_1 c) + (-1)^{|b|(|c|+1)}(a \smile_1 c)\, b \\ \quad + \sum\limits_{1\le i<j\le q} (-1)^{\varepsilon}\, c_1 \cdots c_{i-1}(a \smile_1 c_i)\, c_{i+1} \cdots c_{j-1}(b \smile_1 c_j)\, c_{j+1} \cdots c_q, & q \ge 2, \end{cases} \tag{2.14}$$

where $\varepsilon = (|a|+1)\left(\epsilon^c_{i-1} + i + 1\right) - (|b|+1)\left(\epsilon^c_{j-1} + j\right)$.

**Remark 4.** First, Formula (2.14) can be thought of as a generalization of Adams' formula for the $\smile_1$-product in the cobar construction [1, p. 36] from $q = 2$ to any $q \ge 2$. Second, the usage of $R_\varsigma H$ shows that the multiplication $\mu^*_E$ on $H^*(BA) \approx H^*(\bar{V}_\varsigma, \bar{d}_h)$ is in fact determined only by the $\smile_1$-product on $V_\varsigma$.

Note that for any Hirsch resolution of $H$ considered here, and consequently for any filtered Hirsch model, the first two columns in Fig. 2 are the same.

## 3. Some examples and applications

In the discussion that follows we sometimes abuse notation and denote $R_\varsigma H$ by $RH$. As we mentioned in the introduction, certain applications of the above material are given in [31,32]. The applications that appear here are new.

### 3.1. Symmetric Massey products

Recall the definition of the $n$-fold symmetric Massey product $\langle x\rangle^n$ (cf. [23,25]). Let $x \in H(A)$ be an element for a dga $A$, and $x_0 \in A$ be a cocycle with $x = [x_0]$. Given $n \ge 3$, consider a sequence $(x_0, x_1, \ldots, x_{n-2})$ in $A$ such that

$$dx_k = \sum_{i+j=k-1} (-1)^{|x_i|+1} x_i x_j, \quad 1 \le k \le n-2; \tag{3.1}$$

in particular, $dx_1 = -(-1)^{|x_0|}x_0^2$, i.e., $x^2 = 0$. Then $\sum_{i+j=n-2}(-1)^{|x_i|+1}x_i x_j$ is a cocycle, and a subset of $H(A)$ formed by the classes of all such cocycles is denoted by $\langle x\rangle^n$. (In other words, the existence of a sequence $(x_0, x_1, \ldots, x_k, \ldots)$ satisfying (3.1) for all $k$ implies that $c := \sum_{k\ge 0} x_k$ is a *twisting* element in $A$ whenever this sum (possibly infinite) has a sense; an element $c \in A$ is twisting if $dc = \pm c \cdot c$; cf. [3].)

When $A = C^*(X; \mathbb{Z}_p)$ for $p$ to be an odd prime, and $x \in H^{2m+1}(X; \mathbb{Z}_p)$ is odd dimensional, the following formula is established in [23] (for the dual case see [22]):

$$\langle x\rangle^p = -\beta\mathcal{P}_1(x) \tag{3.2}$$

where $\mathcal{P}_1 : H^{2m+1}(X; \mathbb{Z}_p) \to H^{2mp+1}(X; \mathbb{Z}_p)$ is the Steenrod cohomology operation. Thus, the formulas in [23] and [22] involve the connection of the symmetric Massey products with the Steenrod and Dyer–Lashof (co)homology operations in their respective topological settings (cf. [25]). Below Theorem 3 emphasizes the algebraic content of these formulas and generalizes them using a filtered Hirsch model over the integers.

### 3.2. Massey syzygies in the Hirsch resolution

Let $(RH, d)$ be a Hirsch resolution of $H$. Given a sequence of relations of the form $da_i = \lambda b_i$ and

$$\begin{aligned} &du_i = (-1)^{|a_i|+1} a_i a_{i+1} + \lambda v_i, \qquad dv_i = (-1)^{|a_i|} b_i a_{i+1} + a_i b_{i+1}, \\ &a_i, u_i, v_i \in RH,\ \lambda \in \mathbb{Z}\setminus\{-1, 1\},\ 1 \le i < n, \end{aligned} \tag{3.3}$$

in $(RH, d)$, there are elements $u_{a_{i_1},\ldots,a_{i_k}} \in RH$, $3 \le k \le n$, defined in terms of syzygies that mimic the definition of $k$-fold Massey products arising from $k$-tuples $(a_{i_1}, \ldots, a_{i_k})$ [23]. Precisely, $u_{a_1,\ldots,a_n}$ is defined by

$$\begin{aligned} du_{a_1,\ldots,a_n} &= \sum_{0\le i<n} (-1)^{\epsilon_i^a} u_{a_1,\ldots,a_i} u_{a_{i+1},\ldots,a_n} + \lambda v_{a_1,\ldots,a_n}, \\ dv_{a_1,\ldots,a_n} &= \sum_{0\le i<n} ((-1)^{\epsilon_i^a+1} v_{a_1,\ldots,a_i} u_{a_{i+1},\ldots,a_n} + u_{a_1,\ldots,a_i} v_{a_{i+1},\ldots,a_n}), \end{aligned} \tag{3.4}$$

with the convention that $u_{a_i} = a_i$, $u_{a_i,a_{i+1}} = u_i$ and $v_{a_i} = b_i$, $v_{a_i,a_{i+1}} = v_i$. When $b_i = 0$, Eq. (3.4) reduces to

$$du_{a_1,\ldots,a_n} = \sum_{0\le i<n} (-1)^{\epsilon_i^a} u_{a_1,\ldots,a_i} u_{a_{i+1},\ldots,a_n}.$$

We are interested in the special case of (3.3) obtained by setting $a_1 = \cdots = a_n$. More precisely, we consider the following situation (see also Example 6).

Let $A$ be a torsion free Hirsch algebra over $\mathbb{Z}$ and fix a filtered model $f : (RH, d_h) \to A$. For a module $C$ over $\mathbb{Z}$, let $C_{\Bbbk} := C \otimes_{\mathbb{Z}} \Bbbk$ and let $t_{\Bbbk} : C \to C_{\Bbbk}$ be the standard map; then $A_{\Bbbk} = A \otimes_{\mathbb{Z}} \Bbbk$ and $RH_{\Bbbk} = RH \otimes_{\mathbb{Z}} \Bbbk$. Also let $H_{\Bbbk} := H(A_{\Bbbk})$. There is the Hirsch model of $(A_{\Bbbk}, d_{A_{\Bbbk}})$ given by

$$f_{\Bbbk} = f \otimes 1 : (RH_{\Bbbk}, d_h \otimes 1) \to (A_{\Bbbk}, d_{A_{\Bbbk}}).$$

Given an element $x \in H_{\Bbbk}$, let $x_0$ be a *representative* of $x$ in $RH$ so that $[t_{\Bbbk} f(x_0)] = x$. In particular, $x_0 \in R^0H^*$ for $\beta(x) = 0, k \ge 1$, and $x_0 \in R^{-1}H^*$ with $dx_0 = \lambda x_0'$, $x_0' \in R^0H^*$, for $\beta(x) \ne 0$, where $\beta$ denotes the Bockstein cohomology homomorphism associated with the sequence

$$0 \to \mathbb{Z}_\lambda \to \mathbb{Z}_{\lambda^2} \to \mathbb{Z}_\lambda \to 0.$$

If $x \in H = H^*(A)$, then obviously $x_0 \in R^0H^*$. In any case, assuming $x^2 = 0$ we have the corresponding relation in $(RH, d)$:

$$dx_1 = (-1)^{|x_0|+1}x_0^2 + \lambda x_1'$$

with the convention that $x_1' = 0$ whenever $x_0 \in R^0H^*$. This equality is a special case of (3.3), so (3.4) gives the following sequence of relations in $(RH, d)$:

$$dx_n = \sum_{\substack{i+j=n-1\\ i,j\geq 0}} (-1)^{|x_i|+1} x_i x_j + \lambda x_n', \quad n \geq 1, \tag{3.5}$$

where $x_n' = 0$ for $x_0 \in R^0H$.

We have the following description of Massey symmetric products in terms of the sequence $\mathbf{x} = \{x_n\}_{n\geq 0}$ in $(RH, d_h)$. Denote $y_i = t_{\Bbbk} x_i$ in $(RH_{\Bbbk}, d_h)$. If $hy_i = 0$ for $0 \leq i < n$, then (3.5) implies $d_h d(y_n) = dd(y_n) = 0$, and consequently, $[dy_n] = -[hy_n]$. Therefore

$$f_{\Bbbk}^*[dy_n] = -f_{\Bbbk}^*[hy_n] \in \langle x\rangle^{n+1}. \tag{3.6}$$

Furthermore, the elements $x_n$ appear in a family of relations in $(RH, d)$. For example, these relations can be deduced from the following observation. For $x \in H$ with $x^2 = 0$, let $\iota : BH \to B(RH, d)$ be a chain map such that $\iota([\bar{x}|\cdots|\bar{x}]) = (-1)^n[\overline{x_n}]$ for $[\bar{x}|\ldots|\bar{x}] \in B^{n+1}H$, $n \geq 0$. Assuming $BH$ is endowed with the shuffle product $sh_H$, the map $\iota$ will be multiplicative up to a chain homotopy $\mathfrak{b}$. Since $B(RH)$ is cofree, we can choose $\mathfrak{b}$ to be $(\mu_E \circ (\iota \otimes \iota), \iota \circ sh_H)$-coderivation. This observation easily extends to the mod $\lambda$ case when $x_0 \in R^{-1}H$ with $dx_0 = \lambda x_0'$. Now let

$$\bar{\mathfrak{b}}_{k,\ell} := \mathfrak{b}(\overbrace{[\bar{x}|\cdots|\bar{x}]}^{k} \otimes \overbrace{[\bar{x}|\cdots|\bar{x}]}^{\ell})\,|_{\overline{RH}} \quad \text{and} \quad i_{[n]} := i_1 + \cdots + i_n + n;$$

then the equality $\mu_E(\iota \otimes \iota) - \iota \circ sh_H = d_{B(RH)}\mathfrak{b} + \mathfrak{b}d_{BH\otimes BH}$ implies in $(RH, d)$:

For $|x_0|$ odd:

$$\begin{aligned} d\mathfrak{b}_{k,\ell} = {} & (-1)^{k+\ell}\binom{k+\ell}{k} x_{k+\ell-1} \\ & + \sum_{i_{[p]}=k,\ j_{[q]}=\ell} (-1)^{k+\ell+p+q} E_{p,q}(x_{i_1},\ldots,x_{i_p}; x_{j_1},\ldots,x_{j_q}) \\ & - \sum_{\substack{0\leq r<k, 0\leq m<\ell\\ i_{[s]}=r,\ j_{[t]}=m}} (-1)^{r+m} \Big((-1)^{s+t} E_{s,t}(x_{i_1},\ldots,x_{i_s}; x_{j_1},\ldots,x_{j_t})\mathfrak{b}_{k-r,\ell-m} \\ & + \binom{r+m}{r}\mathfrak{b}_{k-r,\ell-m}\, x_{r+m-1}\Big) + \lambda\mathfrak{b}_{k,\ell}' \end{aligned} \tag{3.7}$$

in which $\mathfrak{b}_{k,\ell}' = 0$ for $x_0 \in R^0H$, and the first equalities are:

$$\begin{aligned} & d\mathfrak{b}_{1,1} = 2x_1 + x_0 \smile_1 x_0 + \lambda\mathfrak{b}_{1,1}', \\ & d\mathfrak{b}_{2,1} = -3x_2 + E_{2,1}(x_0, x_0; x_0) - x_1 \smile_1 x_0 - x_0\mathfrak{b}_{1,1} + \mathfrak{b}_{1,1}x_0 + \lambda\mathfrak{b}_{2,1}', \\ & d\mathfrak{b}_{1,2} = -3x_2 + E_{1,2}(x_0; x_0, x_0) - x_0 \smile_1 x_1 - x_0\mathfrak{b}_{1,1} + \mathfrak{b}_{1,1}x_0 + \lambda\mathfrak{b}_{1,2}'. \end{aligned}$$

For $|x_0|$ even:

$$\begin{aligned} d\mathfrak{b}_{k,\ell} = {} & (-1)^{k+\ell}\alpha_{k,\ell}\, x_{k+\ell-1} \\ & + \sum_{i_{[p]}=k,\ j_{[q]}=\ell} (-1)^{k+\ell+p+q} E_{p,q}(x_{i_1},\ldots,x_{i_p}; x_{j_1},\ldots,x_{j_q}) \\ & - \sum_{\substack{0\leq r<k, 0\leq m<\ell\\ i_{[s]}=r,\ j_{[t]}=m}} \Big((-1)^{(k+r+1)m+s+r+t} E_{s,t}(x_{i_1},\ldots,x_{i_s}; x_{j_1},\ldots,x_{j_t})\mathfrak{b}_{k-r,\ell-m} \\ & + (-1)^{k+\ell+r(\ell+m)}\alpha_{r,m}\mathfrak{b}_{k-r,\ell-m}\, x_{r+m-1}\Big) + \lambda\mathfrak{b}_{k,\ell}', \end{aligned} \tag{3.8}$$

$$\alpha_{i,j}=\begin{cases}\binom{(i+j)/2}{i/2}, & i,j \text{ are even},\\ \binom{(i+j-1)/2}{i/2}, & i \text{ is even, } j \text{ is odd},\\ 0, & i,j \text{ are odd},\end{cases}$$

in which $\mathfrak{b}'_{k,\ell}=0$ for $x_0\in R^0H$, and the first equalities are:

$$\begin{aligned}&d\mathfrak{b}_{1,1}=x_0\smile_1 x_0+\lambda\mathfrak{b}'_{1,1}\quad (\text{i.e., } \mathfrak{b}_{1,1}=x_0\cup_2 x_0 \text{ when } x_0\in R^0H^*),\\&d\mathfrak{b}_{2,1}=-x_2+E_{2,1}(x_0,x_0;x_0)-x_1\smile_1 x_0-x_0\mathfrak{b}_{1,1}-\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{2,1},\\&d\mathfrak{b}_{1,2}=-x_2+E_{1,2}(x_0;x_0,x_0)-x_0\smile_1 x_1+x_0\mathfrak{b}_{1,1}+\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{1,2}.\end{aligned}$$

Of course, for the sake of minimality, one can choose only certain $\mathfrak{b}_{k,\ell}$ above to be nontrivial. For example, let $|x|$ be even, let $\mathfrak{b}_{2j+1}:=\mathfrak{b}_{1,2j+1}$, and set $x_{2n}$ in (3.5) as

$$x_{2n}=-x_0\smile_1 x_{2n-1}+\sum_{i+j=n-1}(x_{2i}\mathfrak{b}_{2j+1}-\mathfrak{b}_{2j+1}x_{2i}). \tag{3.9}$$

Thus one can also set $\mathfrak{b}_{1,2n}=0$ and eliminate $\mathfrak{b}_{1,2n}$ from (3.8); in particular, $\mathfrak{b}_{2,1}$ can be identified with $x_0\smile_2 x_1$ for $n=1$.

Note that for an HGA $A$ (e.g. $A=C^*(X;\mathbb{Z})$) we have that $E_{p,q}=0$ for all $q\geq 2$, that the second Hirsch formula up to homotopy from Section 2 becomes strict, and consequently, the formulas above are much simpler (see also Section 2.6).

**Theorem 2.** *Let $A$ be a Hirsch algebra over $\mathbb{Z}$ and let $\Bbbk$ be a field of characteristic $p\geq 0$.*

(i) *Let $x\in H(A)$ with $x^2=0$. If $\langle x\rangle^n$ is defined for $n\geq 3$, it has a finite order.*
(ii) *Let $x\in H_{\Bbbk}$ with $x^2=0$ and $p>0$. Then $\langle x\rangle^n$ is defined for $3\leq n\leq p$ and vanishes whenever $3\leq n<p$.*
(iii) *Let $x\in H_{\Bbbk}$ with $x^2=0$ and $p=0$. Then $\langle x\rangle^n$ is defined and vanishes for all $n$.*

**Proof.** (i) Observe that the inductive construction of the terms $h^r$, $r\geq 2$, of $h$ in $(RH,d_h)$ implies $hx_i=0$ for $0\leq i\leq n-2$ whenever $\langle x\rangle^n$ is defined. Apply formulas (3.7)–(3.8) to verify that $m\langle x\rangle^n=0$ with $m=n$ for $|x|$ odd (take $(k,\ell)=(1,n-1)$ in (3.7)), while $m=r-1$ or $m=r$ for $n=2r$ or $n=2r+1$ (take $(k,\ell)=(2,n-2)$ in (3.8)) for $|x|$ even.

(ii)–(iii) The proof follows an argument similar to that in (i). □

**Remark 5.** First, regarding Theorem 2, item (i), note that formula (3.9) implies that $\langle x\rangle^n=0$ whenever $|x|$ and $n$ are even. Second, if $|x|$ is odd, formulas (3.7) –(3.8) imply that whenever defined, $\langle x\rangle^n$ consists of a single cohomology class independent of the parity of $n$ (see [23,22]).

### *3.3. The Kraines formula*

Let $p:=\lambda$ be an odd prime. Let $a\in A^{2m+1}$ be an element with $da=0$ or $da=pa'$ for some $a'$. Given $n\geq 2$, take (the right most) $n$th-power of $\bar{a}\in\bar{A}$ under the $\mu_E$ product on $BA$ and consider its component in $\bar{A}$. Denote this component by $s^{-1}(a^{\uplus n})$ for $a^{\uplus n}\in A^{2mn+1}$. The element $a^{\uplus n}$ has the form

$$a^{\uplus n}=a^{\smile_1 n}+Q_n(a),$$

where $Q_n(a)$ is expressed in terms of $E_{1,k}$ for $1<k<n$ (for the relations of small degrees involving this power, see also Fig. 2). For example, $Q_2(a)=0$ since $a^{\uplus 2}=a^{\smile_1 2}$ and $Q_3(a)=2E_{1,2}(a;a,a)$. In particular, if $A$ is an HGA, then obviously $a^{\uplus n}=a^{\smile_1 n}$. Thus $da^{\uplus n}$ is divided by an integer $p\geq 2$ if and only if $p$ is a prime and $n=p^i$, some $i\geq 1$. Consequently, the homomorphism

$$\mathcal{P}_1: H^{2m+1}_{\mathbb{Z}_p}\to H^{2mp+1}_{\mathbb{Z}_p},\qquad [t_{\mathbb{Z}_p}(a)]\to[t_{\mathbb{Z}_p}(a^{\uplus p})],\quad a\in A,\ d(t_{\mathbb{Z}_p}(a))=0, \tag{3.10}$$

is well defined.

**Theorem 3.** *Let $A$ be a Hirsch algebra as in Proposition 5. Let $A$ be torsion free and $p$ be an odd prime. Then formula (3.2) holds in $H_{\mathbb{Z}_p}$ for $\mathcal{P}_1$ given by (3.10).*

**Proof.** Given $n \geq 1$, let $\mathfrak{b}_n := \mathfrak{b}_{1,n}$ and set $(k, \ell) = (1, n)$ in (3.7) to obtain

$$d\mathfrak{b}_n = (-1)^{n+1}\left((n+1)x_n - \sum_{\substack{j_{[q]}=n \\ 1\leq q\leq n}} (-1)^q E_{1,q}(x_0; x_{j_1}, \ldots, x_{j_q})\right) + \sum_{i+j=n-1} (-1)^i \left(\mathfrak{b}_j x_i - x_i \mathfrak{b}_j\right) + p\mathfrak{b}'_n. \quad (3.11)$$

By means of the element $x_0$ and the sequence $\{\mathfrak{b}_n\}_{n\geq 1}$, form the sequence $\{c_n\}_{n\geq 1}$ in $RH$ as follows:

$$c_1 = \mathfrak{b}_1 \quad \text{and} \quad c_n = n!\,\mathfrak{b}_n + x_0 \smile_1 c_{n-1}, \quad n \geq 2.$$

For $n = p-1$, relation (3.11) implies a relation of the form

$$dc_{p-1} = -p!\,x_{p-1} + x_0^{\uplus p} + pu_{p-1}, \quad (3.12)$$

where $u_{p-1} \in RH^+ \cdot RH^+$ for $\beta(x) = 0$, while $u_{p-1} = w_{p-1} + (p-1)!\,\mathfrak{b}'_{p-1}$ with $w_{p-1} \in RH^+ \cdot RH^+$ for $\beta(x) \neq 0$. Hence, from $d^2(c_{p-1}) = 0$ we get

$$d(x_0^{\uplus p}) = p!\,dx_{p-1} - p\,du_{p-1} = p((p-1)!\,dx_{p-1} - du_{p-1}).$$

Obviously, $h(x_0^{\uplus p}) = 0$ because $h(x_0) = 0$ (recall that a perturbation $h$ annihilates $R^{(-1)}H$ and is a derivation on $\mathcal{E}$). Consequently,

$$d_h(x_0^{\uplus p}) = p((p-1)!\,dx_{p-1} - du_{p-1}).$$

Taking into account $(p-1)! = -1 \bmod p$, and passing to $H_{\mathbb{Z}_p}$ we obtain

$$\beta\mathcal{P}_1(x) = f^*_{\mathbb{Z}_p}[-dy_{p-1} - dv_{p-1}] = -f^*_{\mathbb{Z}_p}[dy_{p-1}] - f^*_{\mathbb{Z}_p}[dv_{p-1}] \quad \text{for } v_{p-1} := t_{\mathbb{Z}_p}(u_{p-1}).$$

Since $f^*_{\mathbb{Z}_p}[dy_{p-1}] = \langle x\rangle^p$ by (3.6), it remains to show that $f^*_{\mathbb{Z}_p}[dv_{p-1}] = 0$. Indeed, if $\beta(x) = 0$, then $x_0 \in R^0H$, $u_{p-1} \in RH^+ \cdot RH^+$, and $hv_{p-1} = 0$ by the similar argument as in the proof of Theorem 2 (ii). Consequently, $0 = f^*_{\mathbb{Z}_p}[-hv_{p-1}] = f^*_{\mathbb{Z}_p}[dv_{p-1}]$. If $\beta(x) \neq 0$, then $x_0 \in R^{-1}H$, and let $dx_0 = px'_0$. We have that $u_{p-1}$ contains $\mathfrak{b}'_{p-1}$ as a summand, and $hv_{p-1} = -h\mathfrak{b}'_{p-1}$. Denoting $z_0 = g_\varsigma(x_0)$ and $z'_0 = g_\varsigma(x'_0)$ in $(R_\varsigma, d_h)$ where $g_\varsigma$ is given by (2.13), we have that $g_\varsigma(x_0^{\uplus p}) = z_0^{\smile_1 p}$ and $g_\varsigma(h\mathfrak{b}'_{p-1})$ is mod $p$ cohomologous to

$$\sum_{0\leq i<p} z_0^{\smile_1 i} \smile_1 z'_0 \smile_1 z_0^{\smile_1 p-i-1}, \text{ a summand component of } d(z_0^{\smile_1 p}).$$

But this component bounds $\sum_{0\leq i\leq p-2} z_0^{\smile_1 i} \smile_1 (z_0 \cup_2 z'_0) \smile_1 z_0^{\smile_1 p-i-2}$ mod $p$ that finishes the proof. □

**Remark 6.** When $p = 2$ the relation $d(x_0 \smile_1 x_0) = -2x_0^2 + 2(x'_0 \smile_1 x_0 + x_0 \smile_1 x'_0)$ implies the Adem relation $Sq_0(a) = Sq^1 Sq_1(a)$ in $H_{\mathbb{Z}_2}$ thought of as the "Kraines formula" $\langle a\rangle^2 = a^2 = \beta Sq_1(a)$.

**Example 6.** Fix a Hirsch filtered model $f : (RH, d_h) \to A$ with $RH = T(V)$. Suppose that we are given a single relation

$$da = \lambda b, \quad a \in V^{-1,2k+1},\ b \in V^{0,2k+1},\ \lambda \geq 2,\ k \geq 1, \quad (3.13)$$

and deduce the following relations in $(RH, d)$: First, define $c \in V$ by

$$dc = \begin{cases} ab + \dfrac{\lambda}{2} b \smile_1 b, & \lambda \quad \text{is even} \\ 2ab + \lambda b \smile_1 b, & \lambda \quad \text{is odd.} \end{cases} \quad (3.14)$$

When $\lambda$ is odd, denote (cf. (3.3))

$$u_{2a,b} := -c, \qquad u_{b,2a} := c - 2a \smile_1 b \quad \text{and} \quad u_{2b,b} := 2ab + (\lambda - 1) b \smile_1 b$$

and obtain

$$du_{a,a} = -a^2 + \lambda v_{a,a}, \quad v_{a,a} = c - a \smile_1 b,$$
$$du_{a,2b,b} = -au_{2b,b} - u_{a,2b}b + \lambda v_{a,2b,b} = -2a^2 b - (\lambda - 1)a(b \smile_1 b) + cb + \lambda u_{b,2b,b},$$
$$du_{b,2a,b} = bu_{2a,b} - u_{b,2a}b + \lambda v_{b,2a,b} = bc - (c - 2a \smile_1 b)b + \lambda u_{b,2b,b},$$
$$du_{a,2a,b} = -au_{2a,b} + u_{2a,a}b + \lambda v_{a,2a,b},$$

where $v_{a,2b,b} = v_{b,2a,b} = u_{b,2b,b} = 2u_{b,b,b}$. Keeping in mind the fact that $d_h^2 = 0$, there is the following action of the perturbation $h$ on the relations above:

$$dh^2 u_{a,a} = -\lambda h^2 c,$$
$$dh^2 u_{a,2b,b} = -h^2 c \cdot b - \lambda h^2 u_{b,2b,b},$$
$$dh^2 u_{b,2a,b} = b \cdot h^2 c + h^2 c \cdot b - \lambda h^2 u_{b,2b,b},$$
$$dh^2 u_{a,2a,b} = -a \cdot h^2 c - 2h^2 u_{a,a} \cdot b - \lambda h^2 v_{a,2a,b},$$
$$dh^3 u_{a,2a,b} = -h^3 u_{2a,a} \cdot b - \lambda h^3 v_{a,2a,b} - h^2 h^2 u_{a,2a,b}.$$

Below we shall exploit the third equality in list of relations above. First, we have

$$d\left(h^2 u_{b,2a,b} + b \smile_1 h^2 c\right) = -\lambda h^2 u_{b,2b,b}.$$

Suppose that $\Bbbk$ is a ring such that $\nu$ divides $\lambda$ and

$$[t_{\Bbbk}(a)][t_{\Bbbk}(b)] = 0. \tag{3.15}$$

By (3.14) one has $[t_{\Bbbk}(ab)] = -[t_{\Bbbk} h^2 c]$, so that $h^2 c = 0 \bmod \nu$ above. Denoting $[t_{\Bbbk} f(a)] := y$ and $[t_{\Bbbk} f(b)] := x$, we have $xy = 0$ by (3.15). Thus the triple Massey product $\langle x, y, x\rangle$ is defined in $H_{\Bbbk}$ and contains $[t_{\Bbbk} f(bu_{a,b} - u_{b,a}b)]$ $(= -[t_{\Bbbk} f(hu_{b,a,b})])$. Obviously, $\langle x\rangle^3$ is also defined and

$$\beta_\lambda \langle x, y, x\rangle = -\langle x\rangle^3$$

(here $\beta_\lambda$ denotes the Bockstein map associated with $0 \to \mathbb{Z}_\nu \to \mathbb{Z}_{\nu\lambda} \to \mathbb{Z}_\lambda \to 0$). Now let $p = \lambda = 3$ and consider (3.12) for $x$. Then

$$c_2 = 2\mathfrak{b}_2 + x_0 \smile_1 \mathfrak{b}_1, \qquad x_0^{\uplus 3} = x_0^{\smile_1 3} + 2E_{1,2}(x_0; x_0, x_0), \qquad u_2 = \mathfrak{b}_1 x_0 - x_0 \mathfrak{b}_1$$

and

$$dc_2 = -6x_2 + x_0^{\smile_1 3} + 2E_{1,2}(x_0; x_0, x_0) + 3(\mathfrak{b}_1 x_0 - x_0 \mathfrak{b}_1).$$

Since $[x_0]^2 = 0$, one has $h^2 \mathfrak{b}_1 = 0$ and hence

$$hc_2 = 2(h^2 + h^3)\mathfrak{b}_2$$

(for the relations above, see also Fig. 2). In particular, $dh^2 c_2 = 6h^2 x_2$. Let $a := y_0$, $b := x_0$, $u_{b,b} := x_1$ and $u_{b,b,b} := x_2$ and set $h^2 c_2 = -2h^2 u_{x_0,y_0,x_0}$. Furthermore, if we also have $h^3 c_2 = h^3 u_{x_0,y_0,x_0} \bmod 3$, then $[t_{\Bbbk} f(x_0^{\uplus 3})] = -[t_{\Bbbk} f(hc_2)] = -[t_{\Bbbk} f(hu_{x_0,y_0,x_0})]$ and, consequently,

$$\mathcal{P}_1(x) \in \langle x, y, x\rangle. \tag{3.16}$$

For example, let $A = C^*(BF_4; \mathbb{Z}_3)$, the cochain complex of the classifying space $BF_4$ of the exceptional group $F_4$. Then equality (3.15) together with (3.16) holds in $H(BF_4; \mathbb{Z}_3)$. More precisely, let $x_i \in H^i(BF_4; \mathbb{Z}_3)$ be multiplicative generators in notation of [36] and recall the following relations among them: $x_8 x_9 = 0 = x_4 x_{21}$, $\delta x_8 = x_9$, $\delta x_{25} = x_{26}$; also $\mathcal{P}^3(x_9) = x_{21}$ and $\mathcal{P}^1(x_{21}) = x_{25}$; thus $\mathcal{P}^1\mathcal{P}^3(x_9) = \mathcal{P}_1(x_9) = x_{25}$ by an application of the Adem relation. Thus the knowledge of both $H^*(BF_4; \mathbb{Z}_3)$ and $H^*(F_4; \mathbb{Z}_3)$ in low degrees enables us to use the filtered Hirsch model of $BF_4$ to deduce the following: Let $a$ and $b$ be defined in (3.13) by $[t_{\mathbb{Z}_3} f(a)] = x_8$ and $[t_{\mathbb{Z}_3} f(b)] = x_9$. Then $[t_{\mathbb{Z}_3} f(hc_2)] = [t_{\mathbb{Z}_3} f(hu_{b,a,b})] = -x_{25}$ and $[t_{\mathbb{Z}_3} f(h^2 u_{b,b,b})] = x_{26}$ so that

$$\langle x_9\rangle^3 = -\beta\mathcal{P}_1(x_9) \quad \text{with } \mathcal{P}_1(x_9) = \langle x_9, x_8, x_9\rangle.$$

Finally, we remark that the both sides of this formula become trivial under the loop suspension map $\sigma^*$ : $H^*(BF_4;\mathbb{Z}_3) \to H^{*-1}(F_4;\mathbb{Z}_3)$ by a general well-known fact about Massey products [23,24] (compare $\mathcal{P}_1(i_3)$ for $i_3 \in H^3(K(\mathbb{Z}_3;3);\mathbb{Z}_3)$).

### 3.4. Hochschild cohomology with the G-algebra structure

In this section we assume that $\Bbbk$ is a field of characteristic zero. Refer to Example 5 and recall that the HGA structure $E = \{E_{p,q}\}_{p\geq 0;q=0,1}$ on the Hochschild cochain complex $A = C^\bullet(P;P)$ induces an associative product $\mu_E$ on the bar construction $BA$ and hence the product $\mu_E^*$ on $H^*(BA) = Tor_*^A(\Bbbk,\Bbbk)$. Since $Tor_*^A(\Bbbk,\Bbbk)$ is an associative algebra, it can be converted into a Lie algebra in the standard way.

**Theorem 4.** *If the Hochschild cohomology $H^* = H(C^\bullet(P;P))$ is a free algebra, then the Lie algebra structure on $Tor_*^A(\Bbbk,\Bbbk)$ is completely determined by that of the G-algebra $H^*$. Consequently, the product $\mu_E^*$ on $Tor_*^A(\Bbbk,\Bbbk)$ is commutative if and only if the G-product on $H^*$ is trivial.*

**Proof.** For a free algebra $H$, the module $\mathcal{M} \subset V$ has simple form in the (minimal) Hirsch resolution $(RH,d)$, i.e., $\mathcal{M}^{<0,*} = 0$. Indeed, given an odd dimensional multiplicative generator $x \in H$ and a representative $x_0 \in R^0H$ of $x$, the elements $x_n$ in the sequence (3.5) can be defined as $x_n = \frac{(-1)^n}{(n+1)!}x_0^{\smile_1 n+1}$ and hence $x_n \in \mathcal{E}$ for $n \geq 1$. In particular, there is a map of dg algebras $(RH,d) \to A$ and hence an isomorphism of dg coalgebras $H^*(BA) \approx H^*(BH)$ for a dga $A$ with $H = H^*(A)$ (a free $\Bbbk$-algebra $H$ is intrinsically $\Bbbk$-formal). Regarding the filtered Hirsch model $(RH,d_h)$, the perturbation $h$ may be non-zero only on $\mathcal{T}$. More precisely, according to Example 5 the cohomology class $[h(a\cup_2 b)] \in H^*(RH,d_h)$ is defined by $\rho a * \rho b \in H$ for $a,b \in V^{0,*}$. Since $H^*(BH) \approx H^*(BA) \approx H^*(\bar{V},\bar{d}_h)$ (cf. (2.12)), the multiplication $\mu_E^*$ on $H^*(BH)$ is induced by the $\smile_1$-product on $V$ (cf. Remark 3). Therefore, the Lie bracket on $H^*(BH)$ is determined by the bracket

$$[a,b] = a\smile_1 b - (-1)^{(|a|+1)(|b|+1)}b\smile_1 a$$

on $V$. The observation that $s^{-1}[a,b]$ is cohomologous to $s^{-1}h(a\cup_2 b)$ in $\bar{V}$ for all $a,b \in V^{0,*}$ completes the proof. □

**Remark 7.** Note that the transgressive component $h^{tr}$ evaluated on the elements $a_1\cup_2\cdots\cup_2 a_n \in \mathcal{T}$ for $a_i \in V^{0,*}$, $n \geq 3$, determines higher order operations on $Tor^A(\Bbbk;\Bbbk)$ that extend the Lie algebra structure to an $L_\infty$-algebra structure.

For example, a polynomial algebra $P = \Bbbk[x_1,\ldots,x_n]$ provides the case of $H^*$ in the theorem. Indeed, in general, to calculate the Hochschild cohomology of an algebra $P$ construct a small complex $(C_V^\bullet(P),\bar{d})$, which is quasi-isomorphic to $C^\bullet(P;P)$ as follows (compare [15]): Fix an ordinary multiplicative resolution $\rho : RP \to P$ with $RP = T(V)$, view $P$ as an $RP$-bimodule via $\rho$, and let $B(\rho)^\bullet : C^\bullet(P;P) \to C^\bullet(RP;P)$ be a quasi-isomorphism induced by $B(\rho) : B(RP) \to BP$. Set $(C_V^\bullet(P),\bar{d}) = (Hom(\bar{V},P),\bar{d})$ in which $\bar{d}$ is defined for $f \in C_V^\bullet(P)$ by $\bar{d}f = g$,

$$g(\bar{x}) = \sum_{1\leq i\leq k}(-1)^{\nu_i}\rho(v_1)\cdots f(\bar{v}_i)\cdots\rho(v_k), \qquad dx = \sum v_1\cdots v_k,\ v_i \in V, \quad k \geq 1,$$

$\nu_i = (|f|+1)(|v_1|+\cdots+|v_{i-1}|)$, and define a chain map $\chi : C_V^\bullet(P) \to C^\bullet(RP;P)$ by $\chi f = f'$,

$$f'(\bar{x}) = \begin{cases} f(\bar{x}), & x \in V,\\ \displaystyle\sum_{1\leq i\leq n}(-1)^{\nu_i}\rho(v_1)\cdots f(\bar{v}_i)\cdots\rho(v_n), & x = \sum v_1\cdots v_n,\ v_i \in V, n \geq 2.\end{cases}$$

Isomorphism (2.12) implies that $\chi$ is a homology isomorphism. On the other hand, the $\smile$-product on $C^\bullet(P;P)$ induces a $\smile$-product on $C_V^\bullet(P)$; more precisely, we have that $\bar{V}$ is a coalgebra with the coproduct $\bar{\Delta} : \bar{V} \to \bar{V}\otimes\bar{V}$ induced by the standard coproduct of $BP$ and, consequently, $Hom(\bar{V},P)$ is endowed with the standard $\smile$-product. When $P$ is polynomial, the minimal $V^*$ can be thought of as generated by the iterations of a (commutative) $\smile_1$-product [30]; consequently, $(\bar{V}^*,\bar{\Delta})$ is an exterior coalgebra. Dually, $\bar{V}_*$ is an exterior algebra on generators $\bar{x}_1,\ldots,\bar{x}_n$. Furthermore, $\bar{d} = 0$ and hence $H(C_V^\bullet(P),\bar{d}) = C_V^\bullet(P)$. Thus the Hochschild cohomology $H^*$ is isomorphic to the algebra $C^{\bullet}_{V}(P) \approx \bar{V}_{*-1} \otimes P^{*}$, which is the tensor product of an exterior algebra and a polynomial algebra, as required.

### *3.5. Symmetric Massey products in $C^*(X;\Bbbk)$ and powers in the loop homology $H_*(\Omega X;\Bbbk)$*

Let $A_*$ be a dg coalgebra over a field $\Bbbk$ and let $A^* = Hom(A_*, \Bbbk)$ be a dg algebra so that $H(A^*) = Hom(H(A_*), \Bbbk)$. Let

$$\iota : H(BA^*) \to Hom(H(\Omega A_*), \Bbbk)),$$

be the canonical map, where $\Omega A_*$ denotes the cobar construction of the coalgebra $A_*$. Given the suspension map $\sigma^* : H^*(A^*) \to H^{*-1}(BA^*)$, let $x \in H_*(A^*)$ and $y \in H_{*-1}(\Omega A_*)$, where $y$ is a basis element with $\iota(\sigma^* x)(y) = 1 \in \Bbbk$, and $\iota(\sigma^* x)(y') = 0$ for any basis element $y' \neq y$.

Suppose that $\langle x\rangle^n$ is defined for $x$. Let $\{a_i\}_{0\leq i<n}$ be a defining system of $\langle x\rangle^n$ with $a_0 \in A^*$ a representative cocycle of $x$. Then $\bar{a}_0 \in BA^*$ is a cocycle with $[\bar{a}_0] = \sigma^* x$ and $\{a_i\}_{0\leq i<n}$ lifts to a cocycle $a \in BA^*$ so that the cohomology class $[a] \in H^*(BA^*)$ is represented by the $y^n$ (the $n$th-power of $y$) in $H_*(\Omega A_*)$ via the map $\iota$. Then Theorem 2 immediately implies the following:

**Theorem 5.** *Let $X$ be a simply connected space, let $\Bbbk$ be a field of characteristic zero, and let $\sigma_* : H_*(\Omega X;\Bbbk) \to H_{*+1}(X;\Bbbk)$ be the suspension map. If $y \in H_*(\Omega X;\Bbbk)$ such that $y \notin \operatorname{Ker} \sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ in $H_*(\Omega X;\Bbbk)$ for all $n \geq 2$.*

Finally, recalling the connection between symmetric Massey products and twisting elements in $A^*$, which arise from the sequences $\{a_i\}_{i\geq 0}$ above, we remark that the observation above relates the existence of twisting elements in $A^*$ with the existence of polynomial generators in $H_*(\Omega A_*)$.

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