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The loop cohomology of a space with the polynomial cohomology algebra

Samson Saneblidze

A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University 6, Tamarashvili st., Tbilisi 0177, GA, United States

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To Jim Stasheff on the occasion of his 80th birthday

Abstract

Given a simply connected space X with polynomial cohomology $H^*(X;\mathbb{Z}_2)$, we calculate the loop cohomology algebra $H^*(\Omega X;\mathbb{Z}_2)$ by means of the action of the Steenrod cohomology operation Sq_1 on $H^*(X;\mathbb{Z}_2)$. This calculation uses an explicit construction of the minimal Hirsch filtered model of the cochain algebra $C^*(X;\mathbb{Z}_2)$. As a consequence we obtain that $H^*(\Omega X;\mathbb{Z}_2)$ is the exterior algebra if and only if Sq_1 is multiplicatively decomposable on $H^*(X;\mathbb{Z}_2)$. The last statement in fact contains a converse of a theorem of A. Borel (Switzer, 1975, Theorem 15.60).

Keywords: Loop space; Polynomial cohomology; Hirsch algebra; Multiplicative resolution; Steenrod operation

1. Introduction

Let X denote a simply connected topological space. The cohomology $H^*(X)$ is considered with coefficients $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$ unless otherwise specified explicitly. A. Borel gave a condition for $H^*(X)$ to be polynomial in terms of *a simple system of generators* of the loop space cohomology $H^*(\Omega X)$ that are transgressive [1, Theorem 15.60] , [2, p. 88] (see also [3]). This was one of the first nice applications of Leray–Serre spectral sequences [4], and led in particular to calculations of the cohomology of the Eilenberg–MacLane spaces (see [3]). For the converse direction, that is to determine $H^*(\Omega X)$ as an algebra for a given X with $H^*(X)$ polynomial, the first step is the existence of an additive isomorphism $H^*(\Omega X)\approx H^*(BH^*(X))$ where $BH^*(X)$ denotes the bar construction of $H^*(X)$ (cf. [5]). The module $BH^*(X)$ with the shuffle product is a graded differential algebra, but we get no algebra isomorphism above (cf. [6]). In general, a correct product on $BH^*(X)$ is induced by higher order operations on the cochain complex $C^*(X)$ (see below), but when $H^*(X)$ is polynomial we show that these operations reduce to the $\smile_1$-product on $C^*(X)$.

E-mail address: sane@rmi.ge.
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Consequently, the multiplicative structure of $BH^*(X)$ is determined by the Steenrod cohomology operation Sq_1 on $H^*(X)$. This reduction is beyond a spectral sequence argument.

In this paper we completely calculate the algebra $H^*(\Omega X)$ for $H^*(X)$ polynomial by means of Sq_1 on $H^*(X)$ (Theorem 1) and then establish the criterion for $H^*(\Omega X)$ to be exterior (Corollary 1). Namely, given $H^*(X) = H(C^*(X), d)$ with the $\smile_1$-product on $C^*(X)$, let

$$Sq_1 : H^n(X) \to H^{2n-1}(X) \quad [c] \to [c \smile_1 c], \quad c \in C^n(X), dc = 0.$$

Let now $H^*(X) = \mathbb{Z}_2[y_1, \dots, y_k, \dots]$ with $\mathcal{Y} = \{y_k\}$ to be a set of polynomial generators. Define a subset $\mathcal{S} \subseteq \mathcal{Y}$ as

$$\mathcal{S} = \{z_s \in \mathcal{Y} \mid z_s \notin \operatorname{Im} Sq_1 \bmod H^+ \cdot H^+\}.$$

Thus $\mathcal{S} = \mathcal{Y}$ if and only if $Sq_1(y_k) \in H^+ \cdot H^+$ for all k. Let $0 \leq \nu_i < \infty$ be the smallest integer such that $Sq_1^{(\nu_i+1)}(y_i) \in H^+ \cdot H^+$, where $Sq_1^{(m)}$ denotes the m-fold composition $Sq_1 \circ \cdots \circ Sq_1$. The integer ν_i is referred to as the *weak* $\smile_1$*-height* of y_i; when the finite integer ν_i does not exist, we say that y_i has the infinite weak $\smile_1$-height $\nu_i = \infty$. (This notion is motivated by the fact that Sq_1 induces a binary $\smile_1$-product on $(H^*(X), 0)$; cf. Remark 1(a).)

Let $\sigma : H^*(X) \to H^{*-1}(\Omega X)$ be the suspension homomorphism.

Theorem 1. *Let X be a simply connected space with $H^*(X) = \mathbb{Z}_2[y_1, \dots, y_k, \dots]$ and ν_k to be the weak $\smile_1$-height of y_k. Then the algebra $H^*(\Omega X)$ is multiplicatively generated by the elements $\bar{z}_s = \sigma z_s$ satisfying only the relations $\bar{z}_s^{m_s} = 0$ for $m_s = 2^{\nu_s+1}$ and $\bar{z}_{s_1}^{m_1} + \cdots + \bar{z}_{s_r}^{m_r} = 0$ for $Sq^{(n_1)}(z_{s_1}) + \cdots + Sq^{(n_r)}(z_{s_r}) \in H^+ \cdot H^+$, $m_i = 2^{n_i+1}$, $n_i \leq \nu_i$, $r \geq 2$, $z_{s_i} \in \mathcal{S}$.*

Corollary 1. *$H^*(\Omega X) = \Lambda(\bar{y}_1, \dots, \bar{y}_k, \dots)$ is the exterior algebra if and only if y_k is of zero weak $\smile_1$-height, i.e., $Sq_1(y_k) \in H^+ \cdot H^+$ for all k.*

When $\mathcal{Y}$ is chosen such that y_i is uniquely determined by the equality $Sq_1(y_i) = y_k \bmod H^+ \cdot H^+$, we get

Corollary 2. *$H^*(\Omega X) = \mathbb{Z}_2[\bar{z}_1, \dots, \bar{z}_s, \dots]$ is the polynomial algebra if and only if z_s is of the infinite weak $\smile_1$-height for all s.*

Our method of proving the theorem consists of using the *filtered Hirsch* model $(RH^*, d + h) \to C^*(X)$ of X [7] (see Section 2). Note that the underlying differential (bi)graded algebra (RH^*, d) is a non-commutative version of Tate–Jozefiak resolution of the commutative algebra H^* [8,9], while h is a perturbation of d similar to [10]. Furthermore, the tensor algebra $RH^* = T(V)$ is endowed with higher order operations $E = \{E_{p,q}\}$ that extend $\smile_1$-product measuring the non-commutativity of the product on RH^*; and there also is a binary operation $\cup_2$ on RH^* measuring the non-commutativity of the $\smile_1$-product. In general, by means of $(RH^*, d + h)$ one can recognize the cohomology $H(BC^*(X))$ of the bar construction $BC^*(X)$ as an algebra. The case of polynomial H^* is distinguished because of H^* has no multiplicative relations unless that of the commutativity; furthermore, we can equivalently take a small multiplicative resolution $R_\tau H^* = T(V_\tau)$ in which the Hirsch algebra structure is completely determined by commutative and associative $\smile_1$-product on V_τ. This allows an explicit calculation of the algebra $H(BC^*(X))$, and, consequently, of the loop space cohomology $H^*(\Omega X)$ in question.

Obviously the hypothesis of Corollary 1 is satisfied for an evenly graded polynomial algebra $H^*(X)$. Note that our method can be in fact applied to an evenly graded polynomial algebra $H^*(X; \Bbbk)$ for any coefficient ring $\Bbbk$ to establish that $H^*(\Omega X; \Bbbk)$ is exterior. Though, this fact can be also deduced from the Eilenberg–Moore spectral sequence (see, for example, [3]; for further references of spaces with polynomial cohomology rings see also [11,12]).

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2. Hirsch resolutions of polynomial algebras

We adopt the notations and terminology of [7] and briefly recall some facts. A Hirsch algebra $(A, d_A, \{E_{p,q}\})$ is an associative dga (A, d_A) equipped with multilinear maps

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A, \quad p, q \geq 0, \quad p + q > 0,$$

satisfying the following conditions:

(i) $\deg E_{p,q} = 1 - p - q$;
(ii) $E_{1,0} = Id = E_{0,1}$ and $E_{p>1,0} = 0 = E_{0,q>1}$;
(iii) The homomorphism $E : BA \otimes BA \to A$ defined by

$$E\left([\bar{a}_1|\cdots|\bar{a}_p] \otimes [\bar{b}_1|\cdots|\bar{b}_q]\right) = E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$$

is a twisting cochain in the dga $(Hom(BA \otimes BA, A), \nabla, \smile)$, i.e., $\nabla E = -E \smile E$.

A morphism $f : A \to B$ between two Hirsch algebras is a dga map f that commutes with $E_{p,q}$ for all p, q. Condition (iii) implies that $\mu_E : BA \otimes BA \to BA$ is a chain map; thus BA is a dg bialgebra; in particular, $\mu_{E_{10}+E_{01}}$ is the shuffle product on BA.

For a topological space X, there are operations $E = \{E_{p,q}\}$ on the cochain complex $C^*(X)$ making it into a Hirsch algebra. Note that in the simplicial case one can choose $E_{p,q} = 0$ for $q \geq 2$.

A dga (A^*, d) is multialgebra if it is bigraded $A^n = \underset{n=i+j}{\oplus} A^{i,j}$, $i \leq 0$, $j \geq 0$, and $d = d^0 + d^1 + \cdots + d^n + \cdots$ with $d^n : A^{p,q} \to A^{p+n,q-n+1}$. A dga A is bigraded via $A^{0,*} = A^*$ and $A^{i,*} = 0$ for $i \neq 0$; consequently, A is a multialgebra. A multialgebra A is homological if $d^0 = 0$ (hence $d^1 d^1 = 0$) and

$$H^i(\cdots \xrightarrow{d^1} A^{i,*} \xrightarrow{d^1} A^{i+1,*} \xrightarrow{d^1} \cdots \xrightarrow{d^1} A^{0,*}) = 0, \quad i < 0.$$

For a homological multialgebra the sum $d^2 + d^3 + \cdots + d^n + \cdots$ is called a perturbation of d^1. Furthermore, d^1 is denoted by d, d^r is denoted by h^r, and the sum $h^2 + h^3 + \cdots + h^n + \cdots$ is denoted by h. We sometimes denote $d + h$ by d_h.

A multialgebra is quasi-free if it is a tensor algebra over a bigraded $\Bbbk$-module. Given $m \geq 2$, the map $h^m|_{A^{-m,*}} : A^{-m,*} \to A^{0,*}$ is referred to as the transgressive component of h and is denoted by h^{tr}. A multialgebra A with a Hirsch algebra structure

$$E_{p,q} : \otimes_{r=1}^{p} A^{i_r,k_r} \bigotimes \otimes_{n=1}^{q} A^{j_k,\ell_n} \longrightarrow A^{s-p-q+1\,,\,t}$$

with $(s, t) = \left(i_{(p)} + j_{(q)}\,,\ k_{(p)} + \ell_{(q)}\right)$, $p, q \geq 1$, is called Hirsch multialgebra. A multialgebra is quasi-free if it is a tensor algebra over a bigraded $\Bbbk$-module. A quasi-free Hirsch homological multialgebra $(A, d+h, \{E_{p,q}\})$ is a filtered Hirsch algebra if it has the following additional properties:

(i) In $A = T(V)$ a decomposition

$$V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$$

is fixed where $\mathcal{E}^{*,*} = \underset{p,q\geq 1}{\oplus} \mathcal{E}_{p,q}^{<0,*}$ is distinguished by an isomorphism of modules

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \xrightarrow{\approx} \mathcal{E}_{p,q} \subset V, \quad p, q \geq 1;$$

(ii) The restriction of the perturbation h to $\mathcal{E}$ has no transgressive components h^{tr}, i.e., $h^{tr}|_{\mathcal{E}} = 0$.

An important example of a filtered Hirsch algebra is $A = (R^*H^*, d, \{E_{p,q}\})$, an absolute Hirsch resolution of a graded commutative algebra H^*. In particular, $R^*H^* = T(V)$ with

$$V = \bigoplus\nolimits_{j,m\geq 0} V^{-j,m},$$

where $V^{-j,m} \subset R^{-j}H^m$. The total degree of $R^{-j}H^m$ is the sum $-j+m$, d is of bidegree $(1, 0)$ and $\rho : (R^*H^*, d) \to H^*$ is a map of bigraded algebras inducing an isomorphism $\rho^* : H^*(RH, d) \xrightarrow{\approx} H^*$ where H^* is bigraded via $H^{0,*} = H^*$ and $H^{<0,*} = 0$.

Given a Hirsch algebra $(A, d_A, \{E_{p,q}\})$, a submodule $J \subset A$ is a *Hirsch ideal* of A if it is an ideal with $E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}) \in J$ whenever $a_i \in J$ for some i.

Let $\rho_a : (R_a^*H^*, d) \to H^*$ be an absolute Hirsch resolution and $J \subset R_a^*H^*$ be a Hirsch ideal such that $d : J \to J$ and the quotient map $g : R_a^*H^* \to R_a^*H^*/J$ is a homology isomorphism. A Hirsch resolution of H^* is the Hirsch algebra $R^*H^* = R_a^*H^*/J$ with a map $\rho : R^*H^* \to H^*$ such that $\rho_a = \rho \circ g$. Thus an absolute Hirsch resolution is a Hirsch resolution by taking $J = 0$.

Given a Hirsch algebra $(A, d_A, \{E_{p,q}\})$ with $H^* = H^*(A, d_A)$, there is a filtered Hirsch model

$$f : (R^*H^*, d_h) \to (A, d_A),$$

where R^*H^* denotes an absolute Hirsch resolution. There is a (commutative) binary operation $a\cup_2 b$ on R^*H^* satisfying for basis elements $a, b \in R^*H^*$ the equality

$$d(a\cup_2 b) = \begin{cases} a \cup_2 da + a \smile_1 a, & a = b, \\ a \smile_1 da + da \smile_1 a, & da = b, \\ da \cup_2 b + a\cup_2 db + a \smile_1 b + b \smile_1 a, & \text{otherwise.} \end{cases}$$

(Thus, the first two cases differ $\cup_2$ from the Steenrod $\smile_2$-operation.) In $U \subset V$ we distinguish a submodule $\mathcal{T}^{\leq -2,*} \subset U$ defined by

$$\mathcal{T}^{\leq -2,*} = \{a\cup_2 b \in R^*H^* \mid a\cup_2 b \in U\}.$$

For the sake of minimality of U one can express certain elements $a\cup_2 b \in R^*H^*$ in terms of the $\smile$ and $E_{p,q}$ operations. For example, $da\cup_2 da := a\smile_1 da + a \cdot a$, because $d(a \smile_1 da + a \cdot a) = da \smile_1 da$.

When $H^* = \mathbb{Z}_2[y_1, \ldots, y_k, \ldots]$ is polynomial, the module V is much simplified at the cost of U. Namely,

$$V^{*,*} = \mathcal{E}^{<0,*} \oplus U^{*,*} = \mathcal{E}^{<0,*} \oplus \mathcal{T}^{\leq -2,*} \oplus V^{0,*}.$$

In particular, we have that R^0H^* is a graded subalgebra in R^*H^* and $\operatorname{Ker}\rho \cap R^0H^*$ is an ideal in R^0H^*. Denoting the elements of $\mathcal{V}^{0,*}$ by x_k, i.e., $\rho x_k = y_k$, this ideal is generated by expressions of the form $x_ix_j + x_jx_i$ for $i \neq j$; thus, we get

$$V^{-1,*} = \mathcal{E}^{-1,*} = \langle x_i \smile_1 x_j \mid x_k \in \mathcal{V}^{0,*}\rangle \text{ with}$$
$$d(x_i\smile_1 x_j) = d(x_j\smile_1 x_i) = x_ix_j + x_jx_i \text{ for } i \neq j \text{ and } d(x_i\smile_1 x_i) = 0,$$

while

$$\mathcal{T}^{-2,*} = \langle x_i\cup_2 x_j\, (= x_j\cup_2 x_i) \mid x_k \in \mathcal{V}^{0,*}\rangle \text{ with } d(x_i\cup_2 x_j) =$$
$$x_i\smile_1 x_j + x_j \smile_1 x_i \text{ for } i \neq j, \text{ and } d(x_i\cup_2 x_i) = x_i\smile_1 x_i.$$

Here, we can minimize further both an absolute Hirsch resolution R^*H^* and a small Hirsch resolution $R^*_\varsigma H^*$ in [7] to obtain a *minimal* Hirsch resolution $R^*_\tau H^*$; moreover, we give an explicit construction of $R^*_\tau H^*$ below. Namely, set

$$R^*_\tau H^* = R^*H^*/J_\tau$$

where $J_\tau \subset R^*H^*$ is a Hirsch ideal generated by

$$\Big\{E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}),\ dE_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}),\ a\cup_2 b,\ d(a\cup_2 b) \mid$$
$$p + q \geq 3,\ a \neq b \text{ in } \mathcal{V}\Big\}$$

with

$$\begin{array}{lll} a_1, \ldots, a_p \in R^*H^*, \quad a_{p+1} \in V, & \text{for } p \geq 1 & \text{and } q = 1, \\ a_1, \ldots, a_{p+q} \in R^*H^*, & \text{for } p \geq 1 & \text{and } q > 1. \end{array}$$

Because of $d : J_\tau \to J_\tau$, we get a Hirsch algebra map $g_\tau : (R^*H^*, d) \to (R^*_\tau H^*, d)$. Let $\rho_\tau : R^*_\tau H^* \to H^*$ denote a map of bigraded algebras so that the resolution map $\rho : R^*H^* \to H^*$ factors as

$$\rho : (R^*H^*, d) \xrightarrow{g_\tau} (R^*_\tau H^*, d) \xrightarrow{\rho_\tau} H^*.$$

By definition we have $h : \mathcal{E} \to \mathcal{E}$; furthermore, because of the transgressive component h^{tr} of h annihilates $a\cup_2 b$ for $a \neq b$ in $\mathcal{V}$ (cf. [7, Proposition 5]), we get $h : J_\tau \to J_\tau$, too. Thus g_τ extends to a quasi-isomorphism of Hirsch algebras

$$g_\tau : (R^*H^*, d_h) \to (R^*_\tau H^*, d_h),$$

and, hence, A and $R^*_\tau H^*$ are connected via the diagram

$$(A, d_A) \xleftarrow{f} (R^*H^*, d_h) \xrightarrow{g_\tau} (R^*_\tau H^*, d_h).$$

The Hirsch algebra $(R^*_\tau H^*, d_h)$ can be described immediately. Namely, $R^*_\tau H^* = T(V^{*,*}_\tau)$ with $V^{*,*}_\tau = \langle \mathcal{V}^{*,*}_\tau \rangle$,

$$\mathcal{V}_\tau = \Big\{ x_i,\ x_j{}^{\cup_2 q},\ b_{i_1}\smile_1 \cdots \smile_1 b_{i_n} \mid b_{i_r} \in \{x_i, x_j{}^{\cup_2 q}\},\ q = 2^m,\ m \geq 1,\ n \geq 2, \\ x_k \in \mathcal{V}^{0,*}_\tau,\ x^{\cup_2 q} := x \cup_2 \cdots \cup_2 x \Big\}.$$

The $\smile_1$-product is commutative and associative on V_τ and extended on $R^*_\tau H^*$ by the (left) Hirsch formula

$$c \smile_1 ab = (c \smile_1 a)b + a(c \smile_1 b), \qquad a, b, c \in R^*_\tau H^*,$$

and the (right) generalized Hirsch formula

$$ab \smile_1 c = \begin{cases} a(b \smile_1 c) + (a \smile_1 c)b, & a, b \in R^*_\tau H^* \text{ and } c \in \{x_i, x_j{}^{\cup_2 q}\}, \\ & \qquad q = 2^m, m \geq 1, \\ a(b \smile_1 c) + (a \smile_1 c)b & \\ +(a \smile_1 c_1)(b \smile_1 c_2) & \\ +(a \smile_1 c_2)(b \smile_1 c_1), & a, b \in R^*_\tau H^* \text{ and } c = c_1 \smile_1 c_2 \in V_\tau. \end{cases}$$

The differential d on $R^*_\tau H^*$ is defined by

$$dx_k = 0, \quad d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + ab + ba \quad \text{and} \quad d(a \cup_2 a) = a \smile_1 a,$$

while the perturbation h by

$$hx_k = 0, \quad h(a \smile_1 b) = ha \smile_1 b + a \smile_1 hb$$

and

$$h(x_k \cup_2 x_k) = h^{tr}(x_k \cup_2 x_k) = b_k \quad \text{with} \quad b_k \in R^0_\tau H^* \quad \text{defined by} \quad \rho_\tau b_k = Sq_1(y_k).$$

Note that the value of h on $x_j{}^{\cup_2 2^m}$ for $m > 1$ may be non-zero (see Remark 1(b)). In particular, denoting

$$b_{k,1} := b_k,\ \ b_{k,j+1} := h(b_{k,j} \cup_2 b_{k,j}),\ j \geq 1,$$

and

$$c_0 = x_k \cup_2 x_k,\ \ c_j = x_k^{\smile_1 2^j} \smile_1 c_{j-1} + c_{j-1} \smile_1 b_{k,j} + b_{k,j} \cup_2 b_{k,j},\ j \geq 1,$$

one gets

$$d_h(c_{m-1}) = x_k^{\smile_1 2^m} + b_{k,m} \bmod R_\tau H^+ \cdot R_\tau H^+, m \geq 1, \text{ with } \rho_\tau b_{k,m} = Sq_1^{(m)}(y_k). \tag{2.1}$$

To ensure that $\rho_\tau : (R^*_\tau H^*, d) \to H^*$ is a multiplicative resolution of H^*, it suffices to verify the following.

Proposition 1. *The chain complex $(R^*_\tau H^*, d)$ is acyclic in the negative resolution degrees, i.e., $H^{i,*}(R^i_\tau H^*, d) = 0,\ i < 0$.*

Proof. First observe that as a cochain complex $\operatorname{Ker} \rho_\tau$ can be decomposed via $(\operatorname{Ker} \rho_\tau, d) = (A, d) \oplus (B, d)$ in which $(A, d) = \oplus (A(n), d),\ n \geq 2,\ A(n)$ has a basis consisting of all monomials formed by the $\smile$ and $\smile_1$ products evaluated on generators $x_{i_1}, \ldots, x_{i_n} \in V^{0,*}_\tau$ with distinct x_i's and B has a basis consisting of the other monomials in $\operatorname{Ker} \rho_\tau$. In particular, $(A(n), d)$ can be identified with the cellular chains of the permutohedron P_n (cf. [13]); thus A is acyclic and a chain contracting homotopy $s_A : A \to A$ can be chosen. To see that B is also acyclic, define a map $s_B : B \to B$ of degree -1 as follows. For $ba, ac, bac \in B$ with $a \in A$, let $s_B(ba) = bs_A(a), s_B(ac) = s_A(a)c, s_B(bac) = bs_A(a)c$; otherwise, for $b \smile_1 b$ and $b \smile_1 b \smile_1 c$ with $b, c \in V_\tau$, let $s_B(b \smile_1 b) = b \cup_2 b$ and $s_B(b \smile_1 b \smile_1 c) = b \cup_2 b \smile_1 c$, and then for a monomial $u = u_1 \cdots u_m \in B$, set

$$s_B(u) = \begin{cases} u_1 \cdots u_{i-1} \cdot s_B(u_i) \cdot u_{i+1} \cdots u_m, & u_i \in \{b \smile_1 b,\ b \smile_1 b \smile_1 c\} \text{ and} \\ & u_j \notin \{b \smile_1 b,\ b \smile_1 b \smile_1 c\}, 1 \leq j < i, \\ 0, & \text{otherwise.} \end{cases}$$

Then for each element $b \in B$ there is an integer $n(b) \geq 1$ such that $n(b)$th-iteration of the operator $s_B d + d s_B + Id : B \to B$ evaluated on b is zero, i.e., $(s_B d + d s_B + Id)^{(n(b))}(b) = 0$ as desired. $\square$

3. Proof of Theorem 1

Given the Hirsch algebra $(C^*(X), d_C, \{E_{p,q}\})$, there is an algebra isomorphism [14,15]

$$H^*(\Omega X) \approx H(BC^*(X), d_{BC}, \mu_E).$$

(We assume $C^*(X) = C^*(\mathrm{Sing}^1 X)/C^{>0}(\mathrm{Sing}\, x)$, in which $\mathrm{Sing}^1 X \subset \mathrm{Sing}\, X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point x of X.)

Proposition 2. *A morphism $g : A \to A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions*

$$Bg : BA \to BA'$$

and if g is a homology isomorphism, so is Bg.

Proof. The proof is standard by using a spectral sequence comparison argument. □

Denote $\bar{V}_\tau = s^{-1}(V_\tau^{>0}) \oplus \mathbb{Z}_2$ and define the differential $\bar{d}_h := \bar{d} + \bar{h}$ on $\bar{V}_\tau$ by the restriction of $d + h$ to V_τ to obtain the cochain complex $(\bar{V}_\tau, \bar{d}_h)$. Let $\psi : B(R_\tau H) \to \overline{R_\tau H} \to \bar{V}_\tau$ be the standard projection of cochain complexes. We introduce a product on $\bar{V}_\tau$ so that ψ becomes a map of dga's. Namely, for $\bar{a}, \bar{b} \in \bar{V}_\tau$ define

$$\bar{a}\bar{b} = \overline{a \smile_1 b} \quad \text{with} \quad \bar{a}1 = 1\bar{a} = \bar{a}.$$

Then we get the following sequence of algebra isomorphisms

$$H\big(BC^*(X), d_{BC}, \mu_E\big) \underset{\approx}{\overset{Bf^*}{\longleftarrow}} H\big(B(RH^*), d_{B(RH)}, \mu_E\big) \underset{\approx}{\overset{Bg_\tau^*}{\longrightarrow}} H\big(B(R_\tau H^*), d_{B(R_\tau H)}, \mu_{E_\tau}\big) \underset{\approx}{\overset{\psi^*}{\longrightarrow}} H\left(\bar{V}_\tau, \bar{d}_h\right),$$

where the first two isomorphisms are by Proposition 2, while the third isomorphism (additively) is a consequence of a general fact about tensor algebras [16] (see also [5]). Thus the calculation of the algebra $H^*(\Omega X)$ reduces to that of $H^*(\bar{V}_\tau, \bar{d}_h)$. In particular, $[\bar{x}_k] = \sigma(y_k) \in H^*(\Omega X)$. We have that $\bar{h}$ may be non-trivial only on a basis element of the form

$$s^{-1}(x_k{}^{\cup 2q}) \text{ and } s^{-1}(x_k{}^{\cup 2q}\smile_1 a), \quad \text{some } a \in V_\tau,\ q = 2^m,\ m \geq 1.$$

By definition $\bar{x}_k^q = s^{-1}(x_k^{\smile_1 q})$, $q = 2^m$, and taking into account (2.1), the cohomology algebra $H^*(\bar{V}_\tau, \bar{d}_h)$ is as desired.

Remark 1. (a) Refer to Example 4 from [7] and recall that there is a canonical Hirsch algebra structure $Sq = \{Sq_{p,q}\}$ on $H^*(X)$ determined by Sq_1. The isomorphism $H^*(\Omega X) \approx H^*(BH^*(X))$ from the introduction converts into an algebra one when $BH^*(X)$ is endowed with the product μ_{Sq}. Details are left to the interested reader.

(b) In $(\bar{V}_\tau, \bar{d}_h)$ the transgressive terms $\bar{h}^{tr}s^{-1}(x_i^{\cup 2q})$ detect the Symmetric Massey products $\langle\sigma(y_i)\rangle^q \in H^*(\Omega X)$ for $q = 2^m$, $y_i \in H^*(X)$, or, in general, Stasheff's A_∞-algebra structure on $H^*(\Omega X)$ (cf. [17]). A question arises what else other than the action of Sq_1 on $H^*(X)$ is needed to calculate this structure.

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