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INTERSECTION AND STRING TOPOLOGY PRODUCTS IN THE FREE LOOP FIBRATION

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Abstract. For an oriented closed triangulated manifold M we derive a cubical cellular structure on M so that $M = |X|$ is the geometric realisation of a cubical set X. We recover the intersection product on the homology $H_*(M)$ by defining the pairing on the cubical chains $C_*(X)$. We construct the permutahedral model $\widehat{\mathbf{\Lambda}}X$ for the free loop space ΛM, and lift the intersection pairing on the permutahedral chains $C_*^{\diamond}(\widehat{\mathbf{\Lambda}}X)$ to recover the string topology product on the free loop homology $H_*(\Lambda M)$ and to establish the compatibility condition with its standard coproduct.

1. Introduction

Let M be an oriented closed triangulated n-manifold, and fix the ground coefficient ring to be a field. A motivation of the paper is to establish relationship between the string topology product of degree $-n$

$$\mu_* : H_*(\Lambda M) \otimes H_*(\Lambda M) \to H_*(\Lambda M)$$

on the homology $H_*(\Lambda M)$ of the free loop space ΛM defined in [1] and the standard coproduct

$$\Delta_* : H_*(\Lambda M) \to H_*(\Lambda M) \otimes H_*(\Lambda M)$$

in fact defined for any topological space Y instead of ΛM. The problem naturally requires first to establish the compatibility condition between the classical intersection product

$$\cap_* : H_*(M) \otimes H_*(M) \to H_*(M)$$

and the coproduct Δ_* on the homology $H_*(M)$. In this way we first define $\cap_*$ as induced by a pairing of degree $-n$

$$\cap_{\#} : C_p(K^{\square}) \otimes C_q(K^{\square}) \to C_{p+q-n}(K^{\square}),$$

where $K^{\square}$ is a cubical subdivision of M canonically derived from a triangulation K of M. Consequently, without using the Poincaré isomorphism $H_i(M) \overset{\approx}{\to} H^{n-i}(M)$, we establish that $\cap_*$ is a map of $H_*(M)$-bicomodules:

Theorem 1. *The following diagrams*

$$\begin{array}{ccc} H_*(M) \otimes H_*(M) & \xleftarrow{\cap_* \otimes 1} & H_*(M) \otimes H_*(M) \otimes H_*(M) \\ {\scriptstyle \Delta_*} \uparrow & & \uparrow {\scriptstyle (1\otimes T)\circ(\Delta_* \otimes 1)} \\ H_*(M) & \xleftarrow{\cap_*} & H_*(M) \otimes H_*(M) \end{array} \tag{1.1}$$

and

$$\begin{array}{ccc} H_*(M) \otimes H_*(M) & \xleftarrow{1 \otimes \cap_*} & H_*(M) \otimes H_*(M) \otimes H_*(M) \\ {\scriptstyle \Delta_*} \uparrow & & \uparrow {\scriptstyle (T\otimes 1)\circ(1 \otimes \Delta_*)} \\ H_*(M) & \xleftarrow{\cap_*} & H_*(M) \otimes H_*(M) \end{array} \tag{1.2}$$

are commutative.

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Furthermore, by replacing simplicial closed necklaces by cubical ones (see Figure 2), we construct a cubical closed necklical set $\widehat{\mathbf{\Lambda}}K^{\square}$ having likewise a canonical permutahedral set structure such that the geometric realization $|\widehat{\mathbf{\Lambda}}K^{\square}|$ is homotopy equivalent to ΛM. The chain pairing $\cap_{\#}$ indices naturally the pairing of permutahedral cellular chains of degree $-n$

$$\mu_{\#}: C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\otimes C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\to C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|). \tag{1.3}$$

We also define canonical chain maps

$$\nu_{\#}^{l},\nu_{\#}^{r}: C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\otimes C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\to C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|). \tag{1.4}$$

Then by the isomorphism $H_*(|\widehat{\mathbf{\Lambda}}K^{\square}|)\approx H_*(\Lambda M)$ and a diagonal decomposition of $|\widehat{\mathbf{\Lambda}}K^{\square}|$ by means of an explicit diagonal of permutahedra [5] we immediately get

Theorem 2. *The following diagrams*

$$\begin{array}{ccc} H_*(\Lambda M)\otimes H_*(\Lambda M) & \xleftarrow{\mu_*\otimes\nu_*^r} & H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\\ {\scriptstyle\Delta_*}\uparrow & & {\scriptstyle(1\otimes T\otimes 1)\circ(\Delta_*\otimes\Delta_*)}\uparrow\\ H_*(\Lambda M) & \xleftarrow{\mu_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\end{array}$$

and

$$\begin{array}{ccc} H_*(\Lambda M)\otimes H_*(\Lambda M) & \xleftarrow{\nu_*^l\otimes\mu_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\\ {\scriptstyle\Delta_*}\uparrow & & {\scriptstyle(1\otimes T\otimes 1)\circ(\Delta_*\otimes\Delta_*)}\uparrow\\ H_*(\Lambda M) & \xleftarrow{\mu_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\end{array}$$

are commutative.

Note that the maps ν_*^r and ν_*^l are equivalent to the right and left actions of $H_*(\Omega M)$ on $H_*(\Lambda M)$ whenever the standard map $\iota_*: H_*(\Omega M)\to H_*(\Lambda M)$ is monomorphic. The definition of the map $\cap_{\#}$ uses the pairing (cf. [6])

$$C_p(K)\otimes C_q(K^{*})\to C_{p+q-n}(K'), \tag{1.5}$$

where K is a simplicial subdivision of M, K' is its barycentric subdivision and K^{*} is a block dissection of K' by the barycentric stars $D(\sigma)$ of simplices $\sigma\in K$ involved in the Poincaré isomorphism. The problem of constructing the chain-level intersection pairing gave rise to a number of works. A good reference to the subject is the recent book [2].

The map $\cap_{\#}$ is defined as follows. To each pair $\sigma\supset\tau$ of simplices from K we assign the cubical cell $I(\sigma\supset\tau)$ of dimension $\dim(\sigma)-\dim(\tau)$: Namely, if $\sigma_k\supset\cdots\supset\sigma_1$ denotes a barycentric subdivision simplex formed by subsimplices σ_i of σ_k with $(\sigma_k\supset\cdots\supset\sigma_1)\subset\sigma_k$, then

$$I(\sigma\supset\tau)=\bigcup_{\sigma\supset\sigma_i\supset\tau}\sigma\supset\sigma_r\supset\cdots\supset\sigma_1\supset\tau.$$

Thus $I(\sigma\supset\tau)\subset\sigma$, and the triangulated manifold M with the cubical cellular structure formed by the cubes $I(\sigma\supset\tau)$ for all pair of simplices $\sigma\supset\tau$ is just denoted by $K^{\square}$. Then we define a set map (on the set of the structural cells)

$$\sqcap: K\times K^{*}\longrightarrow K^{\square}$$

for a pair $(\sigma,D(\tau))\in K\times K^{*}$ by

$$\sigma\sqcap D(\tau)=\begin{cases} I(\sigma\supset\tau), & \text{if }\ \sigma\supset\tau,\\ *, & \text{otherwise.}\end{cases}$$

We, obviously, have $\partial(\sigma\sqcap D(\tau))=\partial\sigma\sqcap D(\tau)\bigcup\sigma\sqcap\partial D(\tau)$, and, hence, obtain the induced pairing of degree $-n$

$$\sqcap_{\#}: C_p(K)\otimes C_q(K^{*})\to C_{p+q-n}(K^{\square}).$$

There are the canonical chain maps

$$Sd_{\square}: C_*(K)\to C_*(K^{\square}),\ Sd_{\square}^{*}: C_*(K^{*})\to C_*(K^{\square})\ \text{ and }\ Sd_{\square}': C_*(K^{\square})\to C_*(K')$$

based on the fact that the chains of a cell is mapped to the chains of a subdivision of the cell (see Figure 1, below).

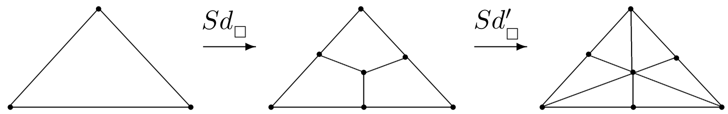

FIGURE 1. The first barycentric cubical and simplicial subdivisions of Δ^2.

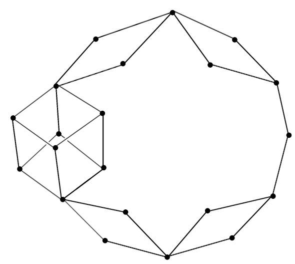

FIGURE 2. A cubical closed necklace $\mathbf{I}^3 \vee I^2 \vee I^2 \vee I^1 \vee I^1 \vee I^2 \vee I^2$ of dimension 7.

These maps have homotopy inverse maps, "cubical displacements"; namely, consider the chain maps

$$\theta_*^{cu} : C_*(K^\square) \to C_*(K) \quad \text{and} \quad \theta_*^{st} : C_*(K^\square) \to C_*(K^{\circledast}),$$

induced in fact by cellular maps $\theta^{cu} : K^\square \to K$ and $\theta^{st} : K^\square \to K^{\circledast}$. In particular, the pairing given by (1.6) is then the composition

$$C_p(K) \otimes C_q(K^{\circledast}) \xrightarrow{\sqcap_\#} C_{p+q-n}(K^\square) \xrightarrow{Sd'_\square} C_{p+q-n}(K'), \tag{1.6}$$

while we define $\cap_\#$ as the composition

$$\cap_\# : C_p(K^\square) \otimes C_q(K^\square) \xrightarrow{\theta_*^{cu} \times \theta_*^{st}} C_p(K) \otimes C_q(K^{\circledast}) \xrightarrow{\sqcap_\#} C_{p+q-n}(K^\square).$$

Proof of Theorem 1. The set of structural cells of $K^\square$ denote by $\mathcal{I}$, and then θ^{cu}, θ^{st} and $\sqcap$ induce a set map of degree $-n$

$$\cap : \mathcal{I} \times \mathcal{I} \to \mathcal{I}.$$

For the moment denote also by $\Delta_I := \{\Delta_{ij}\}$ the components of the Serre diagonal of a cube. Further, let $\mathcal{I} \curlyvee \mathcal{I} \subset \mathcal{I} \times \mathcal{I}$ be the subset such that for each bouquet $I(\sigma_i \supset \tau_i) \vee I(\sigma_j \supset \tau_j) \in \mathcal{I} \curlyvee \mathcal{I}$ there is the overlap cube $I(\sigma \supset \tau)$ with $\Delta_{ij}(I(\sigma \supset \tau)) = I(\sigma_i \supset \tau_i) \vee I(\sigma_j \supset \tau_j)$.

Then the diagram(s)

$$\begin{array}{ccc} \mathcal{I} \curlyvee \mathcal{I} & \xleftarrow{\cap \times 1} & (\mathcal{I} \curlyvee \mathcal{I}) \times \mathcal{I} \\ {\scriptstyle \Delta_{**}} \uparrow & & \uparrow {\scriptstyle (1 \curlyvee T) \circ (\Delta_{**} \times 1)} \\ \mathcal{I} & \xleftarrow{\cap} & \mathcal{I} \times \mathcal{I} \end{array} \tag{1.7}$$

commutes in a sense that each term of the diagonal fixed in $\mathcal{I} \curlyvee \mathcal{I}$ by one side equals to some term obtained by the other side of the diagram. Consequently, it induces the desired commutativity of (1.1). The commutativity of (1.2) follows from the diagram, similar to (1.7). $\square$

The proof of Theorem (2) requires the construction of the permutahedral model $\widehat{\mathbf{\Lambda}} K^\square$ for ΛM. In fact, beside $\widehat{\mathbf{\Lambda}} K^\square$ we also construct the cubical necklical set $\widehat{\mathbf{\Omega}} K^\square$ modeling the based loops ΩM, and, in general, for a cubical set X we have a quasi-fibration $|\widehat{\mathbf{\Omega}} X| \xrightarrow{\iota} |\widehat{\mathbf{\Lambda}} X| \xrightarrow{\zeta} Y$ modeling the free loop fibration $\Omega Y \to \Lambda Y \to Y$, this time Y is the geometric realisation of the cubical set X, while $\widehat{\mathbf{\Omega}} X$ and $\widehat{\mathbf{\Lambda}} X$ are cubical necklical and closed necklical sets, respectively, hence, both are permutahedral sets.

Evidently, this model uses more complicated polytopes rather than the ones in [4], however, since both ΩY and ΛY are modelling by the same type of sets, it has more symmetries.

We note that certain ideas of interactions of different type polytopes come from the works of N. Berikashvili in the obstruction theory to the section problem of a fibration (see also [3]).

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