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ON THE SECONDARY COHOMOLOGY OPERATIONS

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Abstract. The new secondary cohomology operations are constructed. These operations together with the Adams operations are intended to calculate the mod p cohomology algebra of loop spaces. In particular, the kernel of the loop suspension map is explicitly described.

1. Introduction

Let X be a topological space and $H^*(X;\mathbb{Z}_p)$ be the cohomology algebra in the coefficients $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ where $\mathbb{Z}$ is the integers and p is a prime. Given $n \geq 1$, let $P^*_n(X) \subset H^*(X;\mathbb{Z}_p)$ be the subset of elements of finite height

$$P^*_n(X) = \{x \in H^*(X;\mathbb{Z}_p) \mid x^{n+1} = 0,\, n \geq 1\}.$$

Let $\mathcal{P}_1 : H^m(X;\mathbb{Z}_p) \to H^{pm-p+1}(X;\mathbb{Z}_p)$ denote the Steenrod cohomology operation. Given $n, r \geq 1$, we construct the maps

$$\psi_{r,1} : H^{2m+1}(X;\mathbb{Z}_p) \to H^{2mp^{r+1}+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1, \quad p > 2, \tag{1.1}$$

and

$$\psi_{r,n} : P^m_n(X) \to H^{(m(n+1)-2)p^r+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \ \ (m \text{ is even when } p > 2) \tag{1.2}$$

in which $\psi_{1,p^k-1} = \psi_k$ is the Adams secondary cohomology operation for p odd or $p = 2$ and $k > 1$ (cf. [1–3]). Note that when $n > 1$, these maps are linear for $n+1 = p^k$, $k \geq 1$ (e.g., $H^*(X;\mathbb{Z}_p)$ is a Hopf algebra). Let ΩX be the (based) loop space on X. Let $\sigma : H^*(X;\mathbb{Z}_p) \to H^{*-1}(\Omega X;\mathbb{Z}_p)$ be the loop suspension map. Theorem 2 (cf. [3]) explicitly describes $\operatorname{Ker}\sigma$ in terms of the operations $\mathcal{P}_1$ and $\psi_{1,n}$ and higher order Bockstein homomorphisms β_k associated with the short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^{k+1}} \to \mathbb{Z}_{p^k} \to 0.$$

The calculation of the loop space cohomology algebra $H^*(\Omega X;\mathbb{Z}_p)$ in terms of generators and relations will appear elsewhere.

2. The Secondary Cohomology Operations $\psi_{r,n}$

The secondary cohomology operations are constructed by using the integral filtered model of a space X considered in [4].

2.1. **The Hirsch filtered models of a space.** Given a commutative graded algebra (cga) H, there are two kinds of Hirsch resolutions

$$\rho_a : (R_aH, d) \to H \quad \text{and} \quad \rho : (RH, d) \to H,$$

the absolute Hirsch resolution R_aH and the minimal Hirsch resolution RH, respectively. The first R_aH is endowed, besides the Steenrod cochain operation $E_{1,1} = \smile_1$, the cup-one product, with the higher order operations $E_{p,q}$, $p, q \geq 1$, as they usually exist in the cochain complex $C^*(X;\mathbb{Z})$; the second RH is, in fact, endowed only with the cup-one measuring the non-commutativity of the cup product $\cdot := \smile$. In general, the operations $E_{p,q}$ appear to measure the deviations of the cup-one product from being the left and right derivations with respect to the cup product. But in RH the freeness of the multiplicative structure enables us to fix the relationship between the cup and cup-one

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products by explicit formulas, while the relation between RH and the cochain complex $C^*(X;\mathbb{Z})$ is fixed via zig-zag Hirsch maps

$$(RH,d_h) \overset{g}{\leftarrow} (R_aH,d_h) \overset{f}{\to} C^*(X;\mathbb{Z}). \tag{2.1}$$

In fact, $RH = R_aH/J$ for a certain Hirsch ideal $J \subset R_aH$. Thus, the Hirsch algebra (RH,d_h), being generated only by the $\smile_1$-product, becomes an efficient tool for calculating of the loop cohomology algebra.

Denote $H^* = H^*(X;\mathbb{Z})$. Given a prime p, let $t_{\mathbb{Z}_p} : RH \to RH \otimes \mathbb{Z}_p$ be the standard map. For $z = [c] \in H^*(X;\mathbb{Z}_p)$ with $c \in RH \otimes \mathbb{Z}_p$, let $x_0 := t^{-1}_{\mathbb{Z}_p}(c)$. If $c \in P^*_n(X)$, then in RH there is the equality $dx_1 = \lambda x_0^{n+1} = 0 \mod p$, some $x_1 \in RH$, and p does not divide λ. Note that the essential idea can be seen for $n = 1$ (the case $n > 1$ is somewhat technically difficult only). Each $z \in P^*_n$ produces an infinite sequence of elements $(x_m)_{m\geq 0}$ in RH given by the following formulas:

$$dx_{2k+1} = \sum_{i_1+\cdots+i_{n+1}=k} (-1)^{|z|}\,\lambda\, x_{2i_1}\dots x_{2i_{n+1}} + \sum_{i+j=2k-1} x_{2i+1}x_{2j+1} + p\tilde{x}_{2k+1},$$

$$dx_{2k} = \sum_{i+j=2k-1} (-1)^{|x_i|+1} x_i x_j + p\tilde{x}_{2k}, \qquad i_m,i,j,k \geq 0.$$

(The signs are fixed for $|z|$ and $n+1$ to be not simultaneously odd above.) In particular, when z is odd dimensional and $n,\lambda = 1$, one gets for $k,i,j \geq 0$:

$$dx_k = \sum_{i+j=k-1} x_i x_j + p\tilde{x}_k.$$

In turn, the sequence $(x_m)_{m\geq 0}$ by means of the $\smile_1$- product induces four kinds of infinite sequences $b^{i_1,i_2}_{k,\ell} \in \left\{ b^{1,n}_{k,\ell},\, b^{n,1}_{k,\ell},\, b^{n,n}_{k,\ell},\, b^{1,1}_{k,\ell} \right\}$ in RH for $n \geq 1$ (more precisely, one sequence $(b_{k,\ell})_{k,\ell\geq 1}$ when $n = 1$) with $b_{k,\ell} := b^{1,1}_{k,\ell} = b^{1,1}_{\ell,k}$ ($k,l \geq 2$ when $n > 1$, while $k,l \geq 1$ when $n = 1$), $b^{1,n}_{2i,2j} = b^{n,1}_{2i,2j}$, $i,j \geq 1$, defined by the recursive formulas: $b^{1,n}_{1,1} = -(-1)^{|z|} b^{n,1}_{1,1}$ for $(k,\ell) = (1,1)$, and

$$db^{1,n}_{1,1} = \begin{cases} 2x_1 + \lambda\, x_0 \smile_1 x_0^n, & |z| \text{ is odd},\\ x_0 \smile_1 x_0^n, & |z| \text{ is even}, \end{cases}$$

(in the latter case, we, in fact, have $b^{1,n}_{1,1} = \sum\limits_{i+j=n-1} x_0^i (x_0 \cup_2 x_0) x_0^j$),

$b^{n,n}_{1,1} = \sum\limits_{i+j=n-1} x_0^i\, b^{1,n}_{1,1}\, x_0^j$, and for $k,\ell \geq 1$:

$$\begin{aligned} db^{*,*}_{k,\ell} = {}& -(-1)^{|z|}\, \alpha^{*,*}_{k,\ell}\, x^{*,*}_{k+\ell-1} + x^{(*)}_{k-1} \smile_1 x^{(*)}_{\ell-1} \\ &+ \sum_{\substack{0\leq r<k\\ 0\leq m<\ell}} \Big((-1)^{\epsilon_1+|z|}\, \alpha^{*,*}_{r,m}\, b^{*,*}_{k-r,\ell-m}\, x^{*,*}_{r+m-1} - (-1)^{\epsilon_2} \big(x^{(*)}_{r-1} \smile_1 x^{(*)}_{m-1}\big) b^{*,*}_{k-r,\ell-m} \Big) + p\, \tilde{b}^{*,*}_{k,\ell} \end{aligned} \tag{2.2}$$

with the convention $x_{-1} \smile_1 x_m = x_m \smile_1 x_{-1} = -x_m$, and $\alpha_{s,t} := \alpha^{1,1}_{s,t} = \alpha^{n,n}_{s,t}$, $\alpha^{1,n}_{s,t} = \alpha^{n,1}_{s,t}$; in particular, for $|x|$ odd:

$$\alpha_{s,t} = \begin{cases} \binom{s+t}{s}, & n = 1,\\ \binom{(s+t)/2}{s/2}, & n > 1 \text{ and } s,\ell \text{ are even}, \quad \mod p,\\ \binom{(s+t-1)/2}{s/2}, & n > 1 \text{ and } s \text{ is even and } t \text{ is odd}, \quad \mod p,\\ 0, & n > 1 \text{ and } s,t \text{ are odd}, \quad \mod p, \end{cases}$$

and for $|x|$ even:

$$\alpha_{s,t} = \begin{cases} \binom{(s+t)/2}{s/2}, & n \geq 1 \text{ and } s, \ell \text{ are even}, \quad \text{mod } p, \\ \binom{(s+t-1)/2}{s/2}, & n \geq 1 \text{ and } s \text{ is even and } t \text{ is odd}, \quad \text{mod } p, \\ 0, & n \geq 1 \text{ and } s, t \text{ are odd}, \quad \text{mod } p. \end{cases}$$

Therefore, when $|z|$ is odd and $n, \lambda = 1$, formula (2.2) takes the form

$$db_{k,\ell} = \binom{k+\ell}{k} x_{k+\ell-1} + x_{k-1} \smile_1 x_{\ell-1}$$

$$- \sum_{\substack{1 \leq r < k \\ 1 \leq m < \ell}} \left(\binom{r+m}{r} b_{k-r,\ell-m}\, x_{r+m-1} + (x_{r-1} \smile_1 x_{m-1})\, b_{k-r,\ell-m} \right)$$

$$- \sum_{1 \leq r < k,\ell} ((b_{k-r,\ell} + b_{k,\ell-r})x_{r-1} - x_{r-1}(b_{k-r,\ell} + b_{k,\ell-r})) + p\, \tilde{b}_{k,\ell}.$$

The values of the perturbation h on x_q and $b^{*,*}_{k,\ell}$ are, in fact, purely determined by the *transgressive* terms $y_{q+1} := hx_q|_{R^0H \oplus R^{-1}H}$ and $c^{*,*}_{k,\ell} := h(b^{*,*}_{k,\ell})|_{R^0H \oplus R^{-1}H}$, respectively. Namely,

$$hx_q = \sum_{\substack{ir_i = q-m,\, r_i \geq 1 \\ jr_j = q+1, \quad r_j \geq 1 \\ 0 \leq m < q}} -\, x_m \smile_1 y_i^{\cup_2 r_i} + y_j^{\cup_2 r_j} + p\, h\tilde{x}_q$$

and denoting $\gamma_{k,\ell} = \alpha^{*,*}_{\alpha_0,\ell_0} \dots \alpha^{*,*}_{k_s,\ell_s}$ and $m_{[s]} = m_1 + \cdots + m_s$,

$$h\Big(b^{*,*}_{k,\ell}\Big) = \sum_{\substack{1 \leq k_i < k_{i+1} \\ 1 \leq \ell_i < \ell_{i+1}}} -\, \gamma_{k,\ell}\, x^{*,*}_{k_0+\ell_0-1} \smile_1 c^{*,*}_{k_1-k_0,\ell_1-\ell_0} \smile_1 \cdots \smile_1 c^{*,*}_{k-k_s,\ell-\ell_s}$$

$$- \sum_{k=k_{[t]}\,;\,\ell=\ell_{[t]}} c^{*,*}_{k_1,\ell_1} \smile_1 \cdots \smile_1 c^{*,*}_{k_t,\ell_t} + \sum_{\substack{1 \leq r < k \\ 1 \leq m < \ell}} b^{*,*}_{r,m}\, h\Big(b^{*,*}_{k-r,\ell-m}\Big) + c^{*,*}_{k,\ell} + p\, h\Big(\tilde{b}^{*,*}_{k,\ell}\Big). \tag{2.3}$$

Furthermore, by means of $b_{k,\ell}$, we define the elements $\mathfrak{b}_{k,\ell} \in RH$ as follows. Fix the integer $k \geq 1$. Denote $\mathfrak{b}_{k,k} = b_{k,k}$ and $\varrho_{k,k} = 1$. If $\mathfrak{b}_{k,mk}$ has already been constructed for $1 \leq m < q$ and $\varrho_{k,qk} := \alpha_{k,(q-1)k} \dots \alpha_{k,2k}\, \alpha_{k,k}$, let

$$\mathfrak{b}_{k,qk} = \varrho_{k,qk} b_{k,qk} - x_{k-1} \smile_1 \mathfrak{b}_{k,(q-1)k} = \varrho_{k,qk}\, b_{k,qk}$$

$$- \varrho_{k,(q-1)k}\, x_{k-1} \smile_1 b_{k,(q-1)k} - \cdots - \varrho_{k,2k}\, x_{k-1}^{\smile_1 q} \smile_1 b_{k,2k} - x_{k-1}^{\smile_1 (q+1)} \smile_1 b_{k,k}.$$

Then

$$\begin{aligned} d_h \mathfrak{b}_{k,qk} &= \varrho_{k,qk}\, x_{k+qk-1} + x_{k-1}^{\smile_1 (q+1)} + u_{k,qk} + p\, \tilde{\mathfrak{b}}_{k,qk} + h\mathfrak{b}_{k,qk} \\ &= \varrho_{k,qk}\, x_{k+qk-1} + x_{k-1}^{\smile_1 (q+1)} + w_{k,qk} + \varrho_{k,qk} c_{k,qk}, \end{aligned} \tag{2.4}$$

where $w_{k,qk} := u_{k,qk} + p\, \tilde{\mathfrak{b}}_{k,qk} + (h\mathfrak{b}_{k,qk} - \varrho_{k,qk}\, c_{k,qk})$ and $u_{k,qk}$ is expressed by x_i and $b_{s,t}$ with $(s,t) \leq (k, qk)$.

a) Let p be odd. Set $k = p^r$ and $q = p-1$ in (2.4), and define (1.1) for $z \in H^{2m+1}(X; \mathbb{Z}_p)$ and $r \geq 1$ by

$$\psi_{r,1}(z) = \Big[t_{Z_p}\big(x_{p^r-1}^{\smile_1 p} + w_{p^r,(p-1)p^r} \big) \Big];$$

b) Let p and m be not odd simultaneously. Set $k = 2p^{r-1}$ and $q = p-1$ in (2.4), and define (1.2) for $z \in P^m_n(X)$ and $r, n \geq 1$ by

$$\psi_{r,n}(z) = \Big[t_{Z_p}\big(x_{2p^{r-1}-1}^{\smile_1 p} + w_{2p^{r-1},2(p-1)p^{r-1}} \big) \Big].$$

Theorem 1. *For any map $f : X \to Y$, the following diagrams*

$$\begin{array}{ccc} H^{2m+1}(X;\mathbb{Z}_p) & \xrightarrow{\psi_{r,1}} & H^{2mp^{r+1}+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \\ f^*\uparrow & & f^*\uparrow \\ H^{2m+1}(Y;\mathbb{Z}_p) & \xrightarrow{\psi_{r,1}} & H^{2mp^{r+1}+1}(Y;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \end{array}$$

and

$$\begin{array}{ccc} P^m_n(X) & \xrightarrow{\psi_{r,n}} & H^{(m(n+1)-2)p^r+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \\ f^*\uparrow & & f^*\uparrow \\ P^m_n(Y) & \xrightarrow{\psi_{r,n}} & H^{(m(n+1)-2)p^r+1}(Y;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \end{array}$$

commute.

Sketch of the proof. Define the cohomology operations on $H^*(C^*(X;\mathbb{Z}_p))$ by means of the canonical operations $\{E_{p,q}\}_{p,q\geq 1}$ on the cochain complex $C^*(X;\mathbb{Z}_p)$ ([4]) that agree with $\psi_{r,n}$ on $H^*(RH,d_h)$ via zig-zag maps (2.1). □

Let $\mathcal{D}^* := H^+(X;\mathbb{Z}_p)\cdot H^+(X;\mathbb{Z}_p) \subset H^*(X;\mathbb{Z}_p)$ be the decomposables and $\mathcal{P}_1^{(m)}$ denote m-fold composition $\mathcal{P}_1\circ\cdots\circ\mathcal{P}_1$.

Theorem 2. *Let $H^*(X;\mathbb{Z}_p)$ be a Hopf algebra. Given $r\geq 1$, let $p(r)$ denote the largest integer such that $p^{p(r)}$ divides the factorial $p^r!$. Let $\mathcal{I}^*\subset H^*(X;\mathbb{Z}_p)$ be the subset of indecomposables defined for $a\in\mathcal{I}^*$, $z\in H^*(X;\mathbb{Z}_p)$ and the integer $\kappa_z\geq 1$ such that $\beta_{p(t)}\,\mathcal{P}_1^{(t)}(z)=\beta_{p(t)}\,\mathcal{P}_1^{(t-1)}\psi_{1,n}(z)=0$ mod $\mathcal{D}^*$ for $t<\kappa_z$ and*
a) *For $p>2$:*

$$a=\begin{cases}\beta_{p(\kappa_z)}\,\mathcal{P}_1^{(\kappa_z)}(z), & n=1 \textit{ and } z \textit{ is odd dimensional},\\ \beta_{p(\kappa_z)}\,\mathcal{P}_1^{(\kappa_z-1)}\psi_{1,n}(z), & n>1 \textit{ and } z \textit{ is even dimensional};\end{cases}$$

b) *For $p=2$:*

$$a=\beta_{2(\kappa_z)}\,Sq_1^{(\kappa_z-1)}\psi_{1,n}(z),\qquad n\geq 1.$$

Then $\operatorname{Ker}\sigma=\mathcal{I}^*\cup\mathcal{D}^*$.

Proof. The map $\tau : RH\otimes\mathbb{Z}_p\to\bar{V}\otimes\mathbb{Z}_p$, $a\otimes 1\to\overline{a|_V}\otimes 1$ realizes the loop suspension map σ as (cf. [4])

$$\sigma : H^m(X;\mathbb{Z}_p)\approx H^m(RH\otimes\mathbb{Z}_p,d_h)\xrightarrow{\tau^*}H^{m-1}(\bar{V}\otimes\mathbb{Z}_p,\bar{d}_h)\approx H^{m-1}(\Omega X;\mathbb{Z}_p).$$

The inclusion $\mathcal{D}^*\subset\operatorname{Ker}\sigma$ immediately follows from the above definition of σ. Let $a\in\operatorname{Ker}\sigma$ be indecomposable. Then for $y\in RH$ with $[t_{\mathbb{Z}_p}(y)]=a$, there is the sequence $(x_m)_{m\geq 0}$ in RH and $r\geq 1$ such that

$$d_h(x_{m-1})=y+u_{m-1}\qquad \text{mod } p,$$
$$d_h(x_i)=u_i\qquad \text{mod } p,\quad u_i\in\mathcal{D}^*,\quad i<m\ \ \text{for}$$

$$m=\begin{cases}p^r, & p \text{ and } |x_0| \text{ are odd},\\ 2p^{r-1}, & \text{otherwise}.\end{cases}$$

Let $z=\frac{p^{p(r)}}{p^r!}[t_{\mathbb{Z}_p}(x_0)]$. Denote $\kappa_z:=r$. Then taking into account (2.3) and the coefficients $\varrho_{k,qk}$ of x_{k+qk-1} in (2.4) for $q=p-1$ and $k=p^t$ and $k=2p^{t-1}$, $1\leq t\leq\kappa_z$, we establish the equalities of Items a) – b) as desired. Hence, $a\subset\mathcal{I}^*$. The implication $\mathcal{I}^*\cup\mathcal{D}^*\subset\operatorname{Ker}\sigma$ is obvious. □

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