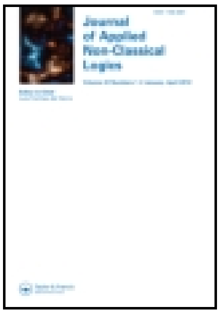

Journal of Applied Non-Classical Logics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tncl20

Description of all functions definable by formulæ of the 2nd order intuitionistic propositional calculus on some linear Heyting algebras

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Published online: 13 Apr 2012.

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ABSTRACT. *Explicit description of maps definable by formulæ of the second order intuitionistic propositional calculus is given on two classes of linear Heyting algebras—the dense ones and the ones which possess successors. As a consequence, it is shown that over these classes every formula is equivalent to a quantifier free formula in the dense case, and to a formula with quantifiers confined to the applications of the successor in the second case.*

KEYWORDS: *linear Heyting algebra, second order intuitionistic propositional logic.*

1. Introduction

In this paper we deal with second order intuitionistic propositional logic. We will exhibit two classes of linearly ordered Heyting algebras which admit explicitly describable interpretation of formulæ of that logic. The main motivation for doing this lies in approaching quantifier elimination for the logic with linearity and some additional axioms added.

Quantifier elimination in broad sense has been considered extremely important (although rare) ever since the seminal paper [TAR 48], and there is a lot of work done on quantifier elimination in classical logic, starting from textbooks and monographs (see [GAB 06] for a recent example) and ending with computer software ([DOL 99, WOL 06]). A modern day evidence for this can be produced just by typing “quantifier elimination” in a Google search window.

Of course it is quite natural to ask about quantifier elimination in non-classical logic. In modal logic, we are aware of [SZA 02]. In the context of the second order intuitionistic logic (in the sense of [GAB 74]), we know of works [BAA 96, BAA 00a, BAA 00b, BAA 06] of Baaz and collaborators.

In the present paper we want to give some indication of possibility of quantifier elimination in presence of linearity axiom in two separate cases—one corresponding to dense linearly ordered Heyting algebras and another corresponding to Heyting algebras which possess successors for all elements (except the top). Namely, we will show that over such algebras semantical interpretation of every formula is the same as for some other formula which does not contain quantifiers in the first case, and only contains quantifiers involved in expressing the successor operator in the second case. We will do this by giving explicit description of all maps which occur as a semantical interpretation of some formula.

2. Setup

2.1. *Basic notions and notation*

The language of the second order intuitionistic propositional logic consists of the logical connectives $\wedge$, $\vee$, $\top$, $\bot$, $\rightarrow$, $\forall$, $\exists$ and a denumerable set of propositional variables.

The set of second order propositional formulæ is defined inductively as follows: $\top$, $\bot$, and any propositional variable is a formula; if A and B are formulæ and p is a propositional variable, then $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\exists p A$, $\forall p A$ are formulæ.

The notions of free and bound variables and the notions of open and closed formulæ are the usual ones. If A is a formula, we will write $A(p_1, ..., p_n)$ to indicate that all free variables of A are among $p_1, ..., p_n$. Moreover $A(A_1/p_{i_1}, ..., A_k/p_{i_k})$ denotes the simultaneous substitution of all the free occurrences of $p_{i_1}, ..., p_{i_k}$ in A with formulæ $A_1, ..., A_k$ respectively.

We will use some abbreviations:

$$\begin{aligned} |V| &:= \text{number of elements in the finite set } V, \\ [n] &:= \{1, 2, ..., n\}, \\ [n]_0 &:= \{0, 1, 2, ..., n\}, \\ \neg A &:= A \rightarrow \bot, \\ A \leftrightarrow B &:= (A \rightarrow B) \wedge (B \rightarrow A) \\ \Box(A) &:= \forall p(p \vee p \rightarrow A), \end{aligned}$$

where p is any variable which is not free in A.

For any second order propositional formula A its *quantifier degree*, a nonnegative integer $\deg(A)$, is defined inductively in the usual way:

– if A is $\top$, $\bot$ or a variable, then $\deg(A) = 0$;
– if A is $\neg B$, then $\deg(A) = \deg(B)$;
– if A is either of $B \wedge C$, $B \vee C$, $B \to C$ or $B \leftrightarrow C$ then $\deg(A) = \max\{\deg(B), \deg(C)\}$;
– if A is $\forall pB$ or $\exists pB$, then $\deg(A) = \deg(B) + 1$.

The logic IpC^2 has the axiomatic description consisting of:

– the usual axioms of the Heyting propositional calculus;
– axiom schemata $\forall pA \to A(B/p)$ and $A(B/p) \to \exists pA$;
– comprehension schema: $\exists p(p \leftrightarrow A)$, where p does not occur freely in A,

with rules

$$\frac{A \to B}{A \to \forall pB}$$

and

$$\frac{B \to A}{\exists pB \to A},$$

where p does not occur freely in A.

We will use the following *congruence property for* $\leftrightarrow$ easily provable in IpC^2. For any formula $A(p_1, ..., p_n)$ one has

(Cong) $$\text{IpC}^2 \vdash (p_1 \leftrightarrow p_1' \wedge \cdots \wedge p_n \leftrightarrow p_n') \to (A(p_1, ..., p_n) \leftrightarrow A(p_1'/p_1, ..., p_n'/p_n)) .$$

2.2. *Semantics*

Semantical content of this logic consists for us in the assignment, for a given Heyting algebra H, to each formula $A(p_1, ..., p_n)$, of a partially defined map

$$\ulcorner A \urcorner : H^n \to H.$$

This assignment, where defined, is uniquely determined by the following (here we denote logical connectives and the Heyting algebra operations corresponding to them by the same symbols):

$$\ulcorner \top \urcorner (h_1, ..., h_n) := \top,$$
$$\ulcorner \bot \urcorner (h_1, ..., h_n) := \bot;$$

for $* \in \{\wedge, \vee, \rightarrow\}$,

$$\ulcorner A * B \urcorner (h_1, ..., h_n) := \ulcorner A \urcorner (h_1, ..., h_n) * \ulcorner B \urcorner (h_1, ..., h_n);$$

$$\ulcorner \forall p_i A \urcorner (h_1, ..., h_{i-1}, h_{i+1}, ..., h_n) := \bigwedge_{h \in H} \ulcorner A \urcorner (h_1, ..., h_{i-1}, h, h_{i+1}, ...h_n),$$

$$\ulcorner \exists p_i A \urcorner (h_1, ..., h_{i-1}, h_{i+1}, ..., h_n) := \bigvee_{h \in H} \ulcorner A \urcorner (h_1, ..., h_{i-1}, h, h_{i+1}, ...h_n).$$

Here the symbol ":=" means that the left hand side is defined to be equal to the right hand side provided the latter is already defined, and is undefined otherwise.

2.3. *Semantical equivalence of formulæ*

A *model* of the second order intuitionistic propositional logic is a Heyting algebra H, such that the above interpretation gives totally defined maps for all formulæ; such Heyting algebras will be called *formula complete*.

If some class $\mathscr{K}$ of models is given, we say that two formulæ A and B are $\mathscr{K}$-*equivalent*, if their interpretations coincide in each model of $\mathscr{K}$.

In Heyting algebras we use the notation $<$ with its usual meaning (i. e. $h_1 < h_2$ means that $h_1 \leqslant h_2$ and $h_1 \neq h_2$). For each $h \in H$ the *successor* of h is the least $h' \in H$ such that $h' > h$ (if it exists), i. e. such $\mathrm{s}(h) \in H$ that $h < \mathrm{s}(h)$, and $\mathrm{s}(h) \leqslant h'$ for every $h' > h$. Of course it is clear, that the top element $\top$ in H can never have a successor.

In every Heyting algebra the interpretation of all logical connectives except quantifiers gives totally defined functions in a well-known way. In any linear Heyting algebra H these are as follows:

$$h_1 \wedge h_2 = \min \{h_1, h_2\};$$

$$h_1 \vee h_2 = \max \{h_1, h_2\};$$

$$\neg h = \begin{cases} \bot, & \text{if } h > \bot, \\ \top, & \text{if } h = \bot; \end{cases}$$

$$h_1 \rightarrow h_2 = \begin{cases} \top, & \text{if } h_1 \leqslant h_2 \\ h_2, & \text{if } h_1 > h_2; \end{cases}$$

$$h_1 \leftrightarrow h_2 = \begin{cases} \top, & \text{if } h_1 = h_2, \\ \min \{h_1, h_2\}, & \text{if } h_1 \neq h_2 \end{cases}$$

for any $h, h_1, h_2 \in H$.

Moreover, although interpretation of quantifiers gives in general only partially defined maps, interpretation of $\Box$ is also totally defined in any linear Heyting algebra:

- $\Box(h) = \bigwedge \{\max\{h', h' \to h\} \mid h' \in H\} = \begin{cases} \mathrm{s}(h), & \text{if this successor exists,} \\ h, & \text{otherwise.} \end{cases}$

2.4. *The classes $\mathscr{D}$ and $\mathscr{S}$*

We consider two classes of linear Heyting algebras $\mathscr{D}$ and $\mathscr{S}$.

The class $\mathscr{D}$ consists of all dense linear Heyting algebras.

Note that any $H \in \mathscr{D}$ is either trivial (i. e. consists of a single element), or is infinite and in it the interpretations of the formulæ $\Box A$ and A coincide for any formula A.

The class $\mathscr{S}$ consists of all those linear Heyting algebras each of whose elements except $\top$ has a successor.

Note that in Heyting algebras from the class $\mathscr{S}$ interpretation of the formula $\Box A$ for each formula A is as follows:

$$\ulcorner \Box A \urcorner (h_1, ..., h_n) = \begin{cases} \mathrm{s}\left(\ulcorner A \urcorner (h_1, ..., h_n)\right), & \text{if } \ulcorner A \urcorner (h_1, ..., h_n) \neq \top, \\ \top, & \text{if } \ulcorner A \urcorner (h_1, ..., h_n) = \top. \end{cases}$$

It will follow, among other things that both classes consist of formula complete algebras. In fact, it will turn out that

> Every formula is $\mathscr{D}$-equivalent to a quantifier-free formula, i. e. to a formula built from propositional variables and the connectives $\wedge$, $\vee$, $\to$, $\top$, $\bot$ (not quantifiers).

and

> Every formula is $\mathscr{S}$-equivalent to a formula in which quantifiers occur only through application of $\Box$, i. e. to a formula built from propositional variables, the connectives $\wedge$, $\vee$, $\to$, $\top$, $\bot$ (not quantifiers), and $\Box$.

2.5. *Characterization of functions*

In order to describe functions associated to formulæ of IpC^2 with at most n free variables for classes $\mathscr{D}$ and $\mathscr{S}$ we consider certain subdivisions of the product H^n into subsets called *strata*.

First we consider more simple subdivision of H^n which is useful to describe interpretation of formulæ for the class $\mathscr{D}$ (each function associated to a formula of

IpC^2 will be in a certain sense "linear" on every stratum). The method is inspired by [GER 00].

Then we consider a bit more complicated subdivision of H^n which is useful to describe interpretation of formulæ for the class $\mathscr{S}$. In this case also each function associated to a formula of IpC^2 will be "linear" on every stratum.

2.6. *Maps, labelings*

Let us define some more auxiliary terminology and notation.

Consider a map $\varphi : [n] \to [k]_0$, $n > 0$, $0 \leqslant k \leqslant n + 1$.

We call a number $r \in [k]_0$ *labeled*, if $\varphi^{-1}(r) \neq \varnothing$. If $\varphi(i) = r$, then we call i a label of r. The number $r \in [k]_0$ may have several labels.

The notion of smallest, largest, previous and next labeled element are the obvious ones.

Statement 2.6.1

REMARK. — If we consider some subset X of some ordered set Y (e. g. $X = \{x \in Y \mid \alpha(x)\}$) and if this subset X is empty, then $\max X$ will be equal to $\min Y$ and $\min X$ will be equal to $\max Y$. □

Using this remark for each $r \in [k]_0$, define the number $d(r) = r - s$, where

$$s = \max \{t \in [k]_0 \mid t \text{ is labeled and } t < r\}.$$

Thus by this remark one has $d(0) = 0$.

3. Subdivision for $\mathscr{D}$

Statement 3.1

For each $H \in \mathscr{D}$ and each natural $n > 0$ we will introduce a subdivision of H^n taking into account the possible orders between components of each n-tuple $(h_1, ..., h_n) \in H^n$.

Consider the set

$$C_n = \left\{\varphi : [n] \to [k]_0 \;\middle|\; \varphi^{-1}(r) \neq \varnothing \text{ for each } 0 < r < k \text{ and } \varphi^{-1}(k) = \varnothing \right\}.$$

Statement 3.2

Each $\varphi : [n] \to [k]_0$ may be represented as a linearly ordered set V with $k + 1$ elements, some of which are labeled with elements of $[n]$ in such a way that for every $i \in [n]$ there is one and only one point from V which is labeled with i. An element of

V may have several labels. Moreover such a V represents an element of C_n iff the top of V is not labeled and each element of V different from the top and from the bottom is labeled. In this paper we identify elements of C_n with their correspondent labeled linearly ordered sets.

Statement 3.3

EXAMPLE. —

$$\begin{array}{ccccccc} 3 & & 1,4 & & 2 & & \\ \circ & \rightarrow & \circ & \rightarrow & \circ & \rightarrow & \circ \end{array}$$

(where "$\rightarrow$" depicts the ordering) represents an element of C_4 with $k = 3$. The corresponding function $\varphi : [4] \rightarrow [3]_0$ is given by $\varphi(1) = \varphi(4) = 1$, $\varphi(2) = 2$, $\varphi(3) = 0$. □

Statement 3.4

For every $\varphi : [n] \rightarrow [k]_0$ in C_n and a Heyting algebra H define

$$\mathfrak{F}_H^\varphi = \left\{ f : [k]_0 \rightarrow H \;\middle|\; \begin{array}{l} f(0) = \perp,\ f(k) = \top \text{ and for all } 0 \leqslant s < r \leqslant k \\ \text{either } f(s) < f(r) \text{ or } f(s) = f(r) = \top \end{array} \right\}$$

and

$$H^\varphi = \left\{ (h_1, h_2, ..., h_n) \in H^n \;\middle|\; \begin{array}{ll} h_i = \perp & \text{if } \varphi(i) = 0, \\ h_i = h_j & \text{if } \varphi(i) = \varphi(j) \\ \text{and} & \\ \text{either } h_i < h_j \text{ or } h_i = h_j = \top & \text{if } \varphi(i) < \varphi(j) \end{array} \right\}.$$

For every $H \in \mathscr{D}$ and $\varphi \in C_n$, there is a naturally defined map

$$\begin{aligned} c_H^\varphi : \mathfrak{F}_H^\varphi &\rightarrow H^\varphi, \\ f &\mapsto f \circ \varphi. \end{aligned}$$

This map is an order preserving isomorphism (with respect to the pointwise order on the set of functions $\mathfrak{F}_H^\varphi$).

E. g., for any $H \in \mathscr{D}$ the subset of H^4 corresponding to the labeled linear ordered set from 3.3 is:

$$\begin{aligned} H^\varphi &= \left\{ (h_1, h_2, h_3, h_4) \;\middle|\; \begin{array}{l} \perp = h_3 < h_1 = h_4 < h_2, \text{ or} \\ \perp = h_3 < h_1 = h_4 = h_2 = \top, \text{ or} \\ \perp = \top \end{array} \right\} \\ &= \{(h, h', \perp, h) \mid \perp < h < h' \text{ or } h = h' = \top\}. \end{aligned}$$

Statement 3.5

For two elements φ_1 and φ_2 from C_n the sets H^{φ_1} and H^{φ_2} for a nontrivial $H \in \mathscr{D}$ are disjoint if and only if $\varphi_1^{-1}(0) \neq \varphi_2^{-1}(0)$. If $\varphi_1^{-1}(0) = \varphi_2^{-1}(0)$, then H^{φ_1} and H^{φ_2} have nonempty intersection. For example take the element $(h_1, ..., h_n)$ of H^n with $h_i = \bot$ if $\varphi_1(i) = 0$ and $h_i = \top$ otherwise. This n-tuple belongs to both subsets H^{φ_1} and H^{φ_2}.

Note that the union of all H^φ is the whole H^n. Indeed for each $(h_1, ..., h_n) \in H^n$, $n \geqslant 1$, consider the linearly ordered set $U = \{h_1, ..., h_n\} \cup \{\bot\}$, with the induced linear order. Label an element h in U with $i \in [n]$ whenever $h_i = h$.

Let V be a new labeled linearly ordered set, obtained from U by adding new non-labeled top element.

It is then clear that $(h_1, ..., h_n) \in H^\varphi$, where φ is the map corresponding to the labeled linearly ordered set V.

Statement 3.6

DEFINITION. — *For two elements $\varphi_1 : [n] \to [k_1]_0$ and $\varphi_2 : [n] \to [k_2]_0$ of C_n the nonnegative integer $\varphi_1|\varphi_2$ is given by the equality*

$$\varphi_1|\varphi_2 = \min\left\{r \in [k_1]_0 \cap [k_2]_0 \;\middle|\; \varphi_1^{-1}(r) \neq \varphi_2^{-1}(r)\right\}.$$

It is clear, that if $\varphi_1|\varphi_2 = k_i$, $i = 1, 2$, then for any $0 \leqslant r \leqslant k_i$ one has $\varphi_1^{-1}(r) = \varphi_2^{-1}(r)$. k_i is not labeled in $[k_i]_0$ and for $[k_i - 1]_0$ are used all labels for φ_i from $[n]$ and therefore

$$\left\{r \in [k_1]_0 \cap [k_2]_0 \;\middle|\; \varphi_1^{-1}(r) \neq \varphi_2^{-1}(r)\right\} = \varnothing,$$

hence $k_1 = k_2$ and $\varphi_1 = \varphi_2$.

Statement 3.7

For each $\varphi \in C_n$ we construct a formula χ_φ, which we call the *characteristic formula* for the stratum of φ. The value of the function $\ulcorner\chi_\varphi\urcorner$ on the tuple $(h_1, ..., h_n)$ will be equal to $\top$ if and only if $(h_1, ..., h_n) \in H^\varphi$

Such a formula can be constructed using the following “characteristic formulæ” for the relations $=$, $<$ and $\leqslant$. These we define as follows:

$$\chi_=(p, q) := p \leftrightarrow q,$$
$$\chi_<(p, q) := (q \to p) \to q,$$
$$\chi_\leqslant(p, q) := p \to q.$$

Using these formulæ one can construct:

$$\begin{aligned}\chi_\varphi(p_1, p_2, ..., p_n) := & \bigwedge \{\chi_=(p_i, \bot) \mid \varphi(i) = 0\} \\ & \wedge \bigwedge \{\chi_=(p_i, p_j) \mid \varphi(i) = \varphi(j)\} \\ & \wedge \bigwedge \{\chi_<(p_i, p_j) \mid \varphi(i) < \varphi(j)\}.\end{aligned}$$

Statement 3.8

PROPOSITION. — *For any $H \in \mathscr{D}$, any $\varphi_1, \varphi_2 \in C_n$ and any $f \in \mathfrak{F}_H^{\varphi_1}$ the following equality holds:*

$$\ulcorner\chi_{\varphi_2}\urcorner(c_H^{\varphi_1}(f)) = f(\varphi_1|\varphi_2).$$

PROOF. — Let $c_H^{\varphi_1}(f) = (h_1, h_2, ..., h_n)$. Then by definition of χ one has

$$\ulcorner\chi_{\varphi_2}\urcorner(h_1, h_2, ..., h_n)$$

$$= \begin{cases} \bot, & \begin{array}{l}\text{if there exists an } i \\ \text{such that either} \\ h_i = \bot \;\&\; \varphi_2(i) > 0 \\ \text{or} \\ h_i > \bot \;\&\; \varphi_2(i) = 0,\end{array} \\ \min\left\{ h_i \;\middle|\; \begin{array}{l}\text{there exists a } j \\ \text{such that either} \\ h_i < h_j \;\&\; \varphi_2(i) = \varphi_2(j) \\ \text{or} \\ h_i \leqslant h_j \;\&\; \varphi_2(i) > \varphi_2(j)\end{array} \right\}, & \text{otherwise}\end{cases}$$

$$= \begin{cases} \bot & \text{if } \varphi_1^{-1}(0) \neq \varphi_2^{-1}(0), \\ f\left(\min\left\{ \varphi_1(i) \;\middle|\; \begin{array}{l}\text{there exists a } j \\ \text{such that either} \\ \varphi_1(i) < \varphi_1(j) \;\&\; \varphi_2(i) = \varphi_2(j) \\ \text{or} \\ \varphi_1(i) \leqslant \varphi_1(j) \;\&\; \varphi_2(i) > \varphi_2(j)\end{array} \right\}\right) & \text{otherwise}\end{cases}$$

$$= \begin{cases} \bot & \text{if } \varphi_1^{-1}(0) \neq \varphi_2^{-1}(0), \\ f\left(\min\left\{ \varphi_1(i) \;\middle|\; \begin{array}{l}\text{there exists a } j \\ \text{such that either} \\ \varphi_1(i) < \varphi_1(j) \;\&\; \varphi_2(i) \geqslant \varphi_2(j) \\ \text{or} \\ \varphi_1(i) = \varphi_1(j) \;\&\; \varphi_2(i) \neq \varphi_2(j)\end{array} \right\}\right) & \text{otherwise.}\end{cases}$$

Let us denote

$$s = \begin{cases} 0, & \text{if } \varphi_1^{-1}(0) \neq \varphi_2^{-1}(0), \\ \min \left\{ \varphi_1(i) \;\middle|\; \begin{array}{l} \text{there exists a } j \\ \text{such that either} \\ \varphi_1(i) < \varphi_1(j) \ \&\ \varphi_2(i) \geqslant \varphi_2(j) \\ \text{or} \\ \varphi_1(i) = \varphi_1(j) \ \&\ \varphi_2(i) \neq \varphi_2(j) \end{array} \right\}, & \text{otherwise;} \end{cases}$$

we will prove that $\varphi_1|\varphi_2 = s$.

First let us prove $s \geqslant \varphi_1|\varphi_2$. It suffices to prove that there exists $s' \leqslant s$ with $\varphi_1^{-1}(s') \neq \varphi_2^{-1}(s')$. Consider the cases:

- $s = 0$ and $\varphi_1^{-1}(0) \neq \varphi_2^{-1}(0)$, then for s' we can choose 0.
- There exist $i, j \in [n]$ such that $s = \varphi_1(i) < \varphi_1(j)$ and $\varphi_2(i) \geqslant \varphi_2(j)$.

If $\varphi_1(i) = \varphi_2(i)$, we can choose $s' = \varphi_2(j) \leqslant \varphi_2(i) = s$. Then $\varphi_2(j) \leqslant \varphi_2(i) = \varphi_1(i) < \varphi_1(j)$.

If $\varphi_1(i) \neq \varphi_2(i)$, we can choose $s' = s$.

- There exist $i, j \in [n]$ such that $s = \varphi_1(i) = \varphi_1(j)$ and $\varphi_2(i) \neq \varphi_2(j)$.

We can choose $s' = s$. $i, j \in \varphi_1^{-1}(s)$ and either $i \notin \varphi_2^{-1}(s)$, or $j \notin \varphi_2^{-1}(s)$.

Now let us prove that $\varphi_1|\varphi_2 \geqslant s$. Consider the cases:

- There exists $i \in [n]$ such that $\varphi_1(i) = \varphi_1|\varphi_2 < \varphi_2(i)$ (the case $\varphi_2(i) < \varphi_1|\varphi_2$ is not possible, because for all $t < \varphi_1|\varphi_2$ one has $\varphi_1^{-1}(t) = \varphi_2^{-1}(t)$). Let $j \in \varphi_2^{-1}(\varphi_1|\varphi_2)$, then $\varphi_2(j) \leqslant \varphi_2(i)$. Moreover one has $\varphi_2(j) < \varphi_2(i)$ and $\varphi_1(i) \leqslant \varphi_1(j)$, therefore $\varphi_1|\varphi_2 = \varphi_1(i) \geqslant s$.
- There exists $i \in [n]$ such that $\varphi_2(i) = \varphi_1|\varphi_2 < \varphi_1(i)$ (the case $\varphi_1(i) < \varphi_1|\varphi_2$ is not possible, as in the previous case). Let $j \in \varphi_1^{-1}(\varphi_1|\varphi_2)$, then $\varphi_1(j) \leqslant \varphi_2(j)$. We have $\varphi_1(j) < \varphi_1(i)$ and $\varphi_2(j) \geqslant \varphi_2(i)$, therefore $\varphi_1|\varphi_2 = \varphi_1(j) \geqslant s$.

■

Statement 3.9

DEFINITION. — *For any* $\varphi : [n] \to [k]_0 \in C_n$, *its* r-th projection ($0 \leqslant r \leqslant k$) *is the map* $\pi_r^\varphi : H^\varphi \to H$ *defined as follows:*

$$\pi_r^\varphi(f \circ \varphi) = f(r) \text{ for any } f \in \mathfrak{F}_H^\varphi.$$

In other words, one has

$$\pi_0^\varphi(h_1, ..., h_n) = \bot,$$
$$\pi_r^\varphi(h_1, ..., h_n) = h_i \text{ for any } i \in \varphi^{-1}(r), \text{ and}$$
$$\pi_k^\varphi(h_1, ..., h_n) = \top.$$

It is clear that for every r there exists a formula A_r^φ (which may be equal either to $\top$ if $r = k$, or to $\perp$ if $r = 0$, or to the variable p_i if i is a label of r), such that $\ulcorner A_r^\varphi \urcorner|_{H^\varphi} = \pi_r^\varphi$.

Statement 3.10

For $n > 1$ and $1 \leqslant i \leqslant n$ we will denote by $\mathrm{pr}_i : H^n \to H^{n-1}$ the projection which omits the i-th component of $(h_1, ..., h_n)$, i. e.

$$\mathrm{pr}_i(h_1, ..., h_n) = (h_1, ..., h_{i-1}, h_{i+1}, ..., h_n).$$

PROPOSITION. — *For any $n > 1$, any $1 \leqslant i \leqslant n$ and any $H \in \mathscr{D}$ the image of each stratum $H^\varphi \subset H^n$ with $\varphi \in C_n$ under the projection $\mathrm{pr}_i : H^n \to H^{n-1}$ is a stratum $H^{\varphi'} \subset H^{n-1}$ for some $\varphi' \in C_{n-1}$. This element of C_{n-1} may be constructed from the labeled linearly ordered set corresponding to $\varphi \in C_n$ by deleting the label i and, if there exists an element without label different from the top and from the bottom, removing it.*

Conversely, for any stratum $H^{\varphi'} \subset H^{n-1}$ with $\varphi' \in C_{n-1}$, one can construct every $\varphi \in C_n$ with $\mathrm{pr}_i(H^\varphi) = H^{\varphi'}$ either by adding new label i to one of the non-top elements in the labeled linear order corresponding to $\varphi' \in C_{n-1}$, or inserting between two consecutive elements of this set a new element with label i.

PROOF. — Trivial. ■

Statement 3.11

For the proof of the subsequent theorems the following lemma is useful:

LEMMA. — *For any $H \in \mathscr{D}$ and any $n > 0$ if there are given two integers r_1, r_2, two elements $\varphi_1 : [n] \to [k_1]_0$, $\varphi_2 : [n] \to [k_2]_0$ of C_n such that $0 \leqslant r_1 \leqslant k_1$, $0 \leqslant r_2 \leqslant k_2$ and restrictions of the projections $\pi_{r_1}^{\varphi_1}$ and $\pi_{r_2}^{\varphi_2}$ to $H^{\varphi_1} \cap H^{\varphi_2}$ coincide (i. e. $\pi_{r_1}^{\varphi_1}\big|_{H^{\varphi_1} \cap H^{\varphi_2}} = \pi_{r_2}^{\varphi_2}\big|_{H^{\varphi_1} \cap H^{\varphi_2}}$), then either $r_1 = r_2$, or $\varphi_1|\varphi_2 \leqslant \min\{r_1, r_2\}$.*

PROOF. — Let $k := \varphi_1|\varphi_2$ and let H be some nontrivial dense Heyting algebra. Consider the order preserving inclusion $g : [k-1]_0 \to H$, with $g(0) = \perp$ and $g(k-1) < \top$. Such an inclusion exists, because H is dense and nontrivial. Using this g let us construct elements $f_1 \in \mathfrak{F}_H^{\varphi_1}$ and $f_2 \in \mathfrak{F}_H^{\varphi_2}$ by the rules

$$f_i(s) = \begin{cases} g(s), & \text{if } 0 \leqslant s < k \text{ for } i = 1, 2, \\ \top, & \text{otherwise.} \end{cases}$$

If $k = 0$, then the claim of the lemma trivially holds, if not, then $\phi_1^{-1}(0) = \phi_2^{-1}(0)$ and two elements $f_1 \circ \phi_1$ and $f_2 \circ \phi_2$ coincide. denote $\hat{h} := f_1 \circ \phi_1 = f_2 \circ \phi_2$. This $\hat{h}$ belongs to both H^{φ_1} and H^{φ_2}. Hence $\pi_{r_1}^{\varphi_1}(\hat{h}) = \pi_{r_1}^{\varphi_1}(\hat{h})$. Therefore if for example $r_1 < k$, then $\pi_{r_1}^{\varphi_1}(\hat{h}) = g(r_1)$ and if $r_1 \neq r_2$ one has $\pi_{r_1}^{\varphi_1}(\hat{h}) \neq \pi_{r_1}^{\varphi_1}(\hat{h})$ contradiction. The claim of lemma holds. ■

Statement 3.12

THEOREM. — *For any formula $A(p_1, ..., p_n)$, there exists a function $r : C_n \to \mathbf{N}$ such that for each $H \in \mathscr{D}$ and any $[n] \xrightarrow{\varphi} [k]_0 \in C_n$, one has $0 \leqslant r(\varphi) \leqslant k$ and $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$. That is, restriction of the function $\ulcorner A \urcorner$ to the stratum H^φ for any $\varphi \in C_n$ is the $r(\varphi)$-th projection of φ.*

PROOF. — The proof proceeds by induction on the difficulty of the formula A:

• $A = \top$, or $A = \bot$, or $A = p_i$, trivial. For these cases for any $[n] \xrightarrow{\varphi} [k]_0 \in C_n$ one can choose $r(\varphi) = k$, $r(\varphi) = 0$ or $r(\varphi) = \varphi(i)$ respectively.

• $A = A_1 \wedge A_2$ or $A = A_1 \vee A_2$ or $A = A_1 \to A_2$. By induction hypothesis there exist functions $r^1,\ r^2 : C_n \to \mathbf{N}$, such that for any $[n] \xrightarrow{\varphi} [k]_0 \in C_n$, one has $0 \leqslant r^i_\varphi \leqslant k$ and $\ulcorner A_i \urcorner|_{H^\varphi} = \pi^\varphi_{r^i_\varphi}$ for $i = 1, 2$. In these cases we can define $r : C_n \to \mathbf{N}$ by $r(\varphi) = \min\left\{r^1_\varphi, r^2_\varphi\right\}$, $r(\varphi) = \max\left\{r^1_\varphi, r^2_\varphi\right\}$ and

$$r(\varphi) = \begin{cases} k, & \text{if } r^1_\varphi \leqslant r^2_\varphi, \\ r^2_\varphi, & \text{if } r^1_\varphi > r^2_\varphi. \end{cases}$$

respectively.

• $A(p_1, ..., p_{i-1}, p_{i+1}, ..., p_n) = \forall p_i B(p_1, ..., p_{i-1}, p_i, p_{i+1}, ..., p_n)$. By induction hypothesis there exists a function $s : C_n \to \mathbf{N}$ such that for any $[n] \xrightarrow{\psi} [k]_0 \in C_n$, one has $0 \leqslant s(\psi) \leqslant k$ and $\ulcorner B \urcorner|_{H^\psi} = \pi^\psi_{s(\psi)}$. By definition for any $\tilde{h} = (h_1, ..., h_{i-1}, h_{i+1}, ..., h_n) \in H^{n-1}$ one has

$$\begin{aligned} \ulcorner A \urcorner(\tilde{h}) &= \bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in H^n \,\&\, \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\} \\ &= \bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\}. \end{aligned}$$

We must prove, that such $\bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\}$ always exists and that there exists a function $r : C_{n-1} \to \mathbf{N}$ such that for any $[n-1] \xrightarrow{\varphi} [k]_0 \in C_{n-1}$ and $\tilde{h} \in H^\varphi$ one has $0 \leqslant r(\varphi) \leqslant k$ and $\bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\} = \pi^\varphi_{r(\varphi)}(\tilde{h})$.

For every $[n-1] \xrightarrow{\varphi} [k]_0 \in C_{n-1}$ consider the subset $C_\varphi \subset C_n$ of all such $[n] \xrightarrow{\psi} [m]_0 \in C_n$ that $\mathrm{pr}_i(H^\psi) = H^\varphi$.

Then for each $\tilde{h} \in H^\varphi$ one has

$$\begin{aligned} &\bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\} \\ &\qquad = \bigwedge_{[n] \xrightarrow{\psi} [m]_0 \in C_\varphi} \bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in H^\psi \,\&\, \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\}. \end{aligned}$$

We will find for each $[n] \xrightarrow{\psi} [m]_0 \in C_\varphi$ a nonnegative integer $r_{\varphi,\psi}$, such that

$$\bigwedge \left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\} = \pi^\varphi_{r_{\varphi,\psi}}(\tilde{h})$$

for every $\tilde{h} \in H^\varphi$.

Each $\psi \in C_\varphi$ using the Proposition in 3.10 can be represented either by adding label i to one of the non-top elements in the labeled linear order corresponding to $\varphi \in C_{n-1}$, or inserting in this linear order a new element with label i between two consecutive points.

Consider the cases:

1) $\psi \in C_\varphi$ is represented by a labeling with label i on one of the nonmaximal points in the labeled linear ordered set corresponding to $\varphi \in C_{n-1}$. Then for each $\hat{h} \in H^\psi$ one has $\pi^\psi_{s(\psi)}(\hat{h}) = \pi^\varphi_{s(\psi)}(\mathrm{pr}_i(\hat{h}))$. Hence $\left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\}$ $= \left\{ \pi^\varphi_{s(\psi)}(\tilde{h}) \right\}$, and $\bigwedge \left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\} = \pi^\varphi_{s(\psi)}(\tilde{h})$. So we may put $r_{\varphi,\psi} = s(\psi)$.

2) $\psi \in C_\varphi$ is represented by inserting into the labeled linear order corresponding to φ a new element labeled with i between two adjacent elements and $s(\psi)$ is not the number of this new element. In this case define: $r_{\varphi,\psi} = s(\psi)$ if $s(\psi)$ is less than number of this new element and $r_{\varphi,\psi} = s(\psi) - 1$ otherwise. It is clear that for each $\hat{h} \in H^\psi$ one has $\pi^\psi_{s(\psi)}(\hat{h}) = \pi^\varphi_{r_{\varphi,\psi}}(\mathrm{pr}_i(\hat{h}))$. Hence $\left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\}$ $= \left\{ \pi^\varphi_{r_{\varphi,\psi}}(\tilde{h}) \right\}$, and $\bigwedge \left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\} = \pi^\varphi_{r_{\varphi,\psi}}(\tilde{h})$.

3) $\psi \in C_\varphi$ is represented by inserting a new element labeled with i between two consecutive elements into the labeled linear order corresponding to φ and $s(\psi)$ is the number of this new element. Let the numbers of these consecutive elements be q_1 and q_2. Then $\ulcorner B \urcorner (h_1, ... h_{i-1}, h_i, h_{i+1}, ... h_n) = h_i$ for all $\hat{h} = (h_1, ... h_{i-1}, h_i, h_{i+1}, ... h_n) \in H^\psi$. Hence $\left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\} =$ $(f(q_1), f(q_2))$, where $f = {c^\psi_H}^{-1}(\hat{h})$. But $\bigwedge(f(q_1), f(q_2)) = f(q_1)$, because H is dense. Therefore

$$\bigwedge \left\{ \ulcorner B \urcorner (\hat{h}) \;\middle|\; \hat{h} \in H^\psi \;\&\; \mathrm{pr}_i(\hat{h}) = \tilde{h} \right\} = \bigwedge(f(q_1), f(q_2)) = f(q_1)$$

So we can choose $r_{\varphi,\psi} = q_1$.

Hence $\ulcorner A \urcorner (h_1, ..., h_{i-1}, h_{i+1}, ..., h_n)$ is a finite intersection of projections, thus is itself a projection. Let us define $r(\varphi)$ as $\min \{ r_{\varphi,\psi} \mid \psi \in C_\varphi \}$.

- $A = \exists p_i B$. Similar to the previous case.

■

Statement 3.13

Now we will prove the converse theorem.

THEOREM. — *For $H \in \mathscr{D}$ and $n > 0$, suppose given a function $r : C_n \to \mathbf{N}$ such that for each $H \in \mathscr{D}$ and any $\varphi_1 : [n] \to [k_1]_0$, $\varphi_2 : [n] \to [k_2]_0$ from C_n one has $0 \leqslant r(\varphi_1) \leqslant k_1$, $0 \leqslant r(\varphi_2) \leqslant k_2$ and restrictions of the projections $\pi^{\varphi_1}_{r(\varphi_1)}$ and $\pi^{\varphi_2}_{r(\varphi_2)}$ to $H^{\varphi_1} \cap H^{\varphi_2}$ coincide,*

$$\pi^{\varphi_1}_{r(\varphi_1)}\Big|_{H^{\varphi_1} \cap H^{\varphi_2}} = \pi^{\varphi_2}_{r(\varphi_2)}\Big|_{H^{\varphi_1} \cap H^{\varphi_2}}.$$

Then there exists a quantifier-free formula $A(p_1, ..., p_n)$ with at most n free variables such that restriction of the function $\ulcorner A \urcorner$ to every stratum H^φ coincides with the projection $\pi^\varphi_{r(\varphi)}$ (i. e. $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$).

PROOF. — Let us construct the formula A using the function $r : C_n \to \mathbf{N}$ as follows:

$$A = \bigwedge_{\varphi \in C_n} (\chi_\varphi \to A^\varphi_{r(\varphi)})$$

where $A^\varphi_{r(\varphi)}$ is the formula mentioned in 3.9.

Let us prove that restriction of the function $\ulcorner A \urcorner$ to any stratum H^φ coincides with the projection $\pi^\varphi_{r(\varphi)}$ (i. e. $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$).

Suppose $f \in \mathfrak{F}^H_\varphi$ and $f \circ \varphi = \hat{h} = (h_1, h_2, ..., h_n) \in H^\varphi$, then

$$\ulcorner A \urcorner(\hat{h}) = \bigwedge_{\psi \in C_n} \ulcorner \chi_\psi \urcorner(\hat{h}) \to \ulcorner A^\psi_{r(\psi)} \urcorner(\hat{h}).$$

First we will prove, that for any $\psi \in C_n$ the inequality

$$(\ulcorner \chi_\varphi \urcorner(\hat{h}) \to \ulcorner A^\varphi_{r(\varphi)} \urcorner(\hat{h})) \leqslant (\ulcorner \chi_\psi \urcorner(\hat{h}) \to \ulcorner A^\psi_{r(\psi)} \urcorner(\hat{h}))$$

holds, then using this inequality we will have $\ulcorner A \urcorner(\hat{h}) = (\ulcorner \chi_\varphi \urcorner(\hat{h}) \to \ulcorner A^\varphi_{r(\varphi)} \urcorner(\hat{h}))$. But by definition of "characteristic" formulæ one has $\ulcorner \chi_\varphi \urcorner(\hat{h}) = \top$ because $\hat{h} = (h_1, h_2, ..., h_n) \in H^\varphi$. Hence $\ulcorner A \urcorner(\hat{h}) = \ulcorner A^\varphi_{r(\varphi)} \urcorner(\hat{h}) = \pi^\varphi_{r(\varphi)}(\hat{h})$, which means, that the claim of the theorem holds.

Really, because $\ulcorner \chi_\varphi \urcorner(\hat{h}) = \top$, we must prove, that $\ulcorner A^\varphi_{r(\varphi)} \urcorner(\hat{h}) \leqslant (\ulcorner \chi_\psi \urcorner(\hat{h}) \to \ulcorner A^\psi_{r(\psi)} \urcorner(\hat{h}))$. Define $k := \psi|\varphi$. Then of course $\ulcorner A^\varphi_{r(\varphi)} \urcorner(\hat{h}) = f(r(\varphi))$ and using the proposition 3.8, one has $\ulcorner \chi_\psi \urcorner(\hat{h}) = f(k)$, therefore we must prove, that $f(r(\varphi)) \leqslant f(k) \to \ulcorner A^\psi_{r(\psi)} \urcorner(\hat{h})$.

There are two cases:

1) $k \leqslant r(\psi)$, then of course $f(k) \leqslant \ulcorner A^\psi_{r(\psi)} \urcorner(\hat{h})$ and our claim holds trivially.

2) $r(\psi) < k$, then $\ulcorner A^\psi_{r(\psi)} \urcorner(\hat{h}) = f(r(\psi))$ and our claim is to prove $f(r(\varphi)) \leq f(k) \to f(r(\psi))$ inequality. Using the lemma from 3.11 one has either $f(r(\varphi)) = f(r(\psi))$, or $k \leqslant \min(f(r(\varphi)), f(r(\psi)))$; in both cases the claim holds.

■

4. Subdivision for $\mathscr{S}$

Statement 4.1

Now we define another subdivision of H^n, which is useful for describing functions for $H \in \mathscr{S}$. This subdivision takes into account not only ordering between components of n-tuples from H^n. If $H \in \mathscr{S}$, then for two elements $h_1, h_2 \in H$ the case $h_1 < h_2$ may be divided into subcases: when $h_2 = \Box(h_1)$, $h_2 = \Box^2(h_1)$, $h_2 = \Box^3(h_1), \ldots$.

First of all let us fix two natural numbers m and m_1 with $m \leqslant m_1$. Consider a map $\varphi : [n]_0 \to [k]_0$, $0 \leqslant k$, such that

1) $\varphi(0) = 0$;

2) for any m_1 consecutive numbers $0 \leqslant i, i+1, i+2, ..., i+m_1-1 < k$ at least one belongs to the image of φ;

3) none of the m consecutive numbers $k-m+1, k-m+1, ..., k-1, k$ belongs to the image of φ.

Let C_n^{m,m_1} denote the set of all such maps φ. If $m = m_1$, we write C_n^m for short.

In terms of the function d defined in 2.6.1 above, these conditions read, respectively:

1) $\varphi(0) = 0$;

2) for any $r \in [k]_0$ one has $d(r) \leqslant m_1$;

3) $\varphi^{-1}(k) = \varnothing$;

4) $d(k) \geqslant m$.

For $\varphi \in C_n^{m,m_1}$ as above, a number $s \in [k]_0$ will be called *φ-peculiar* if either $\varphi^{-1}(s) \neq \varnothing$ and $d(r) \geqslant m$, or $s = k$. Note that each peculiar element except k is labeled and k is not labeled.

Statement 4.2

As for the class $\mathscr{D}$, we can represent elements of C_n^{m,m_1} by finite, labeled linear orders, where now it is not required anymore that all elements different from top and bottom are labeled. Now the set of labels is $[n]_0$ and the bottom element is necessarily labeled at least with 0. We impose the condition that there are no m_1 consecutive nonlabeled elements, except for the last m_1 elements. Moreover we demand that distance from the largest labeled element to the top is exactly m_1.

It is easy to see that in these terms peculiarity of an element means that this element is different from the bottom, is labeled, and the equality

$$\text{(distance from it to the previous labeled element)} \geqslant m$$

is satisfied. Note that the top element is always peculiar. When drawing such ordered sets, peculiar elements will be depicted by black circles.

The set of all such finite, labeled linear ordered sets will be denoted by R_n^{m,m_1}. Here also if $m = m_1$, we write R_n^m for short instead of $R_n^{m,m}$.

There is a natural isomorphism α_n^{m,m_1} between C_n^{m,m_1} and R_n^{m,m_1}, which sends every $\varphi : [n]_0 \to [k]_0$ to the set $[k]_0$ equipped with the corresponding labeling and peculiarity.

Statement 4.3

EXAMPLES. — The picture

$$\begin{array}{ccccccccccccccccc} 0 & & 2,5 & & 6 & & 1 & & & & 3,4 & & & & & & \\ \circ & \to & \circ & \to & \circ & \to & \circ & \to & \circ & \to & \circ & \to & \circ & \to & \circ & \to & \bullet \end{array}$$

represents an element φ of C_6^3. None of its elements except for the top are peculiar.

As another example, involving some more peculiar elements, take

$$\begin{array}{ccccccccccccccccccccc} 2 & & & & & & & & 3 & & 1 & & & & & & & & & & \\ \circ & \to & \circ & \to & \circ & \to & \circ & \to & \bullet & \to & \circ & \to & \circ & \to & \circ & \to & \circ & \to & \circ & \to & \bullet \end{array}$$

This represents an element of C_3^4, and its element with label 3 is peculiar. $\square$

Statement 4.4

For every $\varphi \in C_n^{m,m_1}$ we consider the subset $H^\varphi \subseteq H^n$, consisting of all n-tuples $(h_1, ..., h_n) \in H^n$ such that for any $i, j \in [n]_0$ and $h_0 = \bot$ one has:

- if $\varphi(i) = \varphi(j)$, then $h_i = h_j$;
- if $\varphi(i)$ is the next labeled element after $\varphi(j)$ in $[k]_0$ and $\varphi(i) - \varphi(j) < m$, then $h_i = \square^{\varphi(i)-\varphi(j)} h_j$;
- if $\varphi(i)$ is the next labeled element after $\varphi(j)$ in $[k]_0$ and $\varphi(i) - \varphi(j) \geqslant m$, then $h_i \geqslant \square^{\varphi(i)-\varphi(j)} h_j$

These conditions can be rewritten in the following equivalent form:

- if $\varphi(i) = \varphi(j)$, then $h_i = h_j$;
- if $\varphi(i) - \varphi(j) = d \geqslant 0$, then $h_i \geqslant \square^d h_j$;
- if $\varphi(i) - \varphi(j) = d \geqslant 0$ and there are no peculiar elements among $\varphi(j+1), \varphi(j+2), ..., \varphi(i)$, then $h_i = \square^d h_j$.

Also we can define:

$$\mathfrak{F}_H^\varphi = \left\{ f : [k]_0 \to H \;\middle|\; \begin{array}{l} f(0) = \bot \text{ and} \\ f(k) = \top \text{ and} \\ \text{for all } i \geqslant j, \quad f(i) \geqslant \square^{i-j} f(j) \text{ and} \\ \text{for all } i \geqslant j \quad \text{with no peculiar elements} \\ \text{in } j+1, ..., i, \;\; f(i) = \square^{i-j} f(j) \end{array} \right\}.$$

For every $H \in \mathscr{S}$ and $\varphi \in C_n^m$, there is a naturally defined map

$$c_H^\varphi : \mathfrak{F}_H^\varphi \longrightarrow H^\varphi,$$
$$f \longmapsto f \circ \varphi|_{[n]}$$

(φ is defined on $[n]_0$, and to determine the map c_H^φ we use $\varphi|_{[n]}$).

The map c_H^φ is an order preserving isomorphism.

For simplicity we denote for each $V \in R_n^{m,m_1}$ the stratum $H^{\alpha^{-1}(V)}$, the set $\mathfrak{F}_H^{\alpha^{-1}(V)}$ and the map $c_H^{\alpha^{-1}(V)}$ by H^V, $\mathfrak{F}_H^V$ and c_H^V respectively.

E. g., the stratum in H^6 corresponding to the first example in 4.3 is

$$\begin{aligned}&\left\{(h_1, ..., h_6) \;\middle|\; h_2 = h_5 = \Box(\bot), \Box(h_5) = h_6, \Box(h_6) = h_1, \Box^2(h_1) = h_3 = h_4\right\}\\ =&\left\{(\Box^3(\bot), \Box(\bot), \Box^5(\bot), \Box^5(\bot), \Box(\bot), \Box^2(\bot))\right\}.\end{aligned}$$

The stratum in H^3 corresponding to the second example in 4.3 is

$$\begin{aligned}&\left\{(h_1, h_2, h_3) \in H^3 \;\middle|\; h_2 = \bot,\ \Box^4(h_2) \leqslant h_3,\ h_1 = \Box(h_3)\right\}\\ =&\left\{(\Box(h), \bot, h) \;\middle|\; \Box^4(\bot) \leqslant h \in H\right\}.\end{aligned}$$

Statement 4.5

For each $\varphi : [n]_0 \to [k]_0 \in C_n^m$, the sets H^φ and $\mathfrak{F}_H^\varphi$ are nonempty, moreover we can choose some so called "standard" elements st_H^φ in $\mathfrak{F}_H^\varphi$. Define:

$$\mathrm{st}_H^\varphi : [k]_0 \longrightarrow H$$
$$r \longmapsto \begin{cases} \Box^r(\bot), & \text{if } 0 \leqslant r < k; \\ \top, & \text{if } r = k \end{cases}$$

The element of H^φ corresponding to st_H^φ under the mapping c_H^φ will be denoted by $\ulcorner \mathrm{st}_H^\varphi \urcorner$. This "standard" element is the least element in $\mathfrak{F}_H^\varphi$.

Note that the union of all H^φ, $\varphi \in C_n^m$, is the whole H^n. Indeed, given any $(h_1, ..., h_n) \in H^n$ put $h_0 = \bot$ and consider the set

$$U = \left\{\Box^d(h) \;\middle|\; 0 \leqslant d \leqslant n \text{ and } (h = h_i \text{ for some } i \in [n])\right\}.$$

Label an element h in V with $i \in [n]$ whenever $h_i = h$.

If h is the top element of U and $k = d(h) < m$, then let V be a new labeled linearly ordered set obtained from U by adding $m - k$ new nonlabeled elements which are greater than each element of U, otherwise let $V = U$. Clearly $(h_1, ..., h_n) \in H^V$.

Statement 4.6

DEFINITION. — *For two elements $\varphi_1 : [n]_0 \to [k_1]_0$ and $\varphi_2 : [n]_0 \to [k_2]_0$ of C_n^m the nonnegative integer $\varphi_1|\varphi_2$ is given by the equality*

$$\varphi_1|\varphi_2 = \min \left\{ r \in [k_1]_0 \cap [k_2]_0 \;\middle|\; \varphi_1^{-1}(r) \neq \varphi_2^{-1}(r) \right\}.$$

Then arguing similarly to the case in 3.6 one can show that if $\varphi_1|\varphi_2 = k_i$, $i = 1, 2$, then $k_1 = k_2$ and $\varphi_1 = \varphi_2$.

Statement 4.7

For each $\varphi \in C_n^{m,m_1}$ we construct the formula χ_φ , which we call *characteristic formula* for the stratum of φ. The value of the function $\ulcorner \chi_\varphi \urcorner$ on the tuple $(h_1, ..., h_n)$ will be equal to $\top$ if and only if $(h_1, ..., h_n) \in H^\varphi$.

Such a formula also can be constructed using the "characteristic formulæ" for the relations $=$, $<$ and $\leqslant$, as in 3.7. Free variables of the formula χ_φ will be among $\{p_1, p_2, ..., p_n\}$. Denote by p_0 the constant $\perp$ and put

$$\begin{aligned} &\chi_\varphi(p_1, p_2, ..., p_n) := \\ &\quad \bigwedge \left\{ \chi_\leqslant(\Box^d(p_i), p_j) \;\middle|\; d = \varphi(j) - \varphi(i) \geqslant 0 \right\} \\ &\quad \wedge \bigwedge \left\{ \chi_\leqslant(p_j, \Box^d(p_i)) \;\middle|\; \begin{array}{l} d = \varphi(j) - \varphi(i) \geqslant 0 \\ \text{and none of } \varphi(i)+1, \varphi(i)+2, ..., \varphi(j) \text{ are peculiar} \end{array} \right\}. \end{aligned}$$

Statement 4.8

PROPOSITION. — *For any $H \in \mathscr{S}$, $m > 0$, $\varphi_1, \varphi_2 \in C_n^m$ and any $f \in \mathfrak{F}_H^{\varphi_1}$ the following equality holds:*

$$\ulcorner \chi_{\varphi_2} \urcorner (c_H^{\varphi_1}(f)) = f(\varphi_1|\varphi_2).$$

PROOF. — Let $c_H^{\varphi_1}(f) = (h_1, h_2, ..., h_n)$, and let $h_0 = \perp$. Then by definition of χ one has

$$
\begin{aligned}
&\ulcorner\chi_{\varphi_2}\urcorner(h_1, h_2, ..., h_n)\\
&\quad = \min\left\{ h_i \;\middle|\; \begin{array}{l} \text{there exists a } j \in [n]_0 \text{ such that} \\ d = \varphi_2(i) - \varphi_2(j) \geqslant 0 \text{ and } h_i < \Box^d(h_j) \end{array} \right\} \\
&\quad \wedge \min\left\{ \Box^d(h_i) \;\middle|\; \begin{array}{l} \text{there exists a } j \in [n]_0 \text{ such that } d = \varphi_2(j) - \varphi_2(i) \geqslant 0 \\ \text{and none of } \varphi_2(i)+1, \varphi_2(i)+2, ..., \varphi_2(j) \text{ are peculiar} \\ \text{and } \Box^d(h_i) < h_j \end{array} \right\} \\
&\quad = f\left(\begin{array}{l} \min\left\{ \varphi_1(i) \;\middle|\; \begin{array}{l} \text{there exists a } j \in [n]_0 \text{ such that} \\ d = \varphi_2(i) - \varphi_2(j) \geqslant 0 \text{ and } \varphi_1(i) < d + \varphi_1(j) \end{array} \right\} \\
\wedge \min\left\{ d + \varphi_1(i) \;\middle|\; \begin{array}{l} \text{there exists a } j \in [n]_0 \text{ such that} \\ \quad d = \varphi_2(j) - \varphi_2(i) \geqslant 0 \\ \text{and none of } \varphi_2(i)+1, \varphi_2(i)+2, ..., \varphi_2(j) \\ \quad \text{are peculiar} \\ \text{and } d + \varphi_1(i) < \varphi_1(j) \end{array} \right\} \end{array} \right)
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
s := \min&\left\{ \varphi_1(i) \;\middle|\; \begin{array}{l} \text{there exists a } j \in [n]_0 \text{ such that} \\ d = \varphi_2(i) - \varphi_2(j) \geqslant 0 \text{ and } \varphi_1(i) < d + \varphi_1(j) \end{array} \right\} \\
\wedge \min&\left\{ d + \varphi_1(i) \;\middle|\; \begin{array}{l} \text{there exists a } j \in [n]_0 \text{ such that } d = \varphi_2(j) - \varphi_2(i) \geqslant 0 \\ \text{and none of } \varphi_2(i)+1, \varphi_2(i)+2, ..., \varphi_2(j) \text{ are peculiar} \\ \text{and } d + \varphi_1(i) < \varphi_1(j) \end{array} \right\};
\end{aligned}
$$

we will prove that $\varphi_1|\varphi_2 = s$.

First let us prove $s \geqslant \varphi_1|\varphi_2$. It suffices to prove that there exists $s' \leqslant s$ with $\varphi_1^{-1}(s') \neq \varphi_2^{-1}(s')$. Consider the cases:

• $s = \varphi_1(i)$ and there exists $j \in [n]_0$ such that $d = \varphi_2(i) - \varphi_2(j) \geqslant 0$ and $\varphi_1(i) < d + \varphi_1(j)$. If $\varphi_1(i) \neq \varphi_2(i)$, we can choose s instead of s'; if not, let us choose $s' = s - d$. Then $\varphi_2(j) = \varphi_2(i) - d$, i. e. $j \in \varphi_2^{-1}(s')$ and $\varphi_1(j) > \varphi_1(i) - d = s'$, hence $j \notin \varphi_1^{-1}(s')$.

• $s = \varphi_1(i) + d$ and there exists a $j \in [n]_0$ such that $d = \varphi_2(j) - \varphi_2(i) \geqslant 0$, there are no peculiar elements among $\varphi_2(i)+1, \varphi_2(i)+2, ..., \varphi_2(j)$, and $d + \varphi_1(i) < \varphi_1(j)$.

If $\varphi_1(i) \neq \varphi_2(i)$, we can choose $\varphi_1(i) \leqslant s$ instead of s'.

If $\varphi_1(i) = \varphi_2(i)$, let us choose $s' := s = \varphi_1(i) + d$. Then $j \in \varphi_2^{-1}(s')$ and $s' = \varphi_1(i) + d > \varphi_1(j)$, hence $j \notin \varphi_1^{-1}(s')$.

Now let us prove that $\varphi_1|\varphi_2 \geqslant s$. Denote $r = \varphi_1|\varphi_2$. Consider the cases:

1) There exists an $l \in [n]$ such that $\varphi_1(l) = r < \varphi_2(l)$ (the case $\varphi_2(l) < r$ is not possible, because for all $t < r$ one has $\varphi_1^{-1}(t) = \varphi_2^{-1}(t)$). Let $i = l$, $j = 0$ and $d = \varphi_2(i)$. Then $d = \varphi_2(i) - \varphi_2(j)$ and $\varphi_1(i) < d + \varphi_1(j) = d = \varphi_2(i)$. Hence $s \leqslant \varphi_1(i) = r$.

2) There exists an $l \in [n]$ such that $\varphi_2(l) = r < \varphi_1(l)$ ($\varphi_1(l) < r$ is not possible, as in the previous case) and r is peculiar for φ_2. Since r is peculiar for φ_2, $r - m$ is labeled with φ_2 and none of $r-m+1, r-m+2, ..., r-1$ are labeled with φ_2. But for all $t < r$ one has $\varphi_1^{-1}(t) = \varphi_2^{-1}(t)$, hence $r - m$ is labeled with φ_1 and none of $r-m+1, r-m+2, ..., r-1$ are labeled with φ_1. It means that r is peculiar for φ_1. Therefore it is labeled under φ_1. Suppose $\varphi_1(l') = r$.

If $\varphi_2(l') > r$, we get $r \geqslant s$ as in the previous case.

If $\varphi_2(l') = r$, then $r = \varphi_2(l) = \varphi_2(l') = \varphi_1(l') < \varphi_1(l)$. Denote $i := l'$, $j := l$ and $d := 0$. Then $d+\varphi_1(i) < \varphi_1(j)$ and $\varphi_2(j)-\varphi_2(i) = d$. Hence $s \leqslant d+\varphi_1(i) = r$.

3) There exists $l \in [n]$ such that $\varphi_2(l) = r < \varphi_1(l)$ and $r = 0$. Denote $i = 0$, $j := l$ and $d = 0$. Then $d = \varphi_2(j) - \varphi_2(i)$, there are no elements peculiar for φ_2 between $0 = \varphi_2(j)$ and $\varphi_2(i) = 0$ and $d + \varphi_1(i) = 0 < \varphi_1(j)$. Hence $s = 0$.

4) There exists $l \in [n]$ such that $\varphi_2(l) = r < \varphi_1(l)$, and $r > 0$ and r is not peculiar for φ_2. Let r' be $\max\left\{t < r \;\middle|\; \varphi_2^{-1}(t) \neq \varnothing\right\}$. Then since $r' < r$ one has $\varphi_1^{-1}(r') = \varphi_2^{-1}(r') \neq \varnothing$. Let i be in $\varphi_1^{-1}(r') = \varphi_2^{-1}(r')$, $j := l$ and $d = r - r'$. Then $d = \varphi_2(j) - \varphi_2(i)$, there are no elements peculiar for φ_2 among $\{\varphi_2(i)+1, \varphi_2(i)+2, ..., \varphi_2(j)\} = \{r'+1, r'+2, ..., r\}$ and $d + \varphi_1(i) = r < \varphi_1(j)$. Hence $s \leqslant r$.

■

Statement 4.9

DEFINITION. — *For any* $\varphi : [n] \to [k]_0 \in C_n^{m,m_1}$*, its* r-th projection *(*$0 \leqslant r \leqslant k$*) is the map* $\pi_r^\varphi : H^\varphi \to H$ *is defined as follows:*

$$\pi_r^\varphi(f \circ \varphi) = f(r) \text{ for any } f \in \mathfrak{F}_H^\varphi.$$

In other words, one has

$$\pi_0^\varphi(h_1, ..., h_n) = \bot,$$

$$\pi_k^\varphi(h_1, ..., h_n) = \top,$$

$$\pi_r^\varphi(h_1, ..., h_n) = \Box^{r-r'} h_i, \text{ if } r' \text{ is the greatest among labeled elements not exceeding } r\text{, and } i \text{ is a label of } r'$$

As in the case of dense algebras, for every $r \in [k]_0$ there exists a formula A_r^φ (which may be equal either to $\top$ if $r = k$, or to $\bot$ if $r = 0$, or to $\Box^{r-r'} p_i$ if r' is the greatest among labeled elements not exceeding r and i is a label of r'), such that $\ulcorner A_r^\varphi \urcorner|_{H^\varphi} = \pi_r^\varphi$.

Statement 4.10

Now for each C_n^{m,m_1}, or equivalently for R_n^{m,m_1} we determine a map β^{m,m_1} : $R_n^{m,m_1} \to R_n^m$ such that, for every $V \in R_n^{m,m_1}$ the stratum H^V will be a subset of $H^{\beta_n^{m,m_1}(V)}$. That is, the subdivision $\left\{H^V \mid V \in R_n^{m,m_1}\right\}$ of H^n is finer than $\left\{H^V \mid V \in R_n^m\right\}$.

For $V \in R_n^{m,m_1}$ let V_1 be the subset of V obtained from V by deleting all those nonlabeled and nontop elements $s \in V$ which satisfy $d(s) \geqslant m$, i. e.

$$V_1 = V \setminus \left\{ s \in V \;\middle|\; \begin{array}{l} s \text{ is not a top element and} \\ s \text{ is not labeled and} \\ d(s) \geqslant m \end{array} \right\}.$$

The finite set V_1 with the ordering and labeling induced from V is an element of R_n^m. Determine $\beta^{m,m_1}(V)$ as V_1. Of course $H^V \subset H^{V_1} = H^{\beta^{m,m_1}(V)}$.

Denote by γ_V or by γ_φ (if φ is the element of C_n^{m,m_1} corresponding to V) the inclusion $V_1 \subseteq V$.

For each $f \in \mathfrak{F}_H^V$ the composite $\gamma_V \circ f$ is an element $f' \in \mathfrak{F}_H^{V_1}$ such that $c_H^V(f) = c_H^{V_1}(f')$.

If there is given $0 \leqslant r < |V_1|$, then restriction of the r-th projection of $H^{V_1} = H^{\beta_n^{m,m_1}(V)}$ to H^V is the $\gamma_V(r)$-th projection of H^V. That is,

$$\pi_r^{\beta^{m,m_1}(V)}\Big|_{H^V} = \pi^V_{\gamma_V(r)}.$$

Statement 4.11

To prove subsequent theorems, the following lemma is useful:

LEMMA. — *If for some formula $A(p_1, ..., p_n)$ and some integer $m > 0$ there exists a function $r : C_n^m \to \mathbf{N}$ such that, for each $H \in \mathscr{S}$ and $\varphi \in C_n^m$, one has $0 \leqslant r(\varphi) \leqslant k$ and $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$, then for any $\varphi_1,\ \varphi_2 \in C_n^m$ either $r(\varphi_1) = r(\varphi_2)$, or $\varphi_1|\varphi_2 \leqslant \min\{r(\varphi_1), r(\varphi_2)\}$.*

PROOF. — let $k := \varphi_1|\varphi_2$ and let H be some infinite Heyting algebra from $\mathscr{S}$. Consider the “standard” elements $\mathrm{st}_H^{\varphi_1} = (h_1^1, h_2^1, ..., h_n^1)$ and $\mathrm{st}_H^{\varphi_2} = (h_1^2, h_2^2, ..., h_n^2)$ in H^n. By the congruence property (Cong) (see the end of 2.1) one has

$$\bigwedge_{i\in[n]} (h_i^1 \leftrightarrow h_i^2) \leqslant \ulcorner A \urcorner(\mathrm{st}_H^{\varphi_1}) \leftrightarrow \ulcorner A \urcorner(\mathrm{st}_H^{\varphi_2}).$$

Let $f^1 \in \mathfrak{F}_H^{\varphi_1}$ and $f^2 \in \mathfrak{F}_H^{\varphi_2}$ correspond to $\mathrm{st}_H^{\varphi_1}$ and $\mathrm{st}_H^{\varphi_2}$ respectively. Then one has $\ulcorner A \urcorner(\mathrm{st}_H^{\varphi_1}) = f^1(r(\varphi_1))$, $\ulcorner A \urcorner(\mathrm{st}_H^{\varphi_2}) = f^2(r(\varphi_2))$ and

$$\bigwedge_{i\in[n]} (h_i^1 \leftrightarrow h_i^2) = \min\left\{\min\{h_i^1, h_i^2\} \;\middle|\; h_i^1 \neq h_i^2\right\} = f^1(\varphi_1|\varphi_2) = f^2(\varphi_1|\varphi_2)$$

(because f^1 and f^2 are "standard" elements). Then

$$f^1(\varphi_1|\varphi_2) = f^2(\varphi_1|\varphi_2) \leqslant f^1(r(\varphi_1)) \leftrightarrow f^2(r(\varphi_2)).$$

Since H is infinite, the maps f^1 and f^2 are inclusions, hence the lemma. ■

Statement 4.12

THEOREM. — *For any formula $A(p_1, ..., p_n)$ and $m = 2^{\deg(A)}$ $(\deg(A)$ means quantifier degree of A, see 2.1$)$ there exists a function $r : C_n^m \to \mathbf{N}$ such that, for each $H \in \mathscr{S}$ and $[n]_0 \xrightarrow{\varphi} [k]_0 \in C_n^m$, one has $0 \leqslant r(\varphi) \leqslant k$ and $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$, i. e. restriction of the function $\ulcorner A \urcorner$ to the stratum H^φ for any $\varphi \in C_n^m$ is the $r(\varphi)$-th projection.*

PROOF. — The proof proceeds by induction on the difficulty of the formula A.

• $A = \top$, or $A = \bot$, or $A = p_i$, trivial. For these cases for any $[n]_0 \xrightarrow{\varphi} [k]_0 \in C_n^m$ one can choose $r(\varphi) = k$, $r(\varphi) = 0$ or $r(\varphi) = \varphi(i)$ respectively.

• $A = A_1 \wedge A_2$ or $A = A_1 \vee A_2$ or $A = A_1 \to A_2$. Suppose $m_1 = \deg(A_1)$, $m_2 = \deg(A_2)$ and $m = \max(m_1, m_2)$, then by definition of quantifier degree, one has $m = \deg(A)$. By induction hypothesis there exist functions $r_1,\ r_2 : C_n \to \mathbf{N}$, such that for $i = 1, 2$ and $\varphi_i : [n]_0 \to [k_i]_0$ in $C_n^{m_i}$ one has $0 \leqslant r_i(\varphi_i) \leqslant k$ and $\ulcorner A_i \urcorner|_{H^{\varphi_i}} = \pi^{\varphi_i}_{r_i(\varphi_i)}$. Every $V \in R_n^m$ can be considered as an element of $R_n^{m_i,m}$ for each $i = 1, 2$ and the stratum H^V is a subset of the stratum $H^{\beta_n^{m,m_i}}(V)$ (see 4.10); moreover if γ_i is the corresponding inclusion of $V_i = \beta_n^{m,m_i}(V)$ into V and if $s_i(V) = \gamma_i(r_i(V_i))$, then it is clear that restriction of $\ulcorner A \urcorner_i$ to H^V is the $s_i(V)$-th projection. In these cases we can define $r : C_n^m \to \mathbf{N}$ by $r(V) = \min\{s_1(V), s_2(V)\}$, $r(V) = \max\{s_1(V), s_2(V)\}$, and

$$r(V) = \begin{cases} \text{the top element of } V, & \text{if } s_1(V) \leqslant s_2(V), \\ s_2(V), & \text{if } s_1(V) > s_2(V) \end{cases}$$

respectively.

• $A(p_1, ..., p_{n-1}) = \forall p_i B(p_1, ..., p_{n-1}, p_n)$ (of course we can suppose that $i = n$, i. e. the bound variable in the formula A is p_n). By induction hypothesis there exists a function $s : C_n^{m/2} \to \mathbf{N}$ (note, that $\deg(A) = \deg(B) + 1$) such that for any $\psi : [n]_0 \xrightarrow{\psi} [k]_0$ in $C_n^{m/2}$ one has $0 \leqslant s(\psi) \leqslant k$ and $\ulcorner B \urcorner|_{H^\psi} = \pi^\psi_{s(\psi)}$.

By definition for any $\tilde{h} = (h_1, ..., h_{n-1}) \in H^{n-1}$ one has

$$\ulcorner A \urcorner(\tilde{h}) = \bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\}.$$

We must prove that such $\bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\}$ always exists and that there exists a function $r : C_{n-1}^m \to \mathbf{N}$ such that, for any $[n-1]_0 \xrightarrow{\varphi} [k]_0 \in C_{n-1}^m$ and $\tilde{h} \in H^\varphi$ one has $0 \leqslant r(\varphi) \leqslant k$ and $\bigwedge \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\} = \pi^\varphi_{r(\varphi)}(\tilde{h})$.

For every $[n-1]_0 \xrightarrow{\varphi} [k]_0 \in C^m_{n-1}$ we will construct a subset $C \subset C^{m/2,m}_n$ with $H^\varphi = \bigcup \left\{ H^\psi \mid \psi \in C \right\}$. Then for each $\psi \in C$, the stratum H^ψ will be a subset of the stratum $H^{\beta^{m/2,m}(\psi)}$ and the restriction of the $s(\beta^{m/2,m}(\psi))$-th projection of $H^{\beta^{m/2,m}(\psi)}$ to H^ψ will be $t_\psi = \gamma_\psi(s(\beta^{m/2,m}(\psi)))$-th projection of H^ψ (see 4.10). Then we will determine a number r_ψ such that for every $\tilde{h} \in H^\varphi$ one has $\bigwedge \left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}^{-1}_n(\tilde{h}) \cap H^\psi \right\} = \pi^\varphi_{r_\psi}(\tilde{h})$.

Then for $r(\varphi) = \min \left\{ r_\psi \mid \psi \in C \right\}$ we will have $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$.

For $[n-1]_0 \xrightarrow{\varphi} [k]_0 \in C^m_{n-1}$, we will determine the subset $C \subset R^{m/2,m}_n$ taking into account the possible orders between a new component h_n and the components $h_1, h_2, ..., h_{n-1}$ of any element of $\tilde{h} = (h_1, ..., h_{n-1}) \in H^\varphi$. All $\psi \in C$ will be of the shape $\psi : [n]_0 \to [k]_0$ (note, that ranges of ψ and φ coincide), moreover ψ will be extension of φ expect of one 11-th case. In this 11-th case elements of C, we will find the corresponding element of 10-th case such that the intersection of the elements of the corresponding 10-th case will be less or equal to intersection of elements of 11-th case.

Now consider the cases:

1) $h_n = \bot$. In this case extend the map φ to $[n]_0$ by $\psi(n) = 0$. Then we will have $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}^{-1}_n(\tilde{h}) \cap H^\psi \right\} = \left\{ \pi^\psi_{t_\psi}(\tilde{h}) \right\}$ and we can choose $r_\psi = t_\psi$.

2) $h_n = \top$. Let $\psi(n) = k - m/2$. Then as in the 1-st case we can choose $r_\psi = t_\psi$.

3) $h_n = h_i$ for $i \in [n-1]_0$. Let $\psi(n) = \varphi(i)$. As in previous cases we can choose $r_\psi = t_\psi$.

4) Let i_1 be such element in $[n-1]_0$, that $\varphi(i_1)$ is the largest labeled element of $[k]_0$ (i. e. $i_1 \in \varphi^{-1}(k-m)$) and $h_n = \Box^d(h_{i_1})$ with $0 < d < m/2$. Let $\psi(n) = k - m + d$. This case is also similar to previous cases and let us choose $r_\psi = t_\psi$.

5) $i_1 \in \varphi^{-1}(k-m)$ and $h_n \geqslant \Box^{m/2}(h_{i_1})$. Let $\psi(n) = k - m/2$. Then if $t_\psi < k - m/2$ or $t_\psi = k$, as in previous cases choose $r_\psi = t_\psi$. If $k - m/2 \leqslant t_\psi < k$, then we will have the equality $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}^{-1}_n(\tilde{h}) \cap H^\psi \right\} = (\Box^{t_\psi-(k-m)}(h_{i_1}), \top)$, hence $\bigwedge \left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}^{-1}_n(\tilde{h}) \cap H^\psi \right\} = \Box^{t_\psi-(k-m)}(h_{i_1})$ and let us again choose $r_\psi = t_\psi$.

6) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$ and $\Box^d(h_i) = h_n < h_j$ with $d < m/2$. Let $\psi(n) = \varphi(i) + d$. Then $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}^{-1}_n(\tilde{h}) \cap H^\psi \right\}$ $= \left\{ \pi^\psi_{t_\psi}(\tilde{h}) \right\}$ and we can choose $r_\psi = t_\psi$.

7) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$ and $\Box^{m/2}(h_i) \leqslant h_n \leqslant h_j$ with $t_\psi < \varphi(i) + m/2$ or $t_\psi \geqslant \varphi(j)$. Let $\psi(n) = \varphi(i) + m/2$. As in the cases 1-4 we can choose $r_\psi = t_\psi$.

8) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$, $\varphi(j)-\varphi(i) < m$, $\Box^{m/2}(h_i) \leqslant h_n$ and $\Box^d(h_n) = h_j$ with $m/2 + d = \varphi(j) - \varphi(i)$ and $\varphi(i) + m/2 \leqslant t_\psi < \varphi(j)$. Let $\psi(n) = \varphi(i) + m/2$. Then $\left\{\pi^{\psi}_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi\right\} = \left\{\Box^{m/2}(h_i)\right\}$ and we can choose $r_\psi = t_\psi$.

9) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$, $\varphi(j)-\varphi(i) < m$, $\Box^{m/2}(h_i) \leqslant h_n$ and $\Box^d(h_n) = h_j$ with $m/2 + d < \varphi(j) - \varphi(i)$ and $\varphi(i) + m/2 \leqslant t_\psi < \varphi(j)$. In this case the codomain of ψ is not $[k]_0$. To obtain ψ let us define $\psi(n) = \varphi(i) + m/2$ and delete nonlabeled elements in $[k]_0$ which are greater or equal to $\varphi(i) + m/2 + d$ and less then $\varphi(i)$. Then $\left\{\pi^{\psi}_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi\right\} = \left\{\Box^{t_\psi - \varphi(i)}(h_i)\right\}$ and we choose $r_\psi = t_\psi$.

10) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$, $\varphi(j)-\varphi(i) = m$, $\Box^{m/2}(h_i) \leqslant h_n$ and $\Box^{m/2}(h_n) \leqslant h_j$ with $\varphi(i) + m/2 \leqslant t_\psi < \varphi(j)$. Let $\psi(n) = \varphi(i) + m/2$. Then

$$\left\{\pi^{\psi}_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi\right\} = \left\{\Box^{t_\psi - m/2 - \varphi(i)}(h_n) \;\middle|\; \Box^{m/2}(h_i) \leqslant h_n \text{ and } \Box^{m/2}(h_n) \leqslant h_j\right\}.$$

This set is not empty. For example $\Box^{t_\psi - \varphi(i)}(h_i)$ belongs to this set and in fact is its smallest element. Therefore we can choose $r_\psi = t_\psi$.

11) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$, $\varphi(j)-\varphi(i) = m$, $\Box^{m/2}(h_i) \leqslant h_n$ and $\Box^d(h_n) = h_j$ with $d < m/2$ and $\varphi(i) + m/2 \leqslant t_\psi < \varphi(j)$. In this case codomain of ψ is not $[k]_0$. To obtain ψ let us define $\psi(n) = \varphi(i) + m/2$ and delete nonlabeled elements in $[k]_0$ which are greater or equal to $\varphi(i) + m/2 + d$ and less than $\varphi(i)$. Then

$$\left\{\pi^{\psi}_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi\right\} = \left\{\Box^{t_\psi - m/2 - \varphi(i)}(h_n) \;\middle|\; \Box^{m/2}(h_i) \leqslant h_n \text{ and } \Box^{d}(h_n) = h_j\right\}.$$

For some $H \in \mathscr{S}$ this set may be empty. But in any case, let us consider the element $\psi' \in C_n^{m/2,m}$ with the same i and j from the previous case 10. Then by the lemma from 4.11 one has $t_\psi = t_{\psi'}$. Therefore

$$\bigwedge \left\{\pi^{\psi'}_{t_{\psi'}} \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^{\psi'}\right\} \leqslant \bigwedge \left\{\pi^{\psi}_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi\right\}$$

and we can choose $r_\psi = k$ since the subset in the meet corresponding to ψ will not influence the meet at all.

Hence $\ulcorner A \urcorner (h_1, ..., h_{n-1})$ is a finite meet of projections, thus is itself a projection. Let us define $r(\varphi)$ as $\min\{r_\psi \mid \psi \in C\}$.

- $A(p_1, ..., p_{n-1}) = \exists p_i B(p_1, ..., p_{n-1}, p_n)$ (here too we suppose that the nonfree variable of A is p_n). By induction hypothesis there exists a function $s : C_n^{m/2} \to \mathbf{N}$

(note, that $\deg(A) = \deg(B) + 1$) such that for any $\psi : [n]_0 \to [k]_0$ in $C_n^{m/2}$ one has $0 \leqslant s(\psi) \leqslant k$ and $\ulcorner B \urcorner|_{H^\psi} = \pi^\psi_{s(\psi)}$.

By definition for any $\tilde{h} = (h_1, ..., h_{n-1}) \in H^{n-1}$ one has

$$\ulcorner A \urcorner(\tilde{h}) = \bigvee \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\}.$$

We must prove that such $\bigvee \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\}$ always exists and that there exists a function $r : C^m_{n-1} \to \mathbf{N}$ such that for any $[n-1]_0 \xrightarrow{\varphi} [k]_0 \in C^m_{n-1}$ and $\tilde{h} \in H^\varphi$ one has $0 \leqslant r(\varphi) \leqslant k$ and $\bigvee \left\{ \ulcorner B \urcorner(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_i^{-1}(\tilde{h}) \right\} = \pi^\varphi_{r(\varphi)}(\tilde{h})$.

Here also as in the previous case we construct elements of $C \in C_n^{m/2,m}$ and for every $\psi \in C$ also denote: $t_\psi = \gamma_\psi(s(\beta^{m/2,m}(\psi)))$ and determine a number r_ψ such that for every $\tilde{h} \in H^\varphi$ one has $\bigvee \left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi \right\} = \pi^\varphi_{r_\psi}(\tilde{h})$.

Then putting $r(\varphi) = \max\{r_\psi \mid \psi \in C\}$ we obtain $\ulcorner A \urcorner|_{H^\varphi} = \pi^\varphi_{r(\varphi)}$.

Now consider the cases:

1) Either one of $h_n = \bot$, $h_n = \top$, or $h_n = h_i$. In all these cases the set $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi \right\}$ has a single element and we can choose $r_\psi = t_\psi$.

2) $h_n = \Box^d(h_{i_1})$ with $i_1 \in \varphi^{-1}(k-m)$, $h_n = \Box^d(h_{i_1})$ and $0 < d < m/2$. Here also the set $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi \right\}$ is a singleton and we can choose $r_\psi = t_\psi$.

3) $i_1 \in \varphi^{-1}(k-m)$ and $h_n \geqslant \Box^{m/2}(h_{i_1})$. If either $t_\psi < k - m/2$ or $t_\psi = k$, then as in previous cases let us choose $r_\psi = t_\psi$. If $k - m/2 \leqslant t_\psi < k$, then one has $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi \right\} = \left\{ \Box^{t_\psi - (k-m)}(h_{i_1}), \top \right\}$ and $\bigvee \left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi \right\} = \top$, so let us choose $r_\psi = k$.

4) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$ and $\Box^d(h_i) = h_n < h_j$ with $d < m/2$. Then putting $\psi(n) = \varphi(i) + d$ one will have $\left\{ \pi^\psi_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^\psi \right\} = \left\{ \pi^\psi_{t_\psi}(\tilde{h}) \right\}$ and we can choose $r_\psi = t_\psi$.

5) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$ and $\Box^{m/2}(h_i) \leqslant h_n \leqslant h_j$ with $t_\psi < \varphi(i) + m/2$ or $t_\psi \geqslant \varphi(j)$. Let $\psi(n) = \varphi(i) + m/2$. Here as in the case 1 we can choose $r_\psi = t_\psi$.

6) Let $\varphi(i)$ and $\varphi(j)$ be consecutive labeled elements in $[k]_0$, $\Box^{m/2}(h_i) \leqslant h_n \leqslant h_j$. Then in any case let $\psi(n) = \varphi(i) + m/2$ and consider the element $\psi' \in C$ from the case 1 with $h_n = h_j$. If we compare ψ with t_ψ-th projection and ψ' with $t_{\psi'}$-th projection from the lemma in 4.11 one has $t_{\psi'} > \varphi(i) + m/2$. But $t_{\psi'}$-th projection of $H^{\psi'}$ is induced from some projection of an element of $C_n^{m/2}$, therefore $d(t_{\psi'}) \leqslant m/2$, hence $t_{\psi'} \geqslant \varphi(j)$. It means that $\bigvee \left\{ \pi^{\psi'}_{t_{\psi'}} \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^{\psi'} \right\} \geqslant$

$\bigvee \left\{ \pi^{\psi}_{t_\psi}(\hat{h}) \;\middle|\; \hat{h} \in \mathrm{pr}_n^{-1}(\tilde{h}) \cap H^{\psi} \right\}$. So we can choose $r_\psi = 0$ since the subset in the join corresponding to ψ will not influence the join at all.

Hence $\ulcorner A \urcorner (h_1, ..., h_{n-1})$ is equal to a finite join of projections, thus is itself a projection. Let us define $r(\varphi)$ as $\max \{r_\psi \mid \psi \in C\}$.

■

Statement 4.13

Now we will prove the converse theorem.

THEOREM. — *For $H \in \mathscr{S}$ and $n,\ m > 0$, suppose given a function $r : C_n^m \to \mathbf{N}$ such that for each $H \in \mathscr{S}$ and any $\varphi_1 : [n] \to [k_1]_0$, $\varphi_2 : [n] \to [k_2]_0$ from C_n^m one has $0 \leqslant r(\varphi_1) \leqslant k_1$, $0 \leqslant r(\varphi_2) \leqslant k_2$ and either $r(\varphi_1) = r(\varphi_2)$, or $\varphi_1|\varphi_2 \leqslant \min\{r(\varphi_1), r(\varphi_2)\}$ Then there exists a formula $A(p_1, ..., p_n)$ with at most n free variables in which quantifiers occur only through application of $\Box$, such that restriction of the function $\ulcorner A \urcorner$ to every stratum H^φ coincides with the projection $\pi^{\varphi}_{r(\varphi)}$ (i. e. $\ulcorner A \urcorner|_{H^\varphi} = \pi^{\varphi}_{r(\varphi)}$).*

PROOF. — Let us construct the formula A using the function $r : C_n^m \to \mathbf{N}$ as follows:

$$A = \bigwedge_{\varphi \in C_n^m} (\chi_\varphi \to A^{\varphi}_{r(\varphi)})$$

where $A^{\varphi}_{r(\varphi)}$ is the formula mentioned in 4.9. Then restriction of the function $\ulcorner A \urcorner$ to any stratum H^φ coincides with the projection $\pi^{\varphi}_{r(\varphi)}$ (i. e. $\ulcorner A \urcorner|_{H^\varphi} = \pi^{\varphi}_{r(\varphi)}$).

The proof of this fact coincides with the proof of the theorem 3.13, with the use of the proposition from 3.8 and of the lemma from 3.11 replaced by the proposition from 4.8 and the lemma from 4.11 respectively. ■

Acknowledgements

This research has been supported by the INTAS grant 04-77-7080.

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