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Crack-Type Boundary Value Problems of Electro-Elasticity

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Dedicated to the memory of Erhard Meister

Abstract. Dirichlet, Neumann and mixed crack-type boundary value problems of statics are considered in three-dimensional bounded domains filled with a homogeneous anisotropic electro-elastic medium. Applying the method of the potential theory and the theory of pseudodifferential equations, we prove the existence and uniqueness theorems in Besov and Bessel potential spaces, and derive full asymptotic expansion of solutions near the crack edge.

Introduction

The recent years have shown an ever-growing interest in the investigation of models of an anisotropic elastic medium which take into account the influence of various physical fields such as thermal, electric, magnetic etc. A rather strong motivation for such studies is the creation of new artificial materials which possess non-standard properties. Among them are piezoelectric materials that form the core of modern structures and instruments.

Mathematical models of piezoelectric (electro-elastic) bodies and relevant boundary value problems have been studied with sufficient completeness (see, e.g., [BG2, No1, Pa1, To1] and the references therein). Of special interest is the case where the considered body contains cracks or cuts with an edge having a bihedral angle 2π (cuspidal edge) (see [Ch2, CD3]). In that case the presence of an electric field essentially changes the pattern of stress distribution near the cut or crack edge (see [DNS1, Pa1]).

In this work, Dirichlet, Neumann and mixed boundary value problems of statics are considered for a homogeneous anisotropic piezoelectric body with a crack. The existence and uniqueness of solutions of the considered problems are proved in Bessel potential $\mathbb{H}_p^s$ (and Besov $\mathbb{B}_{p,q}^s$) spaces. Complete asymptotic expansion of a solution near the crack edge is obtained. These results are important in the analysis of a stress field in electro-elastic bodies with cracks.

1. Formulation of boundary value problems

Let $\Omega = \Omega_0$ and Ω_1 ($\overline{\Omega}_1 \subset \Omega$) be bounded domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with infinitely smooth boundaries $\partial\Omega = \partial\Omega_0$ and $\partial\Omega_1$ respectively (we use the alternative notation $\Omega_0 = \Omega$ for conciseness of forthcoming formulae). The boundary $\partial\Omega_1$ of the domain Ω_1 (called interface) is divided in two parts: $\partial\Omega_1 = \overline{S}_0 \cup S_1$ with a smooth common boundary $\mathcal{E} := \partial S_0 = \partial S_1$. Let $\Omega_2 = \Omega\backslash\overline{\Omega}_1$.

We assume, that the domain $\Omega\backslash S_1$ is filled with homogeneous anisotropic electro-elastic material, having a crack at S_1.

We use the following notation for function spaces: $\mathbb{W}^s_p(\Omega)$, $\mathbb{W}^s_p(\partial\Omega_i)$ for the Sobolev-Slobodetskij spaces; $\mathbb{H}^s_p(\Omega)$, $\mathbb{H}^s_p(\partial\Omega_i)$ for the Bessel potential spaces; $\mathbb{B}^s_{p,q}(\Omega)$, $\mathbb{B}^s_{p,q}(\partial\Omega_i)$ for the Besov spaces ($i = 1, 2$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$), (see [Tr1, Tr2] for the definitions and properties of these spaces). We use the common simplified notation $\mathbb{H}^s(\partial\Omega_i) = \mathbb{H}^s_2(\partial\Omega_i)$. Let further,

$$\mathbb{H}^s_p(S_j) = \{r_{S_j} u : \ u \in \mathbb{H}^s_p(\partial\Omega_1)\},$$
$$\widetilde{\mathbb{H}}^s_p(S_j) = \{u \in \mathbb{H}^s_p(\partial\Omega_1) : \ \operatorname{supp} u \in \overline{S_j}\}, \qquad j = 0, 1,$$

where $r_{S_j}\varphi := \varphi\Big|_{S_j}$ denotes the restriction operator onto the subset S_j. Similarly the Sobolev-Slobodetskii spaces $\mathbb{W}^s_p(S_j)$, $\widetilde{\mathbb{W}}^s_p(S_0)$, $\mathbb{B}^s_{p,q}(S_j)$ and $\widetilde{\mathbb{B}}^s_{p,q}(S_0)$ are defined.

We consider the system of static equations of electro-elasticity for a homogeneous anisotropic medium [No1]

$$\mathbf{A}(D)u(x) + F(x) = 0, \qquad x \in \Omega\backslash S_1, \tag{1.1}$$

where $u = (u_1, u_2, u_3, u_4)$; u_1, u_2, u_3 are displacement vector components, u_4 is an electric potential, F is a mass force. $\mathbf{A}(D)$ is a differential operator of the form

$$\begin{gathered}\mathbf{A}(D) = \|\mathbf{A}_{jk}(D)\|_{4\times 4}, \\ \mathbf{A}_{jk}(D) = c_{ijlk}\partial_i\partial_l, \quad j, k = 1, 2, 3, \\ \mathbf{A}_{j4}(D) = e_{kjl}\partial_k\partial_l, \quad j = 1, 2, 3, \\ \mathbf{A}_{4k}(D) = -e_{ikl}\partial_i\partial_l, \quad k = 1, 2, 3, \\ \mathbf{A}_{44}(D) = \varepsilon_{il}\partial_i\partial_l,\end{gathered} \tag{1.2}$$

where c_{ijlk}, e_{ikl}, ε_{ik} are the elastic, piezoelectric and dielectric constants, respectively.

Here and in what follows we use the standard convention: the summation is carried out over the repeated indices.

The constants in (1.2) satisfy the symmetry conditions

$$\begin{gathered}c_{ijlk} = c_{jilk} = c_{lkij}, \quad e_{kjl} = e_{klj}, \quad \varepsilon_{ik} = \varepsilon_{ki}, \\ i, j, k, l = 1, 2, 3\end{gathered} \tag{1.3}$$