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Original article

Mixed boundary value problems of pseudo-oscillations of generalized thermo-electro-magneto-elasticity theory for solids with interior cracks

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Abstract

We investigate the mixed boundary value problems of the generalized thermo-electro-magneto-elasticity theory for homogeneous anisotropic solids with interior cracks. Using the potential methods and theory of pseudodifferential equations on manifolds with boundary we prove the existence and uniqueness of solutions. We analyse the asymptotic behaviour and singularities of the mechanical, electric, magnetic, and thermal fields near the crack edges and near the curves, where different types of boundary conditions collide. In particular, for some important classes of anisotropic media we derive explicit expressions for the corresponding stress singularity exponents and demonstrate their dependence on the material parameters. The questions related to the so called oscillating singularities are treated in detail as well.

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1. Introduction

The paper deals with three-dimensional boundary value problems (BVP) arising in the generalized thermo-electro-magneto-elasticity (GTEME) theory for homogeneous anisotropic solids with interior cracks.

The theory under consideration is associated with Green–Lindsay's model of thermo-electro-magneto-elasticity which describes full coupling of elastic, electric, magnetic, and thermal fields. Another feature of this model is that

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in contrast to the conventional theory of heat transfer, the heat propagation in Green–Lindsay's theory occurs with a finite speed (see [1,2]).

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. For example, piezoelectric materials (electro-elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal–electric coupling) have been applied in thermal imaging devices; and magnetoelastic coupling effects are used in modern signal detection systems and instrumentation (see [3–9] and the references therein).

Theories of thermoelasticity consistent with a finite speed propagation of heat recently are attracting increasing attention. In contrast to the conventional thermoelasticity theory, these nonclassical theories involve a hyperbolic-type heat transport equation, and are motivated by experiments exhibiting the actual occurrence of wave-type heat transport (second sound). Several authors have formulated these theories on different grounds, and a wide variety of problems revealing characteristic features of the theories has been investigated.

As it is well known from the classical mathematical physics and the classical elasticity theory, in general, solutions to crack type and mixed boundary value problems have singularities near the crack edges and near the lines where the types of boundary conditions change, regardless of the smoothness of given boundary data. Throughout the paper we shall refer to such lines as *exceptional curves*. The same effect can be observed also in the GTEME theory. In this paper, our main goal is a detailed theoretical investigation of regularity and asymptotic properties of thermo-mechanical and electro-magnetic fields near the exceptional curves. By explicit calculations we show that the stress singularity exponents essentially depend on the material parameters, in general.

We draw a special attention to the problem of oscillating singularities which is very important in engineering applications. Such singularities usually lead to some mechanical contradictions, e.g., overlapping of materials (see, e.g., [10] and the references therein). It turned out that there are classes of anisotropic media for which the oscillating singularities near the exceptional curves do not occur. In particular, *calcium phosphate based bioceramics*, such as *hydroxyapatite*, possess the above property. These materials are extensively used in medicine and dentistry [11,12].

Our main tools are the potential methods and the theory of pseudodifferential equations, which proved to be very efficient in deriving the asymptotic formulas. They allow us to calculate effectively the field singularity exponents by means of the characteristics related to the symbol matrices of the corresponding pseudodifferential operators. In our analysis we essentially apply the results obtained in the references [13–16,18,19].

To demonstrate the dependence of the singularity exponents on the material parameters let us compare behaviour of solutions to the crack type mixed boundary value problems near the exceptional curves for the Laplace equation (Zaremba type problem), for equations of the classical elasticity (e.g., the Lamé equations for an isotropic solid) and for the equations to generalized thermo-electro-magneto-elasticity equations for transversely-isotropic media.

Near the crack edge the asymptotic formulae for solutions of all the above three problems have the same form, namely,

$$a_0\, r^{1/2} + a_1\, r^{3/2} + \cdots , \tag{1.1}$$

where r is the distance from the reference point x to the crack edge [20,21].

We have quite a different situation near the exceptional curve, where the different types of boundary conditions (for example, the Dirichlet and Neumann type conditions) collide. Unlike the asymptotic expansion (1.1) of solutions to the Laplace equation the asymptotic expansion of the solutions to Lamé equations has the form

$$b_0\, r^{1/2} + b_1\, r^{1/2+i\,\delta} + b_2\, r^{1/2-i\,\delta} + \mathcal{O}(r^{3/2-\varepsilon}),$$

where ε is an arbitrary positive number, while the asymptotic expansion of a solution to the generalized thermo-electro-magneto-elasticity equations for transversely-isotropic case reads as

$$c_0\, r^{\gamma_1} + c_1\, r^{1/2+i\,\widetilde{\delta}} + c_2\, r^{1/2-i\,\widetilde{\delta}} + c_3\, r^{1/2} \ln r + c_4\, r^{1/2} + \mathcal{O}(r^{\gamma_2}),$$

where $\gamma_1 \in (0,\, 1/2)$, $\gamma_2 > 1/2$, and δ and $\widetilde{\delta}$ are real numbers. Note that $\gamma_1 - 1$ represents the dominant stress singularity exponent. The parameter γ_1 in general depends on the material constants and the geometry of the curve and may take an arbitrary value from the interval $(0,\, 1/2)$ (for details see Section 6). Thus, the stress singularity exponent essentially depends on the material constants and is less than $-1/2$, in general. Consequently, in the classical elasticity, we have oscillating stress singularities, while in the generalized thermo-electro-magneto-elasticity theory we have no oscillating stress singularities for the transversely isotropic case due to the inequality $\gamma_1 < 1/2$.

2. Formulation of the problem

2.1. Field equations

In this subsection, we collect the field equations of the generalized thermo-electro-magneto-elasticity (GTEME) for a general anisotropic case and introduce the corresponding matrix partial differential operators

Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} are the components of the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ are the components of the mechanical strain tensor, $E = (E_1, E_2, E_3)^\top$ and $H = (H_1, H_2, H_3)^\top$ are electric and magnetic fields respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials and

$$E = -\mathrm{grad}\,\varphi\,, \qquad H = -\mathrm{grad}\,\psi\,,$$

ϑ is the temperature change to a reference temperature T_0, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

We employ also the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation. In what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise. Throughout the paper the over bar, applied to numbers and functions, denotes complex conjugation and the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a \cdot b \equiv (a, b) := \sum_{j=1}^N a_j \overline{b_j}$ for $a, b \in \mathbb{C}^N$. Over bar, applied to a subset $\mathcal{M}$ of Euclidean space $\mathbb{R}^N$, denotes the closure of $\mathcal{M}$, i.e. $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$, where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$.

In the GTEME theory we have the following governing equations:

The constitutive relations:

$$\sigma_{rj} = \sigma_{jr} = c_{rjkl}\,\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}(\vartheta + \nu_0\,\partial_t\vartheta), \qquad r, j = 1, 2, 3, \tag{2.1}$$

$$D_j = e_{jkl}\,\varepsilon_{kl} + \varkappa_{jl}\,E_l + a_{jl}\,H_l + p_j(\vartheta + \nu_0\,\partial_t\vartheta), \qquad j = 1, 2, 3, \tag{2.2}$$

$$B_j = q_{jkl}\,\varepsilon_{kl} + a_{jl}\,E_l + \mu_{jl}H_l + m_j(\vartheta + \nu_0\,\partial_t\vartheta), \qquad j = 1, 2, 3, \tag{2.3}$$

$$\varrho\,\mathcal{S} = \lambda_{kl}\,\varepsilon_{kl} + p_l\,E_l + m_l\,H_l + a_0 + d_0\,\vartheta + h_0\,\partial_t\vartheta. \tag{2.4}$$

The equations of motion:

$$\partial_j\sigma_{rj} + \varrho\,F_r = \varrho\,\partial_t^2 u_r, \qquad r = 1, 2, 3. \tag{2.5}$$

The quasi-static equations for electric and magnetic fields:

$$\partial_j D_j = \varrho_e\,, \qquad \partial_j B_j = \varrho_c\,. \tag{2.6}$$

The linearized energy equations:

$$\varrho\,T_0\,\partial_t\mathcal{S} = -\partial_j q_j + \varrho\,Q, \qquad q_j = -T_0\,\eta_{jl}\partial_l\vartheta. \tag{2.7}$$

Here the following notation is used: ϱ—the mass density, ϱ_e—the electric charge density, ϱ_c—the electric current density, $F = (F_1, F_2, F_3)^\top$—the mass force density, Q—the heat source intensity, c_{rjkl}—the elastic constants, e_{jkl}—the piezoelectric constants, q_{jkl}—the piezomagnetic constants, $\varkappa_{jk}$—the dielectric (permittivity) constants, μ_{jk}—the magnetic permeability constants, a_{jk}—the electromagnetic coupling coefficients, p_j, m_j, and λ_{rj}—coupling coefficients connecting dissimilar fields, η_{jk}—the heat conductivity coefficients, T_0—the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, ν_0 and h_0—two relaxation times, a_0 and d_0—constitutive coefficients.

The constants involved in the above equations satisfy the symmetry conditions:

$$\begin{aligned} &c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \\ &\varkappa_{kj} = \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad a_{kj} = a_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad r, j, k, l = 1, 2, 3. \end{aligned} \tag{2.8}$$

From physical considerations it follows that (see, e.g., [22,23,2,1,24]):

$$c_{rjkl}\,\xi_{rj}\,\xi_{kl} \ge \delta_0\,\xi_{kl}\,\xi_{kl},\quad \varkappa_{kj}\,\xi_k\,\xi_j \ge \delta_1\,|\xi|^2,\quad \mu_{kj}\,\xi_k\,\xi_j \ge \delta_2\,|\xi|^2,\quad \eta_{kj}\,\xi_k\,\xi_j \ge \delta_3\,|\xi|^2,$$
$$\text{for all }\ \xi_{kj}=\xi_{jk}\in\mathbb{R}\text{ and for all }\ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3, \tag{2.9}$$

$$\nu_0>0,\quad h_0>0,\quad d_0\nu_0-h_0>0, \tag{2.10}$$

where δ_0, δ_1, δ_2, and δ_3 are positive constants depending on material parameters.

Due to the symmetry conditions (2.8), with the help of (2.9) we easily derive

$$c_{rjkl}\,\zeta_{rj}\,\overline{\zeta_{kl}} \ge \delta_0\,\zeta_{kl}\,\overline{\zeta_{kl}},\quad \varkappa_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_1\,|\zeta|^2,\quad \mu_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_2\,|\zeta|^2,\quad \eta_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_3\,|\zeta|^2,$$
$$\text{for all }\ \zeta_{kj}=\zeta_{jk}\in\mathbb{C}\text{ and for all }\ \zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3. \tag{2.11}$$

More careful analysis related to the positive definiteness of the potential energy and the thermodynamical laws insure that the following 8×8 matrix

$$M=[M_{kj}]_{8\times 8} := \begin{bmatrix} [\varkappa_{jl}]_{3\times3} & [a_{jl}]_{3\times3} & [p_j]_{3\times1} & [\nu_0 p_j]_{3\times1} \\ [a_{jl}]_{3\times3} & [\mu_{jl}]_{3\times3} & [m_j]_{3\times1} & [\nu_0 m_j]_{3\times1} \\ [p_j]_{1\times3} & [m_j]_{1\times3} & d_0 & h_0 \\ [\nu_0 p_j]_{1\times3} & [\nu_0 m_j]_{1\times3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times 8} \tag{2.12}$$

is positive definite. Note that the positive definiteness of M remains valid if the parameters p_j and m_j in (2.12) are replaced by the opposite ones, $-p_j$ and $-m_j$. Moreover, it follows that the matrices

$$\Lambda^{(1)} := \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} \end{bmatrix}_{6\times6}, \qquad \Lambda^{(2)} := \begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times2} \tag{2.13}$$

are positive definite as well, i.e.,

$$\varkappa_{kj}\,\zeta'_k\,\overline{\zeta'_j} + a_{kj}\,(\zeta'_k\,\overline{\zeta''_j} + \overline{\zeta'_k}\,\zeta''_j) + \mu_{kj}\,\zeta''_k\,\overline{\zeta''_j} \ge \kappa_1\,(|\zeta'|^2+|\zeta''|^2) \qquad \forall \zeta',\zeta''\in\mathbb{C}^3, \tag{2.14}$$

$$d_0\,|z_1|^2 + h_0\,(z_1\,\overline{z_2} + \overline{z_1}\,z_2) + \nu_0 h_0\,|z_2|^2 \ge \kappa_2\,(|z_1|^2+|z_2|^2) \qquad \forall z_1,z_2\in\mathbb{C}, \tag{2.15}$$

with some positive constants κ_1 and κ_2 depending on the material parameters involved in (2.13).

With the help of the symmetry conditions (2.9) we can rewrite the constitutive relations (2.1)–(2.4) as follows

$$\sigma_{rj} = c_{rjkl}\,\partial_l u_k + e_{lrj}\,\partial_l\varphi + q_{lrj}\,\partial_l\psi - \lambda_{rj}\,(\vartheta+\nu_0\,\partial_t\vartheta),\quad r,j=1,2,3, \tag{2.16}$$

$$D_j = e_{jkl}\,\partial_l u_k - \varkappa_{jl}\,\partial_l\varphi - a_{jl}\,\partial_l\psi + p_j\,(\vartheta+\nu_0\,\partial_t\vartheta),\quad j=1,2,3, \tag{2.17}$$

$$B_j = q_{jkl}\,\partial_l u_k - a_{jl}\,\partial_l\varphi - \mu_{jl}\,\partial_l\psi + m_j\,(\vartheta+\nu_0\,\partial_t\vartheta),\quad j=1,2,3, \tag{2.18}$$

$$\mathcal{S} = \lambda_{kl}\,\partial_l u_k - p_l\,\partial_l\varphi - m_l\,\partial_l\psi + a_0 + d_0\,\vartheta + h_0\,\partial_t\vartheta. \tag{2.19}$$

In the theory of generalized thermo-electro-magneto-elasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{rj}\,n_j = c_{rjkl}\,n_j\,\partial_l u_k + e_{lrj}\,n_j\,\partial_l\varphi + q_{lrj}\,n_j\,\partial_l\psi - \lambda_{rj}\,n_j\,(\vartheta+\nu_0\,\partial_t\vartheta),\quad r=1,2,3, \tag{2.20}$$

while the normal components of the electric displacement vector, magnetic induction vector and heat flux vector read as

$$D_j\,n_j = e_{jkl}\,n_j\,\partial_l u_k - \varkappa_{jl}\,n_j\,\partial_l\varphi - a_{jl}\,n_j\,\partial_l\psi + p_j\,n_j\,(\vartheta+\nu_0\,\partial_t\vartheta), \tag{2.21}$$

$$B_j\,n_j = q_{jkl}\,n_j\,\partial_l u_k - a_{jl}\,n_j\,\partial_l\varphi - \mu_{jl}\,n_j\,\partial_l\psi + m_j\,n_j\,(\vartheta+\nu_0\,\partial_t\vartheta), \tag{2.22}$$

$$q_j\,n_j = -T_0\,\eta_{jl}\,n_j\,\partial_l\vartheta.$$

For convenience we introduce the following matrix differential operator

$$\mathcal{T} = \mathcal{T}(\partial_x, n, \partial_t) = \big[\, \mathcal{T}_{pq}(\partial_x, n, \partial_t)\,\big]_{6\times 6}$$

$$:= \begin{bmatrix} [c_{rjkl}\, n_j\, \partial_l]_{3\times 3} & [e_{lrj}\, n_j\, \partial_l]_{3\times 1} & [q_{lrj}\, n_j\, \partial_l]_{3\times 1} & [-\lambda_{rj}\, n_j (1+\nu_0\,\partial_t)]_{3\times 1} \\ [-e_{jkl}\, n_j\, \partial_l]_{1\times 3} & \varkappa_{jl}\, n_j\, \partial_l & a_{jl}\, n_j\, \partial_l & -p_j\, n_j (1+\nu_0\,\partial_t) \\ [-q_{jkl}\, n_j\, \partial_l]_{1\times 3} & a_{jl}\, n_j\, \partial_l & \mu_{jl}\, n_j\, \partial_l & -m_j\, n_j (1+\nu_0\,\partial_t) \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\, n_j\, \partial_l \end{bmatrix}_{6\times 6}. \tag{2.23}$$

Evidently, for a smooth six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial_x, n, \partial_t)\, U = (\sigma_{1j}\, n_j,\ \sigma_{2j}\, n_j,\ \sigma_{3j}\, n_j,\ -D_j\, n_j,\ -B_j\, n_j,\ -T_0^{-1} q_j\, n_j)^\top. \tag{2.24}$$

Due to the constitutive equations, the components of the vector $\mathcal{T}\, U$ given by (2.24) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the fourth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector respectively with opposite sign, and finally the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector.

Note that the following pairs are called like fields:

(i) $\{u = (u_1, u_2, u_3)^\top,\ (\sigma_{1j}\, n_j,\ \sigma_{2j}\, n_j,\ \sigma_{3j}\, n_j)^\top\}$—pair of mechanical fields,

(ii) $\{\varphi,\ -D_j\, n_j\}$—pair of electric fields,

(iii) $\{\psi,\ -B_j\, n_j\}$—pair of magnetic fields,

(iv) $\{\vartheta,\ -T_0^{-1} q_j\, n_j\}$—pair of thermal fields.

As we see all the thermo-mechanical and electro-magnetic characteristics can be determined by the six functions: three displacement components u_j, $j = 1, 2, 3$, temperature distribution ϑ, and the electric and magnetic potentials φ and ψ. Therefore, all the above field relations and the corresponding boundary value problems we reformulate in terms of these six functions.

First of all, from Eqs. (2.5)–(2.7) with the help of the constitutive relations (2.1)–(2.4) we derive the basic linear system of dynamics of the generalized thermo-electro-magneto-elasticity theory of homogeneous solids

$$\begin{aligned} & c_{rjkl}\, \partial_j \partial_l\, u_k(x,t) + e_{lrj}\, \partial_j \partial_l \varphi(x,t) + q_{lrj}\, \partial_j \partial_l \psi(x,t) - \lambda_{rj}\, \partial_j \vartheta(x,t) - \nu_0 \lambda_{rj} \partial_j \partial_t \vartheta(x,t) \\ & \quad - \varrho\, \partial_t^2\, u_r(x,t) = -\varrho F_r(x,t), \quad r = 1,2,3, \\ & -e_{jkl}\, \partial_j \partial_l u_k(x,t) + \varkappa_{jl}\, \partial_j \partial_l \varphi(x,t) + a_{jl}\, \partial_j \partial_l \psi(x,t) - p_j\, \partial_j \vartheta(x,t) - \nu_0\, p_j\, \partial_j \partial_t \vartheta(x,t) = -\varrho_e(x,t), \\ & -q_{jkl}\, \partial_j \partial_l u_k(x,t) + a_{jl}\, \partial_j \partial_l \varphi(x,t) + \mu_{jl}\, \partial_j \partial_l \psi(x,t) - m_j\, \partial_j \vartheta(x,t) - \nu_0\, m_j\, \partial_j \partial_t \vartheta(x,t) = -\varrho_c(x,t), \\ & -\lambda_{kl}\, \partial_t \partial_l u_k(x,t) + p_l\, \partial_l \partial_t \varphi(x,t) + m_l\, \partial_l \partial_t \psi(x,t) + \eta_{jl}\, \partial_j \partial_l \vartheta(x,t) - d_0\, \partial_t \vartheta(x,t) \\ & \quad - h_0\, \partial_t^2 \vartheta(x,t) = -T_0^{-1} \varrho\, Q(x,t). \end{aligned} \tag{2.25}$$

Let us introduce the matrix differential operator generated by the left hand side expressions in Eqs. (2.25),

$$A(\partial_x, \partial_t) = \big[A_{pq}(\partial_x, \partial_t)\big]_{6\times 6}$$

$$:= \begin{bmatrix} [c_{rjkl}\, \partial_j \partial_l - \varrho\, \delta_{rk}\, \partial_t^2\,]_{3\times 3} & [e_{lrj}\, \partial_j \partial_l]_{3\times 1} & [q_{lrj}\, \partial_j \partial_l]_{3\times 1} & [-\lambda_{rj}\, \partial_j (1+\nu_0 \partial_t)]_{3\times 1} \\ [-e_{jkl}\, \partial_j \partial_l]_{1\times 3} & \varkappa_{jl}\, \partial_j \partial_l & a_{jl}\, \partial_j \partial_l & -p_j\, \partial_j (1+\nu_0 \partial_t) \\ [-q_{jkl}\, \partial_j \partial_l]_{1\times 3} & a_{jl}\, \partial_j \partial_l & \mu_{jl}\, \partial_j \partial_l & -m_j\, \partial_j (1+\nu_0 \partial_t) \\ [-\lambda_{kl}\, \partial_l \partial_t]_{1\times 3} & p_l\, \partial_l \partial_t & m_l\, \partial_l \partial_t & \eta_{jl}\, \partial_j \partial_l - d_0\, \partial_t - h_0\, \partial_t^2 \end{bmatrix}_{6\times 6}. \tag{2.26}$$

Then Eqs. (2.25) can be rewritten in matrix form

$$A(\partial_x, \partial_t)\, U(x,t) = \Phi(x,t)\,, \tag{2.27}$$

where

$$U = (u_1, u_2, u_3, u_4, u_5, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$$

is the sought for vector function and

$$\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-\varrho F_1, -\varrho F_2, -\varrho F_3, -\varrho_e, -\varrho_c, -\varrho T_0^{-1} Q)^\top \tag{2.28}$$

is the given vector function.

If all the functions involved in these equations are harmonic time dependent, that is they can be represented as the product of a function of the spatial variables (x_1, x_2, x_3) and the multiplier $\exp\{\tau\, t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the generalized thermo-electro-magneto-elasticity theory. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau = i\omega$ is a pure imaginary number, with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$, i.e., the functions involved in Eqs. (2.25) are independent of t, we get the *equations of statics*.

Recall that for a smooth function $v(t)$ which is exponentially bounded,

$$e^{-\sigma_0 t}\big[\, |v(t)| + |\partial v(t)| + |\partial_t^2 v(t)|\,\big] = \mathcal{O}(1) \ \text{ as } \ t \to +\infty, \quad \sigma_0 \geqslant 0, \tag{2.29}$$

the corresponding Laplace transform

$$\widehat{v}(\tau) \equiv L_{t\to\tau}[v(t)] := \int_0^{+\infty} e^{-\tau t}\, v(t)\, dt, \quad \tau = \sigma + i\omega, \quad \sigma > \sigma_0, \tag{2.30}$$

possesses the following properties

$$L_{t\to\tau}[\partial_t v(t)] := \int_0^{+\infty} e^{-\tau t}\, \partial_t v(t)\, dt = -v(0) + \tau\, \widehat{v}(\tau), \tag{2.31}$$

$$L_{t\to\tau}[\partial_t^2 v(t)] := \int_0^{+\infty} e^{-\tau t}\, \partial_t^2 v(t)\, dt = -\partial_t v(0) - \tau v(0) + \tau^2\, \widehat{v}(\tau). \tag{2.32}$$

Provided that all the functions involved in the dynamical equations (2.25) are exponentially bounded and applying the Laplace transform to the system (2.25), we obtain the following pseudo-oscillation equations:

$$\begin{aligned}
& c_{rjkl}\, \partial_j \partial_l\, \widehat{u}_k(x,\tau) - \varrho\, \tau^2\, \widehat{u}_r(x,\tau) + e_{lrj}\, \partial_j \partial_l \widehat{\varphi}(x,\tau) + q_{lrj}\, \partial_j \partial_l \widehat{\psi}(x,\tau) \\
& \quad - (1+\nu_0\tau)\lambda_{rj}\, \partial_j \widehat{\vartheta}(x,\tau) = -\varrho \widehat{F}_r(x,\tau) + \Psi_r^{(0)}(x,\tau), \quad r = 1,2,3, \\
& -e_{jkl}\, \partial_j \partial_l \widehat{u}_k(x,\tau) + \varkappa_{jl}\, \partial_j\, \partial_l \widehat{\varphi}(x,\tau) + a_{jl}\, \partial_j\, \partial_l \widehat{\psi}(x,\tau) - (1+\nu_0\tau) p_j\, \partial_j \widehat{\vartheta}(x,\tau) \\
& \quad = -\widehat{\varrho}_e(x,\tau) + \Psi_4^{(0)}(x,\tau), \\
& -q_{jkl}\, \partial_j \partial_l \widehat{u}_k(x,\tau) + a_{jl}\, \partial_j\, \partial_l \widehat{\varphi}(x,\tau) + \mu_{jl}\, \partial_j\, \partial_l \widehat{\psi}(x,\tau) - (1+\nu_0\tau) m_j\, \partial_j \widehat{\vartheta}(x,\tau) \\
& \quad = -\widehat{\varrho}_c(x,\tau) + \Psi_5^{(0)}(x,\tau), \\
& -\tau\, \lambda_{kl}\, \partial_l \widehat{u}_k(x,\tau) + \tau\, p_l\, \partial_l \widehat{\varphi}(x,\tau) + \tau\, m_l\, \partial_l \widehat{\psi}(x,\tau) + \eta_{jl}\, \partial_j \partial_l \widehat{\vartheta}(x,\tau) \\
& \quad - (\tau d_0 + \tau^2 h_0)\widehat{\vartheta}(x,\tau) = -T_0^{-1} \varrho\, \widehat{Q}(x,\tau) + \Psi_6^{(0)}(x,\tau),
\end{aligned} \tag{2.33}$$

where the overset "hat" denotes the Laplace transform of the corresponding function with respect to t (see (2.30)) and

$$\Psi^{(0)}(x,\tau) = \big(\Psi_1^{(0)}(x,\tau), \dots, \Psi_6^{(0)}(x,\tau)\big)^\top$$
$$:= \begin{bmatrix}
-\varrho\, \tau\, u_1(x,0) - \varrho\, \partial_t u_1(x,0) - \nu_0\, \lambda_{1j}\, \partial_j \vartheta(x,0) \\
-\varrho\, \tau\, u_2(x,0) - \varrho\, \partial_t u_2(x,0) - \nu_0\, \lambda_{2j}\, \partial_j \vartheta(x,0) \\
-\varrho\, \tau\, u_3(x,0) - \varrho\, \partial_t u_3(x,0) - \nu_0\, \lambda_{3j}\, \partial_j \vartheta(x,0) \\
\nu_0\, p_j\, \partial_j \vartheta(x,0) \\
\nu_0\, m_j\, \partial_j \vartheta(x,0) \\
-\lambda_{kl}\, \partial_l u_k(x,0) + p_j\, \partial_l \varphi(x,0) + m_j\, \partial_l \psi(x,0) - (d_0 + \tau\, h_0)\vartheta(x,0) - h_0 \partial_t \vartheta(x,0)
\end{bmatrix}. \tag{2.34}$$

Note that the relations (2.30)–(2.32) can be extended to the spaces of generalized functions (see e.g., [25]).

In matrix form these pseudo-oscillation equations can be rewritten as

$$A(\partial_x,\tau)\,\widehat{U}(x,\tau) = \Psi(x,\tau),$$

where

$$\widehat{U} = (\widehat{u}_1,\widehat{u}_2,\widehat{u}_3,\widehat{u}_4,\widehat{u}_5,\widehat{u}_6)^\top := (\widehat{u},\widehat{\varphi},\widehat{\psi},\widehat{\vartheta})^\top$$

is the sought for vector function,

$$\Psi(x,\tau) = \big(\Psi_1(x,\tau),\ldots,\Psi_6(x,\tau)\big)^\top = \widehat{\Phi}(x,\tau) + \Psi^{(0)}(x,\tau)$$

with $\widehat{\Phi}(x,\tau)$ being the Laplace transform of the vector function $\Phi(x,t)$ defined in (2.28) and $\Psi^{(0)}(x,\tau)$ given by (2.34), and $A(\partial_x,\tau)$ is the pseudo-oscillation matrix differential operator generated by the left hand side expressions in Eq. (2.33),

$$A(\partial_x,\tau) = \big[A_{pq}(\partial_x,\tau)\big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\tau^2\delta_{rk}]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)\,p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1\times3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l - \tau^2 h_0 - \tau d_0 \end{bmatrix}_{6\times6}. \tag{2.35}$$

It is evident that the operator

$$A^{(0)}(\partial_x) := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [0]_{3\times1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0 \\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0 \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times6}$$

is the principal part of the operators $A(\partial_x,\tau)$. Clearly, the symbol matrix $A^{(0)}(-i\,\xi)$, $\xi\in\mathbb{R}^3$, of the operator $A^{(0)}(\partial_x)$ is the principal homogeneous symbol matrix of the operator $A(\partial_x,\tau)$ for all $\tau\in\mathbb{C}$,

$$A^{(0)}(-i\,\xi) := \begin{bmatrix} [-c_{rjkl}\,\xi_j\,\xi_l]_{3\times3} & [-e_{lrj}\,\xi_j\xi_l]_{3\times1} & [-q_{lrj}\,\xi_j\xi_l]_{3\times1} & [0]_{3\times1} \\ [e_{jkl}\,\xi_j\xi_l]_{1\times3} & -\varkappa_{jl}\,\xi_j\xi_l & -a_{jl}\,\xi_j\xi_l & 0 \\ [q_{jkl}\,\xi_j\xi_l]_{1\times3} & -a_{jl}\,\xi_j\xi_l & -\mu_{jl}\,\xi_j\xi_l & 0 \\ [0]_{1\times3} & 0 & 0 & -\eta_{jl}\,\xi_j\,\xi_l \end{bmatrix}_{6\times6}.$$

From the symmetry conditions (2.8), inequalities (2.9) and positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.13) it follows that there is a positive constant C_0 depending only on the material parameters, such that

$$\mathrm{Re}\,\big(-A^{(0)}(-i\,\xi)\zeta\cdot\zeta\big) = \mathrm{Re}\left(-\sum_{k,j=1}^{6} A^{(0)}_{kj}(-i\,\xi)\,\zeta_j\,\overline{\zeta_k}\right) \geq C_0\,|\xi|^2\,|\zeta|^2$$
$$\text{for all}\quad \xi\in\mathbb{R}^3 \quad\text{and}\quad \text{for all}\quad \zeta\in\mathbb{C}^6.$$

Therefore, $-A(\partial_x,\tau)$ is a non-selfadjoint strongly elliptic differential operator. We recall that the over bar denotes complex conjugation and the central dot denotes the scalar product in the respective complex vector space. By $A^*(\partial_x,\tau) := [\overline{A(-\partial_x,\tau)}]^\top = A^\top(-\partial_x,\overline{\tau})$ we denote the operator formally adjoint to $A(\partial_x,\tau)$,

$$A^*(\partial_x,\tau) = \Big[A^*_{pq}(\partial_x,\tau)\Big]_{6\times6}$$
$$:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\,\overline{\tau}^2\,\delta_{rk}]_{3\times3} & [-e_{lrj}\partial_j\partial_l]_{3\times1} & [-q_{lrj}\partial_j\partial_l]_{3\times1} & [\overline{\tau}\lambda_{kl}\partial_l]_{3\times1} \\ [e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -\overline{\tau}\,p_l\,\partial_l \\ [q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -\overline{\tau}\,m_l\,\partial_l \\ [(1+\nu_0\overline{\tau})\lambda_{rj}\partial_j]_{1\times3} & (1+\nu_0\overline{\tau})\,p_j\partial_j & (1+\nu_0\overline{\tau})m_j\partial_j & \eta_{jl}\partial_j\partial_l - \overline{\tau}^2\,h_0 - \overline{\tau}\,d_0 \end{bmatrix}_{6\times6}. \tag{2.36}$$

Applying the Laplace transform to the dynamical constitutive relations (2.1)–(2.3) and (2.7) we get

$$\begin{aligned}
\widehat{\sigma_{rj}}(x,\tau) =& c_{rjkl}\,\widehat{\varepsilon_{kl}}(x,\tau) + e_{lrj}\partial_l\widehat{\varphi}(x,\tau) + q_{lrj}\partial_l\widehat{\psi}(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\,\widehat{\vartheta}(x,\tau)\\
&+\nu_0\lambda_{rj}\vartheta(x,0), \qquad r,j=1,2,3,\\
\widehat{D_j}(x,\tau) =& e_{jkl}\,\widehat{\varepsilon_{kl}}(x,\tau) - \varkappa_{jl}\,\partial_l\widehat{\varphi}(x,\tau) - a_{jl}\,\partial_l\widehat{\psi}(x,\tau) + (1+\nu_0\tau)p_j\widehat{\vartheta}(x,\tau)\\
&-\nu_0 p_j\vartheta(x,0), \qquad j=1,2,3,\\
\widehat{B_j}(x,\tau) =& q_{jkl}\,\widehat{\varepsilon_{kl}}(x,\tau) - a_{jl}\,\partial_l\widehat{\varphi}(x,\tau) - \mu_{jl}\partial_l\widehat{\psi}(x,\tau) + (1+\nu_0\tau)m_j\widehat{\vartheta}(x,\tau)\\
&-\nu_0 m_j\vartheta(x,0), \qquad j=1,2,3,\\
\widehat{q_j}(x,\tau) =& -T_0\,\eta_{jl}\partial_l\widehat{\vartheta}(x,\tau).
\end{aligned}$$

With the help of these equalities, the Laplace transform of the stress vector $\mathcal{T}(\partial_x,n,\partial_t)\,U(x,t)$ defined in (2.24) can be represented as follows

$$L_{t\to\tau}[\mathcal{T}(\partial_x,n,\partial_t)\,U(x,t)] = \mathcal{T}(\partial_x,n,\tau)\,\widehat{U}(x,\tau) + F^{(0)}(x),$$

where

$$\mathcal{T}(\partial_x,n,\tau)\,\widehat{U}(x,\tau) = \big(\widehat{\sigma_{1j}}n_j, \widehat{\sigma_{2j}}n_j, \widehat{\sigma_{3j}}n_j, -\widehat{D}_jn_j, -\widehat{B}_jn_j, -T_0^{-1}\widehat{q}_jn_j\big) - F^{(0)}(x),$$

$$F^{(0)}(x) := \begin{bmatrix} \nu_0\lambda_{1j}\,n_j\vartheta(x,0)\\ \nu_0\lambda_{2j}\,n_j\vartheta(x,0)\\ \nu_0\lambda_{3j}\,n_j\vartheta(x,0)\\ \nu_0 p_j\,n_j\vartheta(x,0)\\ \nu_0 m_j\,n_j\vartheta(x,0)\\ 0\end{bmatrix},$$

and the boundary operator $\mathcal{T}(\partial_x,n,\tau)$ reads as (cf. (2.23))

$$\begin{aligned}
\mathcal{T}(\partial_x,n,\tau) &= \big[\,\mathcal{T}_{pq}(\partial_x,n,\tau)\,\big]_{6\times6}\\
&:= \begin{bmatrix} [c_{rjkl}\,n_j\partial_l]_{3\times3} & [e_{lrj}\,n_j\partial_l]_{3\times1} & [q_{lrj}\,n_j\partial_l]_{3\times1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times1}\\
[-e_{jkl}\,n_j\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\partial_l & a_{jl}\,n_j\partial_l & -(1+\nu_0\tau)p_jn_j\\
[-q_{jkl}\,n_j\partial_l]_{1\times3} & a_{jl}\,n_j\partial_l & \mu_{jl}\,n_j\partial_l & -(1+\nu_0\tau)m_jn_j\\
[0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l\end{bmatrix}_{6\times6}.
\end{aligned} \tag{2.37}$$

Below, in Green's formulas there appears also the boundary operator $\mathcal{P}(\partial_x,n,\tau)$ associated with the adjoint differential operator $A^*(\partial_x,\tau)$,

$$\begin{aligned}
\mathcal{P}(\partial_x,n,\tau) &= \big[\,\mathcal{P}_{pq}(\partial_x,n,\tau)\,\big]_{6\times6}\\
&= \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times3} & [-e_{lrj}\,n_j\,\partial_l]_{3\times1} & [-q_{lrj}\,n_j\,\partial_l]_{3\times1} & [\overline{\tau}\,\lambda_{rj}\,n_j]_{3\times1}\\
[e_{jkl}\,n_j\,\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -\overline{\tau}\,p_j\,n_j\\
[q_{jkl}\,n_j\,\partial_l]_{1\times3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -\overline{\tau}\,m_j\,n_j\\
[0]_{1\times3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l\end{bmatrix}_{6\times6}.
\end{aligned} \tag{2.38}$$

2.2. Green's formulas for the pseudo-oscillation model

Let $\Omega=\Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $S=\partial\Omega^+$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega^+}$, $\overline{\Omega^+}=\Omega^+\cup S$. By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to $S=\partial\Omega$ from Ω.

The symbols $\{\,\cdot\,\}^+_S$ and $\{\,\cdot\,\}^-_S$ denote one sided limits on S from Ω^+ and Ω^- respectively. We drop the subscript in $\{\,\cdot\,\}^\pm_S$ if it does not lead to misunderstanding.

By L_p, $L_{p,loc}$, $L_{p,comp}$, W^r_p, $W^r_{p,loc}$, $W^r_{p,comp}$, H^s_p, and $B^s_{p,q}$ (with $r\ge0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$) we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [26,27]). Recall that $H^r_2=W^r_2=B^r_{2,2}$, $H^s_2=B^s_{2,2}$, $W^t_p=B^t_{p,p}$, and $H^k_p=W^k_p$, for any $r\ge0$, for any

$s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we essentially employ also the spaces:

$$\widetilde{H}^s_p(\mathcal{M}) := \{f \,:\, f \in H^s_p(\mathcal{M}_0),\ \operatorname{supp} f \subset \overline{\mathcal{M}}\},$$
$$\widetilde{B}^s_{p,q}(\mathcal{M}) := \{f \,:\, f \in B^s_{p,q}(\mathcal{M}_0),\ \operatorname{supp} f \subset \overline{\mathcal{M}}\},$$
$$H^s_p(\mathcal{M}) := \{r_{\mathcal{M}} f \,:\, f \in H^s_p(\mathcal{M}_0)\},$$
$$B^s_{p,q}(\mathcal{M}) := \{r_{\mathcal{M}} f \,:\, f \in B^s_{p,q}(\mathcal{M}_0)\},$$

where $\mathcal{M}_0$ is a closed manifold without boundary and $\mathcal{M}$ is an open proper submanifold of $\mathcal{M}_0$ with nonempty boundary $\partial\mathcal{M} \neq \varnothing$; $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$. Below, sometimes we use also the abbreviations $H^s_2 = H^s$ and $W^s_2 = W^s$.

If a function $f \in B^s_{p,q}(\mathcal{M})$, where $\mathcal{M}$ is a proper part of a closed surface $\mathcal{M}_0$, can be extended by zero to the whole $\mathcal{M}_0$ preserving the space, we write $f \in \widetilde{B}^s_{p,q}(\mathcal{M})$ instead of $f \in r_{\mathcal{M}}\widetilde{B}^s_{p,q}(\mathcal{M})$.

For arbitrary vector functions

$$U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top \in \big[C^2(\overline{\Omega})\big]^6 \text{ and } U' = (u'_1, u'_2, u'_3, \varphi', \psi', \vartheta')^\top \in \big[C^2(\overline{\Omega})\big]^6$$

we can derive the following Green's identities with the help of the Gauss integration by parts formula:

$$\int_\Omega \Big[A(\partial_x, \tau)\, U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = \int_S \{\mathcal{T}(\partial_x, n, \tau)U\}^+ \cdot \{U'\}^+ dS\,, \tag{2.39}$$

$$\int_\Omega \Big[U \cdot A^*(\partial_x, \tau)\, U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = \int_S \{U\}^+ \cdot \{\mathcal{P}(\partial_x, n, \tau)U'\}^+ dS\,, \tag{2.40}$$

$$\int_\Omega \Big[A(\partial_x, \tau)\, U \cdot U' - U \cdot A^*(\partial_x, \tau)\, U'\Big] dx$$
$$= \int_S \Big[\{\mathcal{T}(\partial_x, n, \tau)U\}^+ \cdot \{U'\}^+ - \{U\}^+ \cdot \{\mathcal{P}(\partial_x, n, \tau)\, U'\}^+\Big] dS, \tag{2.41}$$

where the operators $A(\partial_x, \tau)$, $\mathcal{T}(\partial_x, n, \tau)$, $A^*(\partial_x, \tau)$ and $\mathcal{P}(\partial_x, n, \tau)$ are given in (2.35), (2.37), (2.36), and (2.38) respectively,

$$\begin{aligned}\mathcal{E}_\tau(U, \overline{U'}) := {}& c_{rjkl}\, \partial_l u_k\, \overline{\partial_j u'_r} + \varrho\, \tau^2\, u_r\, \overline{u'_r} + e_{lrj}\, (\partial_l \varphi\, \overline{\partial_j u'_r} - \partial_j u_r\, \overline{\partial_l \varphi'}) \\ & + q_{lrj}\, (\partial_l \psi\, \overline{\partial_j u'_r} - \partial_j u_r\, \overline{\partial_l \psi'}) + \varkappa_{jl}\, \partial_l \varphi\, \overline{\partial_j \varphi'} + a_{jl}\, (\partial_l \varphi\, \overline{\partial_j \psi'} + \partial_j \psi\, \overline{\partial_l \varphi'}) \\ & + \mu_{jl}\, \partial_l \psi\, \overline{\partial_j \psi'} + \lambda_{kj}\, [\tau\, \overline{\vartheta'}\, \partial_j u_k - (1 + \nu_0 \tau)\vartheta\, \overline{\partial_j u'_k}] - p_l\, [\tau\, \overline{\vartheta'}\, \partial_l \varphi + (1 + \nu_0 \tau)\vartheta\, \overline{\partial_l \varphi'}] \\ & - m_l\, [\tau\, \overline{\vartheta'}\, \partial_l \psi + (1 + \nu_0 \tau)\vartheta\, \overline{\partial_l \psi'}] + \eta_{jl}\, \partial_l \vartheta\, \overline{\partial_j \vartheta'} + \tau\, (h_0 \tau + d_0)\, \vartheta\, \overline{\vartheta'}.\end{aligned} \tag{2.42}$$

Note that the above Green's formula (2.39) by standard limiting procedure can be generalized to Lipschitz domains and to vector functions $U \in [W^1_p(\Omega)]^6$ and $U' \in [W^1_{p'}(\Omega)]^6$ with

$$A(\partial_x, \tau)U \in \big[L_p(\Omega)\big]^6, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

With the help of Green's formula (2.39) we can correctly determine a *generalized trace vector* $\{\mathcal{T}(\partial_x, n, \tau)U\}^+ \in [B^{-\frac{1}{p}}_{p,p}(S)]^6$ for a function $U \in [W^1_p(\Omega)]^6$ with $A(\partial_x, \tau)U \in [L_p(\Omega)]^6$ by the following relation (cf. [28–30])

$$\big\langle \{\mathcal{T}(\partial_x, n, \tau)U\}^+\,,\ \{U'\}^+ \big\rangle_S := \int_\Omega [\, A(\partial_x, \tau)\, U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\,]\, dx, \tag{2.43}$$

where $U' \in [W^1_{p'}(\Omega)]^6$ is an arbitrary vector function. Here the symbol $\langle \cdot\,, \cdot \rangle_S$ denotes the duality between the function spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^6$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^6$ which extends the usual L_2 inner product for complex valued vector functions,

$$\langle f, g\rangle_S = \int_S \sum_{j=1}^{6} f_j(x)\, \overline{g_j(x)}\, dS \quad \text{for} \quad f, g \in [L_2(S)]^6.$$

Evidently we have the following estimate

$$\|\{\mathcal{T}(\partial_x, n, \tau)U\}^+\|_{[B_{p,p}^{-1/p}(S)]^6} \le c_0 \left\{\|A(\partial_x, \tau)\, U\|_{[L_p(\Omega)]^6} + (1 + |\tau|^2)\|U\|_{[W_p^1(\Omega)]^6}\right\},$$

where c_0 does not depend on U; in general c_0 depends on the material parameters and on the geometrical characteristics of the domain Ω.

Let us introduce a sesquilinear form on $[H_2^1(\Omega)]^6 \times [H_2^1(\Omega)]^6$

$$\mathcal{B}(U, V) := \int_\Omega \mathcal{E}_\tau(U, \overline{V})\, dx,$$

where $\mathcal{E}_\tau(U, \overline{V})$ is defined by (2.42). With the help of the relations (2.9) and (2.42), positive definiteness of the matrix (2.13) and the well known Korn's inequality we deduce the following estimate

$$\operatorname{Re} \mathcal{B}(U, U) \ge c_1\, \|U\|^2_{[H_2^1(\Omega)]^6} - c_2\, \|U\|^2_{[H_2^0(\Omega)]^6} \tag{2.44}$$

with some positive constants c_1 and c_2 depending on the material parameters (cf. [17,29]), which shows that the sesquilinear form defined in (2.44) is coercive.

From the Green formulas (2.39)–(2.41) by standard limiting procedure we derive similar formulas for the exterior domain Ω^- provided the vector functions $U,\ U' \in \mathbf{Z}(\Omega^-)$, where the class $\mathbf{Z}(\Omega^-)$ is defined as a set of functions U possessing the following asymptotic properties at infinity as $|x| \to \infty$:

$$\begin{array}{ll} u_k(x) = \mathcal{O}(|x|^{-2}), & \partial_j u_k(x) = \mathcal{O}(|x|^{-2}),\\ \varphi(x) = \mathcal{O}(|x|^{-1}), & \partial_j \varphi(x) = \mathcal{O}(|x|^{-2}),\\ \psi(x) = \mathcal{O}(|x|^{-1}), & \partial_j \psi(x) = \mathcal{O}(|x|^{-2}),\\ \vartheta(x) = \mathcal{O}(|x|^{-2}), & \partial_j \vartheta(x) = \mathcal{O}(|x|^{-2}),\\ k, j = 1, 2, 3. & \end{array} \tag{2.45}$$

Assume that $A^*(\partial_x, \tau)\, U'$ is compactly supported as well and U' satisfies the conditions of type (2.45). Then the following Green formulas hold for the exterior domain Ω^-:

$$\int_{\Omega^-} \Big[A(\partial_x, \tau)\, U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = -\int_S \{\mathcal{T}(\partial_x, n, \tau)U\}^- \cdot \{U'\}^- dS\,,$$

$$\int_{\Omega^-} \Big[U \cdot A^*(\partial_x, \tau)\, U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = -\int_S \{U\}^- \cdot \{\mathcal{P}(\partial_x, n, \tau)U'\}^- dS\,,$$

$$\int_{\Omega^-} \Big[A(\partial_x, \tau)\, U \cdot U' - U \cdot A^*(\partial_x, \tau)\, U'\Big] dx \ = \ -\int_S \Big[\{\mathcal{T}(\partial_x, n, \tau)U\}^- \cdot \{U'\}^- - \{U\}^- \cdot \{\mathcal{P}(\partial_x, n, \tau)\, U'\}^-\Big] dS,$$

where $\mathcal{E}_\tau$ is defined by (2.42). We recall that the direction of the unit normal vector to $S = \partial\Omega^-$ is outward with respect to the domain $\Omega = \Omega^+$.

As we shall see below the fundamental matrix of the operator $A(\partial_x, \tau)$ with $\tau = \sigma + i\,\omega$, $\sigma > \sigma_0$, possesses the decay properties (2.45)

2.3. Boundary value problems for pseudo-oscillation equations

Throughout the paper we assume that the origin of the co-ordinate system belongs to Ω. Assume that the domain $\overline{\Omega}$ is occupied by an anisotropic homogeneous material with the above described generalized thermo-electro-magneto-elastic properties (henceforth such type of materials will be referred to as GTEME materials).

Further, we assume that $\partial\Omega$ is divided into two disjoint parts $S_D \neq \varnothing$ and S_N: $\partial\Omega = S = \overline{S}_D \cup \overline{S}_N$, $\overline{S}_D \cap \overline{S}_N = \varnothing$. Set $\partial S_D = \partial S_N =: \ell_m$. In what follows, for simplicity we assume that S, S_D, S_N, ℓ_m are C^∞-smooth.

Here we preserve the notation introduced in the previous subsections and formulate the boundary value problems for the pseudo-oscillation equations of the GTEME theory. The operators $A(\partial_x, \tau)$ and $\mathcal{T}(\partial_x, n, \tau)$ involved in the formulations below are determined by the relations (2.35) and (2.37) respectively. In what follows we always assume that

$$\tau = \sigma + i\omega, \quad \sigma > \sigma_0 \geqslant 0, \quad \omega \in \mathbb{R},$$

if not otherwise stated.

The Dirichlet pseudo-oscillation problem (D) $^+_\tau$: *Find a solution*

$$U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega)]^6, \quad 1 < p < \infty$$

to the pseudo-oscillation equations of the GTEME theory,

$$A(\partial_x, \tau)\, U(x) = \Phi(x), \qquad x \in \Omega, \tag{2.46}$$

satisfying the Dirichlet type boundary condition

$$\{U(x)\}^+ = f(x), \quad x \in S, \tag{2.47}$$

i.e.

$$\{u_r(x)\}^+ = f_r(x), \quad x \in S, \quad r = 1, 2, 3, \tag{2.48}$$

$$\{\varphi(x)\}^+ = f_4(x), \quad x \in S, \tag{2.49}$$

$$\{\psi(x)\}^+ = f_5(x), \quad x \in S, \tag{2.50}$$

$$\{\vartheta(x)\}^+ = f_6(x), \quad x \in S, \tag{2.51}$$

where $\Phi = (\Phi_1, \dots, \Phi_6)^\top \in [L_p(\Omega)]^6$, *and* $f = (f_1, \dots, f_6)^\top \in [B^{1-1/p}_{p,p}(S)]^6$, $1 < p < \infty$ *are given functions from the appropriate spaces.*

In the case when U satisfies the homogeneous equation

$$A(\partial_x, \tau)\, U(x) = 0, \qquad x \in \Omega, \tag{2.52}$$

we denote the corresponding problem by $(\mathrm{D})^+_{\tau,0}$.

The Neumann pseudo-oscillation problem $(N)^+_\tau$: *Find a regular solution*

$$U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega)]^6, \quad 1 < p < \infty$$

to the pseudo-oscillation equations of the GTEME theory (2.46) *satisfying the Neumann type boundary condition*

$$\{\mathcal{T}(\partial_x, n, \tau) U(x)\}^+ = F(x), \qquad x \in S, \tag{2.53}$$

i.e.

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_r\}^+ \equiv \{\sigma_{rj}\, n_j\}^+ = F_r(x), \quad x \in S, \quad r = 1, 2, 3, \tag{2.54}$$

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_4\}^+ \equiv \{-D_j\, n_j\}^+ = F_4(x), \quad x \in S, \tag{2.55}$$

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_5\}^+ \equiv \{-B_j\, n_j\}^+ = F_5(x), \quad x \in S, \tag{2.56}$$

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_6\}^+ \equiv \{-T_0^{-1} q_j\, n_j\}^+ = F_6(x), \quad x \in S, \tag{2.57}$$

where $F = (F_1, \dots, F_6) \in [B^{-1/p}_{p,p}(S)]^6$, $1 < p < \infty$ *is a given vector function.*

In the case when U satisfies the homogeneous equation (2.52) we denote the corresponding problem by $(\mathrm{N})^+_{\tau,0}$.

Mixed boundary value problems for solids with interior cracks. Let us assume that a GTEME type solid occupying the simply connected domain $\overline{\Omega}$ contains an interior crack. We identify the crack surface as a two-dimensional, two-sided manifold Σ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a submanifold of a closed surface $S_0 \subset \Omega$ surrounding a domain $\overline{\Omega}_0 \subset \Omega$ and that Σ, S_0, and ℓ_c are C^∞-smooth. Denote $\Omega_\Sigma := \Omega \setminus \overline{\Sigma}$.

We write $v \in W_p^1(\Omega_\Sigma)$ if $v \in W_p^1(\Omega_0)$, $v \in W_p^1(\Omega \setminus \overline{\Omega_0})$, and $r_{S_0 \setminus \overline{\Sigma}}\{v\}^+ = r_{S_0 \setminus \overline{\Sigma}}\{v\}^-$.

Recall that throughout the paper $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega$ and $S_0 = \partial\Omega_0$. This agreement defines the positive direction of the normal vector on the crack surface Σ.

We will consider the following problem $(\mathbf{MC})_\tau$:

(i) the magneto-piezoelectric elastic solid under consideration is mechanically fixed along the subsurface S_D, and at the same time there are given the temperature and the electric and magnetic potential functions (i.e., on S_D there are given the components of the vector $\{U\}^+$-Dirichlet conditions);

(ii) on the subsurface S_N there are prescribed the mechanical stress vector and the normal components of the heat flux, the electric displacement and magnetic induction vectors (i.e., on S_N there are given the components of the vector $\{\mathcal{T}U\}^+$-Neumann conditions);

(iii) the crack surface Σ is mechanically traction free and we assume that the temperature, electric and magnetic potentials, and the normal components of the heat flux, the electric displacement and magnetic induction vectors are continuous across the crack surface.

Reducing the nonhomogeneous differential equations (2.46) to the corresponding homogeneous ones, we can formulate the above problem mathematically as follows:

Find a vector $U = (u, \varphi, \psi, \theta)^\top = (u_1, u_2, u_3, u_4, u_5, u_6)^\top \in \left[W_p^1(\Omega_\Sigma)\right]^6$ *with* $1 < p < \infty$, *satisfying the homogeneous pseudo-oscillation differential equation of the GTEME theory*

$$A(\partial_x, \tau)\, U = 0 \ \text{ in } \ \Omega_\Sigma, \quad \tau = \sigma + i\,\omega, \quad \sigma > 0, \tag{2.58}$$

the crack conditions on Σ,

$$\{[\mathcal{T}U]_j\}^+ = F_j^+ \quad \text{on } \ \Sigma, \qquad j = \overline{1,3}, \tag{2.59}$$

$$\{[\mathcal{T}U]_j\}^- = F_j^- \quad \text{on } \ \Sigma, \qquad j = \overline{1,3}, \tag{2.60}$$

$$\{u_4\}^+ - \{u_4\}^- = f_4 \quad \text{on } \ \Sigma, \tag{2.61}$$

$$\{[\mathcal{T}U]_4\}^+ - \{[\mathcal{T}U]_4\}^- = F_4 \quad \text{on } \ \Sigma, \tag{2.62}$$

$$\{u_5\}^+ - \{u_5\}^- = f_5 \quad \text{on } \ \Sigma, \tag{2.63}$$

$$\{[\mathcal{T}U]_5\}^+ - \{[\mathcal{T}U]_5\}^- = F_5 \quad \text{on } \ \Sigma, \tag{2.64}$$

$$\{u_6\}^+ - \{u_6\}^- = f_6 \quad \text{on } \ \Sigma, \tag{2.65}$$

$$\{[\mathcal{T}U]_6\}^+ - \{[\mathcal{T}U]_6\}^- = F_6 \quad \text{on } \ \Sigma, \tag{2.66}$$

and the mixed boundary conditions on $S = \overline{S}_D \cup \overline{S}_N$,

$$\{U\}^+ = g^{(D)} \quad \text{on} \quad S_D, \tag{2.67}$$

$$\{\mathcal{T}U\}^+ = g^{(N)} \quad \text{on} \quad S_N. \tag{2.68}$$

We require that the boundary data possess the natural smoothness properties associated with the trace theorems,

$$\begin{aligned} &F_j^+, F_j^- \in B_{p,p}^{-\frac{1}{p}}(\Sigma), \quad j = 1,2,3; \quad f_4,\ f_5,\ f_6 \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma), \\ &F_4,\ F_5,\ F_6 \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad g^{(D)} \in \left[B_{p,p}^{1-\frac{1}{p}}(S_D)\right]^6, \quad g^{(N)} \in \left[B_{p,p}^{-\frac{1}{p}}(S_N)\right]^6, \\ &1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \end{aligned} \tag{2.69}$$

Moreover, the following compatibility conditions

$$F_j^+ - F_j^- \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad j = 1,2,3,$$

are to be satisfied.

The differential equation (2.58) is understood in the distributional sense, in general. We remark that if $U \in [W^1_p(\Omega_\Sigma)]^6$ solves the homogeneous differential equation then actually we have the inclusion $U \in [C^\infty(\Omega_\Sigma)]^6$ due to the ellipticity of the corresponding differential operators. In fact, U is a complex valued analytic vector function of spatial real variables (x_1, x_2, x_3) in Ω_Σ.

The Dirichlet-type conditions (2.61), (2.63), (2.65) and (2.67) are understood in the usual trace sense, while the Neumann-type conditions (2.59), (2.60), (2.62), (2.64), (2.66) and (2.68) involving boundary limiting values of the components of the vector $\mathcal{T}U$ are understood in the above described generalized functional sense related to Green's formula (2.43).

2.3.1. Uniqueness theorems for the pseudo-oscillation problems

We prove here the following uniqueness theorem for solutions to the pseudo-oscillation problems in the case of $p = 2$.

Theorem 2.1. *Let S be Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$.*

(i) *The basic boundary value problem $(D)^+_\tau$ has at most one solution in the space $\left[W^1_2(\Omega)\right]^6$.*

(ii) *Solutions to the Neumann type boundary value problem $(N)^+_\tau$ in the space $\left[W^1_2(\Omega)\right]^6$ are defined modulo a vector of type $U^{(N)} = (0, 0, 0, b_1, b_2, 0)^\top$, where b_1 and b_2 are arbitrary constants.*

(iii) *Mixed type boundary value problem $(\mathbf{MC})_\tau$ has at most one solution in the space $\left[W^1_2(\Omega_\Sigma)\right]^6$.*

Proof. Due to the linearity of the boundary value problems in question it suffices to consider the corresponding homogeneous problems.

First we demonstrate the proof for the problems stated in the items (i) and (ii) of the theorem. Let $U = (u, \varphi, \psi, \vartheta)^\top \in \left[W^1_2(\Omega)\right]^6$ be a solution to the homogeneous problem $(\mathrm{D})^+_{\tau,0}$ or $(\mathrm{N})^+_{\tau,0}$. For arbitrary $U' = (u', \varphi', \psi', \vartheta')^\top \in \left[W^1_2(\Omega)\right]^6$ from Green's formula (2.43) we have

$$\int_\Omega \mathcal{E}_\tau(U, \overline{U'})\, dx = \big\langle \{\mathcal{T}(\partial_x, n, \tau)U\},\ \{U'\}^+ \big\rangle_{\partial\Omega}, \tag{2.70}$$

where $\mathcal{E}_\tau(U, \overline{U'})$ is given by (2.42).

If in (2.70) we substitute the vectors $(u_1, u_2, u_3, 0, 0, 0)^\top$, $(0, 0, 0, \varphi, 0, 0)^\top$, $(0, 0, 0, 0, \psi, 0)^\top$, and $(0, 0, 0, 0, 0, (1 + \nu_0\tau)[\overline{\tau}]^{-1}\vartheta)^\top$ for the vector U' successively and take into consideration the homogeneous boundary conditions, we get

$$\int_\Omega \Big[c_{rjkl}\partial_l u_k \overline{\partial_j u_r} + \varrho\tau^2 u_r \overline{u_r} + e_{lrj}\,\partial_l\varphi\,\overline{\partial_j u_r} + q_{lrj}\,\partial_l\psi\,\overline{\partial_j u_r} - (1 + \nu_0\tau)\lambda_{kj}\vartheta\,\overline{\partial_j u_k}\,\Big]dx = 0, \tag{2.71}$$

$$\int_\Omega \big[-e_{lrj}\,\partial_j u_r\,\overline{\partial_l\varphi} + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,\partial_j\psi\,\overline{\partial_l\varphi} - (1 + \nu_0\tau)p_l\,\vartheta\,\overline{\partial_l\varphi}\,\big]\,dx = 0, \tag{2.72}$$

$$\int_\Omega \big[-q_{lrj}\,\partial_j u_r\,\overline{\partial_l\psi} + a_{jl}\,\partial_l\varphi\,\overline{\partial_j\psi} + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi} - (1 + \nu_0\tau)m_l\,\vartheta\,\overline{\partial_l\psi}\,\big]\,dx = 0, \tag{2.73}$$

$$\int_\Omega \Big\{(1 + \nu_0\overline{\tau})[\lambda_{kj}\overline{\vartheta}\partial_j u_k - p_l\overline{\vartheta}\partial_l\varphi - m_l\overline{\vartheta}\partial_l\psi + (h_0\tau + d_0)|\vartheta|^2] + \frac{1 + \nu_0\overline{\tau}}{\tau}\,\eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\Big\}dx = 0. \tag{2.74}$$

Add to Eq. (2.71) the complex conjugate of Eqs. (2.72)–(2.74) and take into account the symmetry properties (2.8) to obtain

$$\begin{aligned}
&\int_\Omega \Big\{c_{rjkl}\partial_l u_k\,\overline{\partial_j u_r} + \varrho\,\tau^2|u|^2 + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\,\overline{\partial_j\psi}\\
&\quad - 2\,\mathrm{Re}\big[p_l(1 + \nu_0\tau)\vartheta\,\overline{\partial_l\varphi}\,\big] - 2\,\mathrm{Re}\big[m_l(1 + \nu_0\tau)\vartheta\,\overline{\partial_l\psi}\,\big] + (1 + \nu_0\tau)(h_0\overline{\tau} + d_0)\,|\vartheta|^2\\
&\quad + \frac{1 + \nu_0\tau}{\overline{\tau}}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta}\Big\}\,dx = 0.
\end{aligned} \tag{2.75}$$

Due to the relations (2.11) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.13), we find that

$$\begin{aligned} &c_{ijlk}\, \partial_i u_j\, \overline{\partial_l u_k} \geq 0, \quad \eta_{jl}\, \partial_l \vartheta\, \overline{\partial_j \vartheta} \geq 0, \\ &[\varkappa_{jl}\, \partial_l \varphi\, \overline{\partial_j \varphi} + a_{jl}\,(\partial_l \psi\, \overline{\partial_j \varphi} + \partial_j \varphi\, \overline{\partial_l \psi}) + \mu_{jl}\, \partial_l \psi\, \overline{\partial_j \psi}] \geq \lambda_0(|\nabla \varphi|^2 + |\nabla \psi|^2), \end{aligned} \tag{2.76}$$

where λ_0 is a positive constant. Use the equalities

$$\begin{aligned} &\tau^2 = \sigma^2 - \omega^2 + 2i\sigma\omega, \quad \frac{1+\nu_0\tau}{\overline{\tau}} = \frac{\sigma + \nu_0(\sigma^2 - \omega^2)}{|\tau|^2} + i\,\frac{\omega(1+2\sigma\nu_0)}{|\tau|^2}, \\ &(1+\nu_0\tau)(h_0\overline{\tau} + d_0) = d_0 + \nu_0 h_0 |\tau|^2 + (h_0 + \nu_0 d_0)\sigma + i\,\omega(\nu_0 d_0 - h_0), \end{aligned}$$

and separate the imaginary part of (2.75) to deduce

$$\omega \int_\Omega \Big\{ 2\,\varrho\,\sigma\, |u|^2 + (\nu_0 d_0 - h_0)|\vartheta|^2 + \frac{1+2\sigma\nu_0}{|\tau|^2}\, \eta_{jl}\, \partial_l \vartheta\, \overline{\partial_j \vartheta} \Big\}\, dx = 0.$$

By the inequalities in (2.10) and since $\sigma > \sigma_0 \geqslant 0$, we conclude $u = 0$ and $\vartheta = 0$ in Ω for $\omega \neq 0$. From (2.75) we then have

$$\int_\Omega \big[\varkappa_{jl}\, \partial_l \varphi\, \overline{\partial_j \varphi} + a_{jl}\,(\partial_l \psi\, \overline{\partial_j \varphi} + \partial_j \varphi\, \overline{\partial_l \psi}) + \mu_{jl}\, \partial_l \psi\, \overline{\partial_j \psi}\,\big]\, dx = 0.$$

Whence, in view of the last inequality in (2.76), we get $\partial_l \varphi = 0$, $\partial_l \psi = 0$, $l = 1, 2, 3$, in Ω. Thus, if $\omega \neq 0$,

$$u = 0, \quad \varphi = b_1 = \text{const}, \quad \psi = b_2 = \text{const}, \quad \vartheta = 0 \text{ in } \Omega. \tag{2.77}$$

If $\omega = 0$, then $\tau = \sigma > 0$ and (2.75) can be rewritten in the form

$$\begin{aligned} &\int_\Omega \Big\{ c_{rjkl} \partial_l u_k\, \overline{\partial_j u_r} + \varrho\,\sigma^2 |u|^2 + \frac{1+\nu_0\sigma}{\sigma}\, \eta_{jl}\, \partial_l \vartheta\, \overline{\partial_j \vartheta} \Big\}\, dx \\ &\quad + \int_\Omega \Big\{ \varkappa_{jl}\, \partial_l \varphi\, \overline{\partial_j \varphi} + a_{jl}(\partial_l \psi\, \overline{\partial_j \varphi} + \partial_j \varphi\, \overline{\partial_l \psi}) + \mu_{jl} \partial_l \psi\, \overline{\partial_j \psi} - 2p_l(1+\nu_0\sigma)\, \mathrm{Re}\big[\vartheta\, \overline{\partial_l \varphi}\,\big] \\ &\quad - 2m_l(1+\nu_0\sigma)\, \mathrm{Re}\big[\vartheta\, \overline{\partial_l \psi}\,\big] + (1+\nu_0\sigma)(h_0\sigma + d_0)\, |\vartheta|^2 \Big\}\, dx = 0. \end{aligned} \tag{2.78}$$

The integrand in the first integral is nonnegative. Let us show that the integrand in the second integral is also nonnegative. To this end, let us set

$$\zeta_j := \partial_j \varphi, \quad \zeta_{j+3} := \partial_j \psi, \quad \zeta_7 := -\vartheta, \quad \zeta_8 := -\sigma\vartheta, \quad j = 1, 2, 3,$$

and introduce the vector

$$\Theta := (\zeta_1, \zeta_2, \ldots, \zeta_8)^\top.$$

It can be easily checked that (summation over repeated indices is meant from 1 to 3)

$$\begin{aligned} &\varkappa_{jl}\, \partial_l \varphi\, \overline{\partial_j \varphi} + a_{jl}(\partial_l \psi\, \overline{\partial_j \varphi} + \partial_j \varphi\, \overline{\partial_l \psi}) + \mu_{jl} \partial_l \psi\, \overline{\partial_j \psi} - 2p_l(1+\nu_0\sigma)\, \mathrm{Re}\big[\vartheta\, \overline{\partial_l \varphi}\,\big] \\ &\qquad - 2m_l(1+\nu_0\sigma)\, \mathrm{Re}\big[\vartheta\, \overline{\partial_l \psi}\,\big] + (1+\nu_0\sigma)(h_0\sigma + d_0)\, |\vartheta|^2 \\ &\quad = [\varkappa_{jl}\, \partial_l \varphi + a_{jl}\, \partial_l \psi + p_j(-\vartheta) + \nu_0 p_j(-\sigma\vartheta)]\, \overline{\partial_j \varphi} \\ &\qquad + [a_{jl}\, \partial_l \varphi + \mu_{jl}\, \partial_l \psi + m_j(-\vartheta) + \nu_0 m_j(-\sigma\vartheta)]\, \overline{\partial_j \psi} \\ &\qquad + [p_l \partial_l \varphi + m_l\, \partial_l \psi + d_0(-\vartheta) + h_0(-\sigma\vartheta)](-\overline{\vartheta}) \\ &\qquad + [\nu_0 p_l \partial_l \varphi + \nu_0 m_l\, \partial_l \psi + h_0(-\vartheta) + \nu_0 h_0(-\sigma\vartheta)](-\sigma\overline{\vartheta}) \\ &\qquad + \sigma(d_0\nu_0 - h_0)\, |\vartheta|^2 \\ &\quad = [\varkappa_{jl}\, \zeta_l + a_{jl}\, \zeta_{l+3} + p_j \zeta_7 + \nu_0 p_j \zeta_8]\, \overline{\zeta}_j \\ &\qquad + [a_{jl}\, \zeta_l + \mu_{jl}\, \zeta_{l+3} + m_j \zeta_7 + \nu_0 m_j \zeta_8]\, \overline{\zeta}_{j+3} \\ &\qquad + [p_l \zeta_l + m_l\, \zeta_{l+3} + d_0 \zeta_7 + h_0 \zeta_8]\, \overline{\zeta}_7 \\ &\qquad + [\nu_0 p_l \zeta_l + \nu_0 m_l\, \zeta_{l+3} + h_0 \zeta_7 + \nu_0 h_0 \zeta_8]\, \overline{\zeta}_8 \\ &\qquad + \sigma(d_0\nu_0 - h_0)\, |\vartheta|^2 \\ &\quad = \sum_{p,q=1}^{8} M_{pq} \zeta_q\, \overline{\zeta}_p + \sigma(d_0\nu_0 - h_0)\, |\vartheta|^2 = M\,\Theta \cdot \Theta + \sigma(d_0\nu_0 - h_0)\, |\vartheta|^2 \geq C_0\, |\Theta|^2 \end{aligned} \tag{2.79}$$

with some positive constant C_0 due to the positive definiteness of the matrix M defined by (2.12).

Therefore, from (2.78) we see that the relations (2.77) hold for $\omega = 0$ as well.

Thus the equalities (2.77) hold for arbitrary $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$, whence the items (i) and (ii) of the theorem follow immediately, since the homogeneous Dirichlet conditions for φ and ψ imply $b_1 = b_2 = 0$, while a vector $U^{(\mathcal{N})} = (0, 0, 0, b_1, b_2, 0)^\top$, where b_1 and b_2 are arbitrary constants, solves the homogeneous Neumann BVP $(N)^+_{\tau,0}$.

To prove the third item of the theorem we have to add together two Green's formulas of type (2.70) for the domains $\Omega \setminus \overline{\Omega_0}$ and Ω_0, where Ω_0 is the above introduced auxiliary domain $\Omega_0 \subset \Omega$. We recall that the crack surface Σ is a proper part of the boundary $S_0 = \partial\Omega_0 \subset \Omega$ and any solution to the homogeneous differential equation $A(\partial_x, \tau)\, U = 0$ of the class $\big[W^1_2(\Omega_\Sigma)\big]^6$ and its derivatives are continuous across the surface $S_0 \setminus \overline{\Sigma}$. If U is a solution to the homogeneous crack type BVP by the same approach as above, we arrive at the relation

$$\begin{aligned}\int_{\Omega_\Sigma} \Big\{ & c_{rjkl}\partial_l u_k\, \overline{\partial_j u_r} + \varrho\, \tau^2 |u|^2 + \varkappa_{jl}\, \partial_l\varphi\, \overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\, \overline{\partial_j\varphi} + \partial_j\varphi\, \overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\, \overline{\partial_j\psi} \\ & - 2\,\mathrm{Re}\big[p_l(1+\nu_0\tau)\vartheta\, \overline{\partial_l\varphi}\,\big] - 2\,\mathrm{Re}\big[m_l(1+\nu_0\tau)\vartheta\, \overline{\partial_l\psi}\,\big] + (1+\nu_0\tau)(h_0\overline{\tau} + d_0)\,|\vartheta|^2 \\ & + \frac{1+\nu_0\tau}{\overline{\tau}}\, \eta_{jl}\, \partial_l\vartheta\, \overline{\partial_j\vartheta} \Big\}\, dx = 0. \end{aligned} \tag{2.80}$$

The surface integrals vanish due to the homogeneous boundary and crack type conditions and the above mentioned continuity of solutions and its derivatives across the auxiliary surface $S_0 \setminus \overline{\Sigma}$. Therefore, the proof of item (iii) can be verbatim performed. □

3. Properties of potentials and boundary operators

The full symbol of the pseudo-oscillation differential operator $A(\partial_x, \tau)$ is elliptic provided $\mathrm{Re}\,\tau \neq 0$, i.e.,

$$\det A(-i\,\xi, \tau) \neq 0, \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\,\xi, \tau)$ are locally integrable functions decaying at infinity as $\mathcal{O}(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x, \tau) = [\Gamma_{kj}(x, \tau)]_{6\times 6}$ of the operator $A(\partial_x, \tau)$ by means of the Fourier transform technique,

$$\Gamma(x, \tau) = \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi, \tau)\big]. \tag{3.1}$$

The properties of the fundamental matrix $\Gamma(x, \tau)$ near the origin and at infinity, and the properties of corresponding layer potentials are studied in [31]. Here we collect some results which are necessary for our further analysis. Detailed proofs of the theorems below are similar to the proofs of their counterparts in [32,30,33–35].

Let us introduce the single and double layer potentials:

$$V(h)(x) = \int_S \Gamma(x-y, \tau)\, h(y)\, d_yS,$$
$$W(h)(x) = \int_S \Big[\mathcal{P}(\partial_y, n(y), \overline{\tau})\big[\Gamma(x-y, \tau)\big]^\top\Big]^\top h(y)\, d_yS,$$

where $h = (h_1, h_2, \ldots, h_6)^\top$ is a density vector function.

Theorem 3.1. *Let $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. Then the single and double layer potentials can be extended to the following continuous operators*

$$\begin{aligned} V &: \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+1+\frac1p}_{p,q}(\Omega)\big]^6, & W &: \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+\frac1p}_{p,q}(\Omega)\big]^6, \\ &: \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+1+\frac1p}_{p,q,\,loc}(\Omega^-)\big]^6, & &: \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+\frac1p}_{p,q,\,loc}(\Omega^-)\big]^6, \\ &: \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+1+\frac1p}_{p}(\Omega)\big]^6, & &: \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+\frac1p}_{p}(\Omega)\big]^6, \\ &: \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+1+\frac1p}_{p,\,loc}(\Omega^-)\big]^6, & &: \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+\frac1p}_{p,\,loc}(\Omega^-)\big]^6. \end{aligned}$$

Theorem 3.2. *Let*

$$h^{(1)} \in \big[B^{-1+s}_{p,q}(S)\big]^6,\ h^{(2)} \in \big[B^{s}_{p,q}(S)\big]^6,\ 1 < p < \infty,\ 1 \le q \le \infty,\ s > 0.$$

Then

$$\big\{V(h^{(1)})(z)\big\}^{\pm} = \int_S \Gamma(z-y,\tau)h^{(1)}(y)\,d_yS \quad on \ \ S,$$
$$\big\{W(h^{(2)})(z)\big\}^{\pm} = \pm\,\frac{1}{2}\,h^{(2)}(z) + \int_S \Big[\mathcal{P}(\partial_y, n(y), \overline{\tau})\big[\Gamma(z-y,\tau)\big]^{\top}\Big]^{\top} h^{(2)}(y)\,d_yS \quad on \ \ S.$$

The equalities are understood in the sense of the space $[B^{s}_{p,q}(S)\big]^6$.

Theorem 3.3. *Let* $h^{(1)} \in \big[B^{-\frac{1}{p}}_{p,q}(S)\big]^6$, $h^{(2)} \in \big[B^{1-\frac{1}{p}}_{p,q}(S)\big]^6$, $1 < p < \infty$, $1 \le q \le \infty$. *Then*

$$\big\{\mathcal{T}V(h^{(1)})(z)\big\}^{\pm} = \mp\,\frac{1}{2}\,h^{(1)}(z) + \int_S \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z-y,\tau)\,h^{(1)}(y)\,d_yS \quad on \ \ S,$$
$$\big\{\mathcal{T}W(h^{(2)})(z)\big\}^{+} = \big\{\mathcal{T}W(h^{(2)})(z)\big\}^{-} \quad on \ \ S,$$

where the equalities are understood in the sense of the space $[B^{-\frac{1}{p}}_{p,q}(S)\big]^6$.

We introduce the following notation for the boundary operators generated by the single and double layer potentials:

$$\mathcal{H}(h)(z) = \int_S \Gamma(z-y,\tau)\,h(y)\,d_yS, \quad z \in S, \tag{3.2}$$
$$\mathcal{K}(h)(z) = \int_S \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z-y,\tau)\,h(y)\,d_yS, \quad z \in S, \tag{3.3}$$
$$\mathcal{N}(h)(z) = \int_S \Big[\mathcal{P}(\partial_y, n(y), \overline{\tau})\big[\Gamma(z-y,\tau)\big]^{\top}\Big]^{\top} h(y)\,d_yS, \quad z \in S,$$
$$\mathcal{L}(h)(z) = \big\{\mathcal{T}W(h)(z)\big\}^{+} = \big\{\mathcal{T}W(h)(z)\big\}^{-}, \quad z \in S.$$

Actually, $\mathcal{H}$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators (pseudodifferential operator of order 0), and $\mathcal{L}$ is a singular integro-differential operator (pseudodifferential operator of order 1). These operators possess the following mapping and Fredholm properties (see [31]).

Theorem 3.4. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. *Then the operators*

$$\begin{aligned}
\mathcal{H} : \big[B^{s}_{p,q}(S)\big]^6 &\to \big[B^{s+1}_{p,q}(S)\big]^6,\\
: \big[H^{s}_{p}(S)\big]^6 &\to \big[H^{s+1}_{p}(S)\big]^6,\\
\mathcal{K},\ \mathcal{N} : \big[B^{s}_{p,q}(S)\big]^6 &\to \big[B^{s}_{p,q}(S)\big]^6,\\
: \big[H^{s}_{p}(S)\big]^6 &\to \big[H^{s}_{p}(S)\big]^6,\\
\mathcal{L} : \big[B^{s}_{p,q}(S)\big]^6 &\to \big[B^{s-1}_{p,q}(S)\big]^6,\\
: \big[H^{s}_{p}(S)\big]^6 &\to \big[H^{s-1}_{p}(S)\big]^6
\end{aligned}$$

are continuous.

The operators $\mathcal{H}$ *and* $\mathcal{L}$ *are strongly elliptic pseudodifferential operators, while the operators* $\pm\frac{1}{2}\,I_6 + \mathcal{K}$ *and* $\pm\frac{1}{2}\,I_6 + \mathcal{N}$ *are elliptic, where* I_6 *stands for the* 6×6 *unit matrix.*

Moreover, the operators $\mathcal{H}$, $\frac{1}{2}\,I_6 + \mathcal{N}$ *and* $\frac{1}{2}\,I_6 + \mathcal{K}$ *are invertible, whereas the operators* $-\frac{1}{2}\,I_6 + \mathcal{K}$, $-\frac{1}{2}\,I_6 + \mathcal{N}$ *and* $\mathcal{L}$ *are Fredholm operators with zero index.*

There hold the following operator equalities

$$\mathcal{L}\mathcal{H} = -\frac{1}{4}\,I_6 + \mathcal{K}^2, \quad \mathcal{H}\mathcal{L} = -\frac{1}{4}\,I_6 + \mathcal{N}^2\,. \tag{3.4}$$

4. Existence and regularity of solutions to mixed BVP for solids with interior crack

If not otherwise stated, throughout this section we assume that

$$1 < p < \infty, \quad q \geq 1, \quad s \in \mathbb{R}.$$

Before we start analysis of the mixed problem we present here existence results for the basic Dirichlet and Neumann boundary value problems. Using Theorem 3.4 and the fact that the null spaces of strongly elliptic pseudodifferential operators acting in Bessel potential $H^s_p(S)$ and Besov $B^s_{p,q}(S)$ spaces actually do not depend on the parameters s, p, and q, we arrive at the following existence results (for details see [31]).

Theorem 4.1. *Let $1 < p < \infty$ and $f \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$. Then the pseudodifferential operator*

$$2^{-1} I_6 + \mathcal{N} \; : \; \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6 \to \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$$

is continuously invertible, the interior Dirichlet BVP (2.52), (2.47)–(2.51) is uniquely solvable in the space $[W^1_p(\Omega)]^6$ and the solution is representable in the form of double layer potential $U = W(h)$ with the density vector function $h \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$ being a unique solution of the equation

$$[\,2^{-1} I_6 + \mathcal{N}\,]\, h = f \quad \text{on} \quad S.$$

Theorem 4.2. (i) *Let a vector function $U \in [W^1_p(\Omega)]^6$, $1 < p < \infty$ solves the homogeneous differential equation $A(\partial, \tau)U = 0$ in Ω. Then it is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^+\big)(x), \quad x \in \Omega,$$

where $\{U\}^+$ is the trace of U on S from Ω and belongs to the space $\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$.

(ii) *Let a vector function $U \in [W^1_{p,loc}(\Omega^-)]^6$, $1 < p < \infty$ satisfy the decay conditions (2.45), and solve the homogeneous differential equation $A(\partial, \tau)U = 0$ in Ω^-. Then it is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^-\big)(x), \quad x \in \Omega^-,$$

where $\{U\}^-$ is the trace of U on S from Ω^- and belongs to the space $\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$.

Theorem 4.3. *Let $1 < p < \infty$ and $F = (F_1, \ldots, F_6)^\top \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6$.*

(i) *The operator*

$$-2^{-1} I_6 + \mathcal{K} \; : \; \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6 \to \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6 \tag{4.1}$$

is an elliptic pseudodifferential operator with zero index and has a two-dimensional null space $\Lambda(S) := \ker(-2^{-1} I_6 + \mathcal{K}) \subset [C^\infty(S)]^6$, which represents a linear span of the vector functions

$$h^{(1)}, \; h^{(2)} \in \Lambda(S),$$

such that

$$V(h^{(1)}) = \Psi^{(1)} := (0, 0, 0, 1, 0, 0)^\top \text{ and } V(h^{(2)}) = \Psi^{(2)} := (0, 0, 0, 0, 1, 0)^\top \text{ in } \quad \Omega. \tag{4.2}$$

(ii) *The null space of the operator adjoint to (4.1),*

$$-2^{-1} I_6 + \mathcal{K}^* \; : \; \big[B^{\frac{1}{p}}_{p',p'}(S)\big]^6 \to \big[B^{\frac{1}{p}}_{p',p'}(S)\big]^6, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

is a linear span of the vectors $(0, 0, 0, 1, 0, 0)^\top$ and $(0, 0, 0, 0, 1, 0)^\top$.

(iii) *The equation*

$$[\,-2^{-1} I_6 + \mathcal{K}\,]\, h = F \text{ on } \; S, \tag{4.3}$$

is solvable if and only if

$$\int_S F_4(x)\,dS = \int_S F_5(x)\,dS = 0. \tag{4.4}$$

(iv) *If the conditions (4.4) hold, then solutions to Eq. (4.3) are defined modulo a linear combination of the vector functions $h^{(1)}$ and $h^{(2)}$.*

(v) *If the conditions (4.4) hold, then the interior Neumann type boundary value problem (2.52), (2.53)–(2.57) is solvable in the space $[W^1_p(\Omega)]^6$, $1 < p < \infty$ and its solution is representable in the form of single layer potential $U = V(h)$, where the density vector function $h \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6$ is defined by Eq. (4.3). A solution to the interior Neumann BVP in Ω is defined modulo a linear combination of the constant vector functions $\Psi^{(1)}$ and $\Psi^{(2)}$ given by (4.2).*

Remark 4.4. If boundary data of Dirichlet and Neumann boundary value problems $(D)^+_{\tau,0}$ and $(N)^+_{\tau,0}$ are sufficiently smooth, then the problems have regular solutions (see [31]).

Now we start investigation of the mixed boundary value problems for solids with interior cracks.

First let us note that the boundary conditions on the crack faces Σ, (2.59) and (2.60), can be transformed equivalently as

$$\{[\mathcal{T}U]_j\}^+ - \{[\mathcal{T}U]_j\}^- = F_j^+ - F_j^- \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma), \qquad j=\overline{1,3},$$

$$\{[\mathcal{T}U]_j\}^+ + \{[\mathcal{T}U]_j\}^- = F_j^+ + F_j^- \in B^{-\frac{1}{p}}_{p,p}(\Sigma), \qquad j=\overline{1,3}.$$

Thus, the boundary conditions (2.59)–(2.68) of the problem under consideration can be rewritten as

$$\{\mathcal{T}U\}^+ = g^{(N)} \quad \text{on } S_N, \tag{4.5}$$

$$\{U\}^+ = g^{(D)} \quad \text{on } S_D, \tag{4.6}$$

$$\{[\mathcal{T}U]_j\}^+ + \{[\mathcal{T}U]_j\}^- = F_j^+ + F_j^- \quad \text{on } \Sigma, \quad j=\overline{1,3}, \tag{4.7}$$

$$\{u_4\}^+ - \{u_4\}^- = f_4 \quad \text{on } \Sigma, \tag{4.8}$$

$$\{u_5\}^+ - \{u_5\}^- = f_5 \quad \text{on } \Sigma, \tag{4.9}$$

$$\{u_6\}^+ - \{u_6\}^- = f_6 \quad \text{on } \Sigma, \tag{4.10}$$

$$\{[\mathcal{T}U]_j\}^+ - \{[\mathcal{T}U]_j\}^- = F_j^+ - F_j^- \quad \text{on } \Sigma, \quad j=\overline{1,3}, \tag{4.11}$$

$$\{[\mathcal{T}U]_4\}^+ - \{[\mathcal{T}U]_4\}^- = F_4 \quad \text{on } \Sigma, \tag{4.12}$$

$$\{[\mathcal{T}U]_5\}^+ - \{[\mathcal{T}U]_5\}^- = F_5 \quad \text{on } \Sigma, \tag{4.13}$$

$$\{[\mathcal{T}U]_6\}^+ - \{[\mathcal{T}U]_6\}^- = F_6 \quad \text{on } \Sigma. \tag{4.14}$$

We look for a solution of the boundary value problem (2.58)–(2.68) in the following form:

$$U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)}) \quad \text{in} \quad \Omega_\Sigma, \tag{4.15}$$

where $\mathcal{H}^{-1}$ is the operator inverse to the integral operator $\mathcal{H}$ defined by (3.2),

$$V_c(h^{(1)})(x) := \int_\Sigma \Gamma(x-y,\tau)\,h^{(1)}(y)\,d_yS,$$

$$W_c(h^{(2)})(x) := \int_\Sigma \big[\mathcal{P}(\partial_y, n(y), \overline{\tau})[\Gamma(x-y,\tau)]^\top\big]^\top\,h^{(2)}(y)\,d_yS,$$

$$V(\mathcal{H}^{-1}h)(x) := \int_S \Gamma(x-y,\tau)\,(\mathcal{H}^{-1}h)(y)\,d_yS,$$

$h^{(i)} = (h_1^{(i)}, \dots, h_6^{(i)})^\top$, $i = 1, 2$, and $h = (h_1, \dots, h_6)^\top$ are unknown densities,

$$h^{(1)} \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)\big]^6, \quad h^{(2)} \in \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)\big]^6, \quad h \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6. \tag{4.16}$$

Due to the above inclusions, clearly, in V_c and W_c we can take the closed surface S_0 as an integration manifold instead of the crack surface Σ. Recall that Σ is assumed to be a proper part of $S_0 = \partial\Omega_0 \subset \Omega$ (see Section 2.3).

The boundary and transmission conditions (4.5)–(4.14) lead to the equations:

$$r_{S_N}\mathcal{A}h + r_{S_N}\big[\mathcal{T}W_c(h^{(2)})\big] + r_{S_N}\big[\mathcal{T}V_c(h^{(1)})\big] = g^{(N)} \quad \text{on } S_N, \tag{4.17}$$

$$r_{S_D}h + r_{S_D}\big[W_c(h^{(2)})\big] + r_{S_D}V_c(h^{(1)}) = g^{(D)} \quad \text{on } S_D, \tag{4.18}$$

$$r_\Sigma\big[\mathcal{T}V\big(\mathcal{H}^{-1}h\big)\big]_j + r_\Sigma\big[\mathcal{L}_c h^{(2)}\big]_j + r_\Sigma\big[\mathcal{K}_c(h^{(1)})\big]_j = 2^{-1}\big(F_j^+ + F_j^-\big), \quad j = 1, 2, 3, \quad \text{on } \Sigma, \tag{4.19}$$

$$h_4^{(2)} = f_4 \quad \text{on } \Sigma, \tag{4.20}$$

$$h_5^{(2)} = f_5 \quad \text{on } \Sigma, \tag{4.21}$$

$$h_6^{(2)} = f_6 \quad \text{on } \Sigma, \tag{4.22}$$

$$h_j^{(1)} = F_j^- - F_j^+, \quad j = 1, 2, 3, \quad \text{on } \Sigma, \tag{4.23}$$

$$h_4^{(1)} = -F_4 \quad \text{on } \Sigma, \tag{4.24}$$

$$h_5^{(1)} = -F_5 \quad \text{on } \Sigma, \tag{4.25}$$

$$h_6^{(1)} = -F_6 \quad \text{on } \Sigma, \tag{4.26}$$

where $\mathcal{A} := \big(-2^{-1}\ I_5 + \mathcal{K}\big)\mathcal{H}^{-1}$ is the Steklov–Poincaré type operator on S, and

$$\mathcal{L}_c(h^{(2)})(z) := \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^+ = \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^- \ \text{ on } \Sigma,$$

$$\mathcal{K}_c(h^{(1)})(z) := \int_\Sigma \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z - y, \tau)\,h^{(1)}(y)\,d_yS \quad \text{ on } \Sigma.$$

As we see the sought for density $h^{(1)}$ and the last two components of the vector $h^{(2)}$ are determined explicitly by the data of the problem. Hence, it remains to find the density h and the first three components of the vector $h^{(2)}$.

The operator generated by the left hand side expressions of the above simultaneous equations, acting upon the unknown vector $(h, h^{(2)}, h^{(1)})$ reads as

$$\mathcal{Q} := \begin{bmatrix} r_{S_N}\mathcal{A} & r_{S_N}\mathcal{T}W_c & r_{S_N}\mathcal{T}V_c \\ r_{S_D}I_6 & r_{S_D}W_c & r_{S_D}V_c \\ r_\Sigma[\mathcal{T}V\big(\mathcal{H}^{-1}\big)]_{3\times 6} & r_\Sigma[\mathcal{L}_c]_{3\times 6} & r_\Sigma[\mathcal{K}_c]_{3\times 6} \\ [\,0\,]_{3\times 6} & r_\Sigma I^*_{3\times 6} & [\,0\,]_{3\times 6} \\ [\,0\,]_{6\times 6} & [\,0\,]_{6\times 6} & r_\Sigma I_6 \end{bmatrix}_{24\times 18},$$

where $[\,M\,]_{3\times 6}$ denotes the first three rows of a 6×6 matrix M, $[\,0\,]_{m\times n}$ stands for the corresponding zero matrix,

$$I^*_{3\times 6} := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}.$$

This operator possesses the following mapping properties

$$\begin{aligned} \mathcal{Q}\ &: [H_p^s(S)]^6 \times [\widetilde{H}_p^s(\Sigma)]^6 \times [\widetilde{H}_p^{s-1}(\Sigma)]^6 \\ &\to [H_p^{s-1}(S_N)]^6 \times [H_p^s(S_D)]^6 \times [H_p^{s-1}(\Sigma)]^3 \times [\widetilde{H}_p^s(\Sigma)]^3 \times [\widetilde{H}_p^{s-1}(\Sigma)]^6, \\ \mathcal{Q}\ &: [B_{p,q}^s(S)]^6 \times [\widetilde{B}_{p,q}^s(\Sigma)]^6 \times [\widetilde{B}_{p,q}^{s-1}(\Sigma)]^6 \\ &\to [B_{p,q}^{s-1}(S_N)]^6 \times [B_{p,q}^s(S_D)]^6 \times [B_{p,q}^{s-1}(\Sigma)]^3 \times [\widetilde{B}_{p,q}^s(\Sigma)]^3 \times [\widetilde{B}_{p,q}^{s-1}(\Sigma)]^6, \\ &1 < p < \infty, \quad 1 \le q \le \infty, \quad s \in \mathbb{R}. \end{aligned} \tag{4.27}$$

Our main goal is to establish invertibility of the operators (4.27).

To this end, by introducing a new additional unknown vector we extend Eq. (4.18) from S_D onto the whole of S. We will do this in the following way. Denote by $g_0^{(D)}$ some fixed extension of $g^{(D)}$ from S_D onto the whole of S preserving the space and introduce a new unknown vector ϕ on S

$$\phi = h + r_s\left[W_c(h^{(2)})\right] + r_S\, V_c(h^{(1)}) - g_0^{(D)}. \tag{4.28}$$

It is evident that $\phi \in \left[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)\right]^6$ in accordance with (4.18), (4.16), (2.69), and the embedding $g_0^{(D)} \in \left[B_{p,p}^{1-\frac{1}{p}}(S)\right]^6$. Moreover, the restriction of Eq. (4.28) on S_D coincide with Eq. (4.18). Therefore, we can replace Eq. (4.18) in the system (4.17)–(4.26) by Eq. (4.28). Finally, we arrive at the following simultaneous equations with respect to unknowns h, ϕ, $h^{(2)}$ and $h^{(1)}$:

$$r_{S_N}\mathcal{A}h + r_{S_N}\left[\mathcal{T}W_c(h^{(2)})\right] + r_{S_N}\left[\mathcal{T}\,V_c(h^{(1)})\right] = g^{(N)} \qquad \text{on } S_N, \tag{4.29}$$

$$h - \phi + r_s\left[W_c(h^{(2)})\right] + r_s\,V_c(h^{(1)}) = g_0^{(D)} \qquad \text{on } S, \tag{4.30}$$

$$r_\Sigma\left[\mathcal{T}V\big(\mathcal{H}^{-1}h\big)\right]_j + r_\Sigma\left[\mathcal{L}_c\,h^{(2)}\right]_j + r_\Sigma\left[\mathcal{K}_c(h^{(1)})\right]_j = 2^{-1}\big(F_j^+ + F_j^-\big), \quad j=1,2,3, \qquad \text{on } \Sigma, \tag{4.31}$$

$$h_4^{(2)} = f_4 \qquad \text{on } \Sigma, \tag{4.32}$$

$$h_5^{(2)} = f_5 \qquad \text{on } \Sigma, \tag{4.33}$$

$$h_6^{(2)} = f_6 \qquad \text{on } \Sigma, \tag{4.34}$$

$$h_j^{(1)} = F_j^- - F_j^+, \quad j=1,2,3, \qquad \text{on } \Sigma, \tag{4.35}$$

$$h_4^{(1)} = -F_4 \qquad \text{on } \Sigma, \tag{4.36}$$

$$h_5^{(1)} = -F_5 \qquad \text{on } \Sigma, \tag{4.37}$$

$$h_6^{(1)} = -F_6 \qquad \text{on } \Sigma. \tag{4.38}$$

In what follows, for the zero vector $g^{(D)} = 0$ on S_D we always choose the fixed extension vector $g_0^{(D)} = 0$ on S.

Rewrite the system (4.29)–(4.38) in the equivalent form

$$\begin{aligned} &r_{S_N}\mathcal{A}\phi + r_{S_N}\mathcal{T}W_c(h^{(2)}) - r_{S_N}\mathcal{A}[r_s\,W_c(h^{(2)})] + r_{S_N}\mathcal{T}\,V_c(h^{(1)}) \\ &\quad - r_{S_N}\mathcal{A}[r_s\,V_c(h^{(1)})] = g^{(N)} - r_{S_N}\mathcal{A}\,g_0^{(D)} \qquad \text{on } S_N, \end{aligned} \tag{4.39}$$

$$-\phi + h + r_s\left[W_c(h^{(2)})\right] + r_{\partial\Omega}\,V_c(h^{(1)}) = g_0^{(D)} \qquad \text{on } S, \tag{4.40}$$

$$r_\Sigma\left[\mathcal{T}V\big(\mathcal{H}^{-1}h\big)\right]_j + r_\Sigma\left[\mathcal{L}_c\,h^{(2)}\right]_j + r_\Sigma\left[\mathcal{K}_c(h^{(1)})\right]_j = 2^{-1}\big(F_j^+ + F_j^-\big), \quad j=1,2,3, \qquad \text{on } \Sigma, \tag{4.41}$$

$$h_4^{(2)} = f_4 \qquad \text{on } \Sigma, \tag{4.42}$$

$$h_5^{(2)} = f_5 \qquad \text{on } \Sigma, \tag{4.43}$$

$$h_6^{(2)} = f_6 \qquad \text{on } \Sigma, \tag{4.44}$$

$$h_j^{(1)} = F_j^- - F_j^+, \quad j=1,2,3, \qquad \text{on } \Sigma, \tag{4.45}$$

$$h_4^{(1)} = -F_4 \qquad \text{on } \Sigma, \tag{4.46}$$

$$h_5^{(1)} = -F_5 \qquad \text{on } \Sigma, \tag{4.47}$$

$$h_6^{(1)} = -F_6 \qquad \text{on } \Sigma. \tag{4.48}$$

Remark 4.5. The systems (4.17)–(4.26) and (4.39)–(4.48) are equivalent in the following sense:

(i) if $(h, h^{(2)}, h^{(1)})^\top$ solves the system (4.17)–(4.26), then $(\phi, h, h^{(2)}, h^{(1)})^\top$ with ϕ given by (4.28) where $g_0^{(D)}$ is some fixed extension of the vector $g^{(D)}$ from S_D onto the whole of S involved in the right hand side of Eq. (4.40), solves the system (4.39)–(4.48);

(ii) if $(\phi, h, h^{(2)}, h^{(1)})^\top$ solves the system (4.39)–(4.48), then $(h, h^{(2)}, h^{(1)})^\top$ solves the system (4.17)–(4.26).

The operator generated by the left hand sides of system (4.39)–(4.48) reads as

$$\mathcal{Q}_1 := \begin{bmatrix} r_{S_N}\,\mathcal{A} & [\,0\,]_{6\times 6} & r_{S_N}\,\mathcal{R}_2 & r_{S_N}\,\mathcal{R}_1 \\ -r_S\, I_6 & r_S\, I_6 & r_S\, W_c & r_S\, V_c \\ [\,0\,]_{3\times 6} & r_\Sigma\,[\mathcal{T}V(\mathcal{H}^{-1})]_{3\times 6} & r_\Sigma\,[\mathcal{L}_c]_{3\times 6} & r_\Sigma\,[\mathcal{K}_c]_{3\times 6} \\ [\,0\,]_{3\times 6} & [\,0\,]_{3\times 6} & r_\Sigma\, I^*_{3\times 6} & [\,0\,]_{3\times 6} \\ [\,0\,]_{6\times 6} & [\,0\,]_{6\times 6} & [\,0\,]_{6\times 6} & r_\Sigma\, I_6 \end{bmatrix}_{24\times 24}, \tag{4.49}$$

where

$$\mathcal{R}_1 = \mathcal{T}\,V_c - \mathcal{A}[r_S\, V_c], \qquad \mathcal{R}_2 = \mathcal{T}\,W_c - \mathcal{A}[r_S\, W_c].$$

Here and in what follows $[\,M\,]_{6\times k}$ with $k < 6$ denotes the first k columns of a 6×6 matrix M, while $[\,M\,]_{k\times 6}$ denotes the first k rows of the same matrix, and $[\,M\,]_{k\times k}$ stands for the upper left $k\times k$ block of M.

This operator possesses the following mapping properties

$$\begin{aligned} \mathcal{Q}_1 :\ & [\widetilde{H}^s_p(S_N)]^6 \times [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^6 \times [\widetilde{H}^{s-1}_p(\Sigma)]^6 \\ & \to [H^{s-1}_p(S_N)]^6 \times [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^3 \times [\widetilde{H}^s_p(\Sigma)]^3 \times [\widetilde{H}^{s-1}_p(\Sigma)]^6, \\ \mathcal{Q}_1 :\ & [\widetilde{B}^s_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [\widetilde{B}^s_{p,q}(\Sigma)]^6 \times [\widetilde{B}^{s-1}_{p,q}(\Sigma)]^6 \\ & \to [B^{s-1}_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [B^{s-1}_{p,q}(\Sigma)]^3 \times [\widetilde{B}^s_{p,q}(\Sigma)]^3 \times [\widetilde{B}^{s-1}_{p,q}(\Sigma)]^6, \\ & 1 < p < \infty, \quad 1 \le q \le \infty, \quad s \in \mathbb{R}. \end{aligned} \tag{4.50}$$

Due to the above agreement about the extension of the zero vector we see that if the right hand side functions of the system (4.17)–(4.26) vanish then the same holds for the system (4.39)–(4.48) and vice versa.

The uniqueness Theorem 2.1 and properties of the single and double layer potentials imply the following assertion.

Lemma 4.6. *The null spaces of the operators $\mathcal{Q}$ and $\mathcal{Q}_1$ are trivial for $s = 1/2$ and $p = 2$.*

Now we start to analyse Fredholm properties of the operator $\mathcal{Q}_1$.

From the structure of the operator $\mathcal{Q}_1$ it is evident that we need only to study Fredholm properties of the operator generated by the upper left 15×15 block of the matrix operator (4.49),

$$\mathcal{M} := \begin{bmatrix} r_{S_N}\,\mathcal{A} & [\,0\,]_{6\times 6} & r_{S_N}\,[\mathcal{R}_2]_{6\times 3} \\ -r_S\, I_6 & r_S\, I_6 & r_S\,[\,W_c\,]_{6\times 3} \\ [\,0\,]_{3\times 6} & r_\Sigma\,[\mathcal{T}V(\mathcal{H}^{-1})]_{3\times 6} & r_\Sigma\,[\,\mathcal{L}_c\,]_{3\times 3} \end{bmatrix}_{15\times 15}.$$

This operator has the following mapping properties:

$$\begin{aligned} \mathcal{M} :\ & [\widetilde{H}^s_p(S_N)]^6 \times [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^3 \\ & \to [H^{s-1}_p(S_N)]^6 \times [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^3, \\ \mathcal{M} :\ & [\widetilde{B}^s_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [\widetilde{B}^s_{p,q}(\Sigma)]^3 \\ & \to [B^{s-1}_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [B^{s-1}_{p,q}(\Sigma)]^3, \\ & 1 < p < \infty, \quad 1 \le q \le \infty, \quad s \in \mathbb{R}. \end{aligned} \tag{4.51}$$

For the principal part $\mathcal{M}_0$ of the operator $\mathcal{M}$ we have

$$\mathcal{M}_0 := \begin{bmatrix} r_{S_N}\,\mathcal{A} & [\,0\,]_{6\times 6} & [\,0\,]_{6\times 3} \\ -r_S\, I_6 & r_S\, I_6 & [\,0\,]_{6\times 3} \\ [\,0\,]_{3\times 6} & [\,0\,]_{3\times 6} & r_\Sigma\,\mathcal{L}^{(1)} \end{bmatrix}_{15\times 15}, \tag{4.52}$$

where

$$\mathcal{L}^{(1)} := \|[\mathcal{L}_c]_{kj}\|_{3\times 3}, \quad \mathcal{L}_c = \|[\mathcal{L}_c]_{kl}\|_{6\times 6}. \tag{4.53}$$

Clearly, the operator $\mathcal{M}_0$ has the same mapping properties as $\mathcal{M}$ and the difference $\mathcal{M}-\mathcal{M}_0$ is compact. Actually, $\mathcal{M}-\mathcal{M}_0$ is an infinitely smoothing operator.

The operators $\mathcal{L}_c$ and $\mathcal{A}$ are strongly elliptic pseudodifferential operators of order 1 (see [31]). From (4.53) we get then that $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator as well. Moreover, we have the following invertibility results.

Theorem 4.7. *Let $1 < p < \infty$, $1 \le q \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$. Then the operators*

$$r_\Sigma\,\mathcal{L}^{(1)}\ :\ \big[\widetilde{H}^s_p(\Sigma)\big]^3 \to \big[H^{s-1}_p(\Sigma)\big]^3, \quad r_\Sigma\,\mathcal{L}^{(1)}\ :\ \big[\widetilde{B}^s_{p,q}(\Sigma)\big]^3 \to \big[B^{s-1}_{p,q}(\Sigma)\big]^3 \tag{4.54}$$

are invertible.

Proof. With the help of the first equality in (3.4) we derive that the principal homogeneous symbol matrix of the strongly elliptic pseudodifferential operator $\mathcal{L}_c$ reads as

$$\begin{aligned}\mathfrak{S}(\mathcal{L}_c; x, \xi) = \mathfrak{S}(\mathcal{L}_{S_0}; x, \xi) &:= \Big(-\frac{1}{4}\,I_6 + \mathfrak{S}^2(\mathcal{K}_{S_0}; x, \xi)\Big)\,[\,\mathfrak{S}(\mathcal{H}_{S_0}; x, \xi)\,]^{-1}\\ &= \Big(-\frac{1}{4}\,I_6 + \mathfrak{S}^2(\mathcal{K}_c; x, \xi)\Big)\,[\,\mathfrak{S}(\mathcal{H}_c; x, \xi)\,]^{-1}, \quad x \in \overline{\Sigma}, \quad \xi \in \mathbb{R}^2 \setminus \{0\},\end{aligned}$$

where $\mathcal{H}_{S_0}$ and $\mathcal{K}_{S_0}$ are integral operators given by (3.2) and (3.3) with S_0 for S.

One can show that the principal homogeneous symbol matrix of the operator $\mathcal{K}_c$ is an odd matrix function in ξ, whereas the principal homogeneous symbol matrix of the operator $\mathcal{H}_c$ is an even matrix function in ξ. Consequently, the matrix $\mathfrak{S}(\mathcal{L}_c; x, \xi)$ is even in ξ (for details see [31]).

From these results it follows that $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator with even principal homogeneous symbol. Therefore the matrix

$$[\,\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)\,]^{-1}\ \mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1)$$

is the unit matrix and the corresponding eigenvalues equal to 1 (see Appendix A). Now, from Theorem A in Appendix A it follows that the operators (4.54) are Fredholm with zero index for $1 < p < \infty$, $1 \le q \le \infty$ and $1/p - 1/2 < s < 1/p + 1/2$. It remains to show that the corresponding null spaces are trivial. In turn, due to the same Theorem A (see Appendix A), it suffices to establish that the operator

$$r_\Sigma\,\mathcal{L}^{(1)}\ :\ \big[\widetilde{H}^{\frac12}_2(\Sigma)\big]^3 \to \big[H^{-\frac12}_2(\Sigma)\big]^3$$

is injective, i.e, we have to prove that the homogeneous equation

$$r_\Sigma\,\mathcal{L}^{(1)}\,\chi = 0 \quad \text{on} \quad \Sigma \tag{4.55}$$

possesses only the trivial solution in the space $\big[\widetilde{H}^{\frac12}_2(\Sigma)\big]^3$.

Let $\chi \in \big[\widetilde{H}^{\frac12}_2(\Sigma)\big]^3$ solve Eq. (4.55) and construct the double layer potential

$$U = (u_1, \dots, u_6)^\top = W_c(\widetilde{\chi}), \quad \widetilde{\chi} = (\chi, 0, 0, 0)^\top\,.$$

In view of properties of the double layer potential and Eq. (4.55), it can easily be verified that the vector $U \in \big[W^1_2(\mathbb{R}^3 \setminus \overline{\Sigma})\big]^6$ is a solution to the following crack type boundary transmission problem:

$$\begin{aligned}
&A(\partial_x, \tau)\,U = 0 && \text{in } \ \mathbb{R}^3 \setminus \overline{\Sigma},\\
&\big\{[\,\mathcal{T}\,U\,]_j\big\}^+ = \big\{[\,\mathcal{T}\,U\,]_j\big\}^- = 0, \quad j = 1, 2, 3, && \text{on } \ \Sigma,\\
&\big\{u_k\big\}^+ - \big\{u_k\big\}^- = 0, \quad k = 4, 5, 6, && \text{on } \ \Sigma,\\
&\big\{[\,\mathcal{T}\,U\,]_k\big\}^+ - \big\{[\,\mathcal{T}\,U\,]_k\big\}^- = 0, \quad k = 4, 5, 6, && \text{on } \ \Sigma
\end{aligned}$$

and satisfy the decay conditions (2.45) at infinity, i.e., $U \in \mathbf{Z}(\mathbb{R}^3 \setminus \overline{\Sigma})$.

Applying Green's identity (2.70) by standard arguments we arrive at the equality $U = 0$ in $\mathbb{R}^3 \setminus \overline{\Sigma}$. Whence $\chi = (\chi_1, \chi_2, \chi_3)^\top = 0$ on Σ follows due to the equalities $\big\{u_j\big\}^+ - \big\{u_j\big\}^- = \chi_j$ on Σ, $j = 1, 2, 3$. This completes the proof. □

Let λ_k, $k = \overline{1,6}$, be the eigenvalues of the matrix

$$a_0(x) := [\,\mathfrak{S}(\mathcal{A}; x, 0, +1)\,]^{-1}\,\mathfrak{S}(\mathcal{A}; x, 0, -1), \quad x \in \ell_m,$$

where $\mathfrak{S}(\mathcal{A}; x, \xi)$ with $x \in \overline{S}_N$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$.

We can show that $\lambda = 1$ is an eigenvalue of the matrix $a_0(x)$. It follows from the following technical lemma.

Lemma 4.8. *Let* $\mathbf{Q}$ *be the set of all non-singular* $k \times k$ *square matrices with complex-valued entries and having the structure*

$$\begin{bmatrix} [Q_{lj}]_{(k-1)\times(k-1)} & \{0\}_{(k-1)\times 1} \\ \{0\}_{1\times(k-1)} & Q_{kk} \end{bmatrix}_{k\times k}, \qquad k \in \mathbb{N}.$$

If $X, Y \in \mathbf{Q}$, *then* $XY \in \mathbf{Q}$ *and* $X^{-1} \in \mathbf{Q}$. *Moreover, if in addition* $X = [X_{jl}]_{k\times k}$ *and* $Y = [Y_{jl}]_{k\times k}$ *are strongly elliptic, i.e.*

$$\mathrm{Re}\,(X\zeta\dot{\zeta}) > 0, \quad \mathrm{Re}\,(Y\zeta\dot{\zeta}) > 0 \quad \textit{for all} \quad \zeta \in \mathbb{C}^k \setminus \{0\},$$

and X_{kk} *and* Y_{kk} *are real numbers, then* $\lambda = X_{kk}Y_{kk} > 0$ *is an eigenvalue of the matrix* XY.

In particular if $X_{kk} = Y_{kk}^{-1}$, *then* $\lambda = 1$ *is an eigenvalue of the matrix* XY.

Let us introduce the notation

$$\delta' = \inf_{\substack{1\le j\le 6 \\ x\in\ell_m}} \frac{1}{2\pi}\arg\lambda_j(x), \quad \delta'' = \sup_{\substack{1\le j\le 6 \\ x\in\ell_m}} \frac{1}{2\pi}\arg\lambda_j(x). \tag{4.56}$$

Due to strong ellipticity of the operator $\mathcal{A}$ and since one eigenvalue, say λ_6 equals 1, we easily derive that

$$-\frac{1}{2} < \delta' \le 0 \le \delta'' < \frac{1}{2}.$$

Applying again Theorem A in Appendix A, we get (see [31], Lemma 5.20).

Theorem 4.9. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $1/p - 1/2 + \delta'' < s < 1/p + 1/2 + \delta'$ *with* δ' *and* δ'' *given by* (4.56). *Then the Steklov–Poincaré operators*

$$r_{S_N}\mathcal{A} : \big[\widetilde{H}_p^s(S_N)\big]^6 \to \big[H_p^{s-1}(S_N)\big]^6,$$
$$r_{S_N}\mathcal{A} : \big[\widetilde{B}_{p,q}^s(S_N)\big]^6 \to \big[B_{p,q}^{s-1}(S_N)\big]^6,$$

are invertible.

These assertions imply

Theorem 4.10. *Let*

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad 1/p - 1/2 + \delta'' < s < 1/p + 1/2 + \delta' \tag{4.57}$$

with δ' *and* δ'' *given by* (4.56). *Then the operators* (4.51) *are Fredholm with index 0.*

Proof. From Theorems 4.7 and 4.9 we conclude that for arbitrary p, q and s satisfying the conditions (4.57), the operators

$$\begin{aligned} \mathcal{M}_0 &: [\widetilde{H}_p^s(S_N)]^6 \times [H_p^s(S)]^6 \times [\widetilde{H}_p^s(\Sigma)]^3 \\ &\quad \to [H_p^{s-1}(S_N)]^6 \times [H_p^s(S)]^6 \times [H_p^{s-1}(\Sigma)]^3, \\ \mathcal{M}_0 &: [\widetilde{B}_{p,q}^s(S_N)]^6 \times [B_{p,q}^s(S)]^6 \times [\widetilde{B}_{p,q}^s(\Sigma)]^3 \\ &\quad \to [B_{p,q}^{s-1}(S_N)]^6 \times [B_{p,q}^s(S)]^6 \times [B_{p,q}^{s-1}(\Sigma)]^3, \end{aligned}$$

with $\mathcal{M}_0$ defined in (4.52) are invertible. Therefore the operators (4.51) are Fredholm operators with index 0. □

Now we are in a position to prove the invertibility of the operator $\mathcal{Q}_1$.

Theorem 4.11. *Let conditions (4.57) be satisfied. Then the operators (4.50) are invertible.*

Proof. From Theorem 4.10 it follows that the operator $\mathcal{Q}_1$ is Fredholm with index zero if (4.57) holds. By Lemma 4.6 we conclude then that for $s = 1/2$ and $p = 2$ it is invertible. The null-spaces and indices of the operators (4.50) are the same for all values of the parameter $q \in [1, +\infty]$, provided p and s satisfy the inequalities (4.57) (see [36, Ch. 3., Proposition 10.6]). Therefore, for these values of the parameters p and s they are invertible. In particular, the nonhomogeneous system (4.39)–(4.48) is uniquely solvable in the corresponding spaces. Moreover, it can be easily shown that the solution vectors $h, h^{(2)}, h^{(1)}$ do not depend on the extension of the vector $g^{(D)}$, while ϕ does. However, the sum $\phi + g_0^{(D)}$ is defined uniquely. $\square$

Due to Remark 4.5 we conclude that the operators (4.27) are invertible if p, q and s satisfy the conditions (4.57).

With the help of this theorem we arrive at the following existence result for the original mixed BVP.

Theorem 4.12. *Let*

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'} \,. \tag{4.58}$$

with δ' and δ'' given by (4.56). Then the BVP (2.58)–(2.68) has a unique solution U in the space $\big[W_p^1(\Omega_\Sigma)\big]^6$, which can be represented as $U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)})$ in Ω_Σ, where h, $h^{(2)}$ and $h^{(1)}$ are defined by the system (4.17)–(4.25).

Proof. The condition (4.58) follows from the inequality (4.57) with $s = 1 - 1/p$. Now existence of a solution $U \in [W_p^1(\Omega_\Sigma)]^6$ with p satisfying (4.58) follows from Theorem 4.6. Due to the inequalities $-\frac{1}{2} < \delta' \le \delta'' < \frac{1}{2}$ we have $p = 2 \in \left(\frac{4}{3-2\delta''}\,,\ \frac{4}{1-2\delta'}\right)$. Therefore the unique solvability for $p = 2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (4.58) we proceed as follows. Let a vector $U \in [W_p^1(\Omega_\Sigma)]^6$ with p satisfying (4.58) be a solution to the homogeneous boundary value problem (2.58)–(2.68).

Then, it is evident that

$$\begin{aligned}
&\{U\}_S^+ \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, && \{\mathcal{T}U\}_S^+ \in [B_{p,p}^{-\frac{1}{p}}(S)]^6,\\
&\{U\}_\Sigma^\pm \in [B_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6, && \{\mathcal{T}U\}_\Sigma^\pm \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6,\\
&\{U\}_\Sigma^+ - \{U\}_\Sigma^- \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6, && \{\mathcal{T}U\}_\Sigma^+ - \{\mathcal{T}U\}_\Sigma^- = 0 \ \text{ on } \ \Sigma.
\end{aligned}$$

By the general integral representation formula the vector U can be represented in Ω_Σ as

$$U = W_c(\{U\}_\Sigma^+ - \{U\}_\Sigma^-) - V_c(\{\mathcal{T}U\}_\Sigma^+ - \{\mathcal{T}U\}_\Sigma^-) + W(\{U\}_S^+) - V(\{\mathcal{T}U\}_S^+)\,,$$

i.e.,

$$U = U^* + W_c(h^{(2)}) + V_c(h^{(1)}) \quad \text{in} \quad \Omega_\Sigma, \tag{4.59}$$

where

$$h^{(1)} = \{\mathcal{T}U\}_\Sigma^+ - \{\mathcal{T}U\}_\Sigma^-, \quad h^{(2)} := \{U\}_\Sigma^+ - \{U\}_\Sigma^- \quad \text{on} \quad \Sigma,$$

$$U^* := W(\{U\}_S^+) - V(\{\mathcal{T}U\}_S^+) \in \left[W_p^1(\Omega)\right]^6.$$

Note that U^* solves the homogeneous equation

$$A(\partial, \tau)U^* = 0 \ \text{ in } \ \Omega.$$

Denote $h := \{U^*\}_S^+$. Clearly, $h \in \left[B_{p,p}^{1-1/p}(S)\right]^6$. Since the Dirichlet problem possesses a unique solution in the space $[W_p^1(\Omega)]^6$ for arbitrary $p \in [1, +\infty)$, we can represent U^* uniquely in the form of a single layer potential, $U^* = V(\mathcal{H}^{-1}h)$ in Ω (for details see [31, Ch. 5, Section 5.6]). Therefore from (4.59) we get

$$U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)}) \quad \text{in} \quad \Omega_\Sigma.$$

Now, the homogeneous boundary and transmission conditions for U lead to the homogeneous system (cf. (4.17)–(4.25)) $\mathcal{Q}\Psi = 0$, where $\Psi = (h, h^{(2)}, h^{(1)})^\top$. Whence, $\Psi = 0$ follows immediately due to invertibility of $\mathcal{Q}$ (see Theorem 4.11). Consequently, $U = 0$ in Ω_Σ. □

Let us now present some regularity results for solutions of the mixed boundary value problem (2.58)–(2.68).

Theorem 4.13. *Let* $1 < t < \infty$, $1 \le q \le \infty$,

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'}, \quad \frac{1}{t} - \frac{1}{2} + \delta'' < s < \frac{1}{t} + \frac{1}{2} + \delta',$$

with δ' *and* δ'' *given by* (4.56), *and let* $U \in [W^1_p(\Omega_\Sigma)]^6$ *be the solution of the boundary value problem* (2.58)–(2.68). *Then the following regularity results hold:*

(i) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,t}(\Sigma), \quad F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,t}(\Sigma), \quad j = 1,2,3, \quad F_k \in \widetilde{B}^{s-1}_{t,t}(\Sigma),$$
$$f_k \in \widetilde{B}^{s}_{t,t}(\Sigma), \quad k = 4,5,6, \quad g^{(D)} \in \big[B^s_{t,t}(S_D)\big]^6, \quad g^{(N)} \in \big[B^{s-1}_{t,t}(S_N)\big]^6,$$

then $U \in \big[H_t^{s+\frac{1}{t}}(\Omega_\Sigma)\big]^6$;

(ii) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,q}(\Sigma), \quad F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,q}(\Sigma), \quad j = 1,2,3, \quad F_k \in \widetilde{B}^{s-1}_{t,q}(\Sigma),$$
$$f_k \in \widetilde{B}^{s}_{t,q}(\Sigma), \quad k = 4,5,6, \quad g^{(D)} \in \big[B^s_{t,q}(S_D)\big]^6, \quad g^{(N)} \in \big[B^{s-1}_{t,q}(S_N)\big]^6,$$

then $U \in \big[B_{t,q}^{s+\frac{1}{t}}(\Omega_\Sigma)\big]^6$;

(iii) *If* $\alpha > 0$ *and*

$$F_j^+, F_j^- \in B^{\alpha-1}_{\infty,\infty}(\Sigma), \quad F_j^+ - F_j^- \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \quad j = 1,2,3,$$
$$F_k \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \quad f_k \in C^\alpha(\overline{\Sigma}), \quad r_{\ell_c} f_k = 0, \quad k = 4,5,6,$$
$$g^{(D)} \in \big[C^\alpha(\overline{S}_D)\big]^6, \quad g^{(N)} \in \big[B^{\alpha-1}_{\infty,\infty}(S_N)\big]^6,$$

then

$$U \in \bigcap_{\alpha'<\gamma} C^{\alpha'}(\overline{\Omega}_j), \quad j = 0,1,$$

where $\gamma = \min\{\alpha,\ 1/2 + \delta'\}$, $-1/2 < \delta' \le 0$ *and* Ω_0 *is an arbitrary proper subdomain of* Ω *such that* $\Sigma \subset \partial\Omega_0 = S_0 \in C^\infty$ *and* $\Omega_1 = \Omega \setminus \overline{\Omega}_0$.

Moreover, in one-sided interior and exterior neighbourhoods of the surface S_0 *the vector* U *has* $C^{\gamma'-\varepsilon}$*-smoothness with* $\gamma' = \min\{\alpha,\ 1/2\}$*, while in a one-sided interior neighbourhood of the surface* S *the vector* U *possesses* $C^{\gamma''-\varepsilon}$*-smoothness with* $\gamma'' = \min\{\alpha,\ 1/2 + \delta'\}$*; here* ε *is an arbitrarily small positive number.*

Proof. The proof is exactly the same as that of Theorem 5.22 in [31]. □

Remark 4.14. Theorem 4.13 describes global smoothness properties of solutions. Below, in Section 6.1, with the help of the asymptotic analysis, we will show that actually in a neighbourhood of the crack edge ℓ_c the functions u, φ and ψ have $C^{1/2}$ regularity while the temperature function ϑ possesses $C^{3/2}$ smoothness.

5. Asymptotic expansion of solutions

Here we investigate the asymptotic behaviour of solutions to the problem (2.58)–(2.68) near the exceptional curves ℓ_c and ℓ_m. For simplicity of description of the method applied below, we assume that the boundary data of the problem are infinitely smooth,

$$F_j^+,\ F_j^- \in C^\infty(\overline{\Sigma}), \quad F_j^+ - F_j^- \in C_0^\infty(\overline{\Sigma}), \quad j = 1,2,3, \quad f_k,\ F_k \in C_0^\infty(\overline{\Sigma}), \quad k = 4,5,6,$$
$$g^{(D)} \in \big[C^\infty(\overline{S}_D)\big]^6, \quad g^{(N)} \in \big[C^\infty(\overline{S}_N)\big]^6,$$

where $C_0^\infty(\overline{\Sigma})$ denotes a space of functions vanishing along with all tangential (to Σ) derivatives on $\ell_c = \partial\Sigma$.

In Section 4, we have shown that the boundary value problem (2.58)–(2.68) is uniquely solvable and the solution U can be represented by (4.15), where the densities are defined by Eqs. (4.17)–(4.26) or by the equivalent system (4.39)–(4.48).

Let $\Phi := (\phi, h, h^{(2)}, h^{(1)})^\top$ be a solution of the system (4.39)–(4.48):

$$\mathcal{Q}_1\, \Phi = G,$$

where G is the vector constructed by the right hand sides of the system,

$$G \in [C^\infty(\overline{S}_N)]^6 \times [C^\infty(S)]^6 \times [C^\infty(\overline{\Sigma})]^3 \times [C_0^\infty(\overline{\Sigma})]^9.$$

To establish the asymptotic behaviour of the vector U near the curves ℓ_c and ℓ_m, we rewrite (4.15) as follows

$$U = V\left(\mathcal{H}^{-1}\,\phi\right) + W_c\,(\widetilde{\chi}) + \mathcal{R}, \tag{5.1}$$

where

$$\mathcal{R} := -V\Big(\mathcal{H}^{-1}\Big[r_S\, W_c(h^{(2)}) + r_S\, V_c(h^{(1)}) - g_0^{(D)}\Big]\Big) + W_c(f_0) + V_c(h^{(1)}),$$

with $f_0 = (0, 0, 0, f_4, f_5, f_6)^\top$. Note that $r_{\overline{\Omega}_j}\,\mathcal{R} \in \big[C^\infty(\overline{\Omega}_j)\big]^6$, where Ω_j, $j = 0, 1$, are as in Theorem 4.13, item (iii), since

$$r_S\, W_c(h^{(2)}) + r_S\, V_c(h^{(1)}) - g_0^{(D)} \in [C^\infty(S)]^6,$$
$$h^{(1)} = \big(F_1^- - F_1^+, F_2^- - F_2^+, F_3^- - F_3^+, -F_4, -F_5, -F_6\big) \in [C_0^\infty(\overline{\Sigma})]^6,$$
$$h_4^{(2)} = f_4 \in C_0^\infty(\overline{\Sigma}), \quad h_5^{(2)} = f_5 \in C_0^\infty(\overline{\Sigma}), \quad h_6^{(2)} = f_6 \in C_0^\infty(\overline{\Sigma}).$$

Further, the vector $\widetilde{\chi}$ involved in (5.1) is defined as follows: $\widetilde{\chi} = (\chi, 0, 0, 0)^\top$, where $\chi = (\chi_1, \chi_2, \chi_3)^\top \equiv (h_1^{(2)}, h_2^{(2)}, h_3^{(2)})^\top$, and χ solves the pseudodifferential equation

$$r_\Sigma \mathcal{L}^{(1)}\,\chi = \Psi^{(1)} \quad \text{on} \quad \Sigma \tag{5.2}$$

with $\Psi^{(1)} = (\Psi_1^{(1)}, \Psi_2^{(1)}, \Psi_3^{(1)})^\top$. Evidently,

$$\Psi_j^{(1)} = 2^{-1}\,(F_j^+ + F_j^-) - r_\Sigma\,[T\,V\big(\mathcal{H}^{-1}\,h\big)]_j - r_\Sigma\,[\mathcal{K}_c(h^{(1)})]_j \in C^\infty(\overline{\Sigma}), \qquad j = 1, 2, 3.$$

Finally, the vector ϕ involved in (5.1) solves the pseudodifferential equation

$$r_{S_N}\,\mathcal{A}\,\phi = \Psi^{(2)} \quad \text{on} \quad S_N, \tag{5.3}$$

where

$$\Psi^{(2)} = g^{(N)} - r_{S_N}\,\mathcal{A}\,g_0^{(D)} - r_{S_N}\,T\,W_c(h^{(2)}) + r_{S_N}\,\mathcal{A}\,[r_S\,W_c(h^{(2)})]$$
$$\qquad - r_{S_N}\,T\,V_c(h^{(1)}) + r_{S_N}\,\mathcal{A}[r_S\,V_c(h^{(1)})] \in [C^\infty(\overline{S}_N)]^6\,.$$

The principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)}; x, \xi)$, $x \in \overline{\Sigma}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even with respect to the variable ξ and, therefore, the matrix

$$\big[\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)\big]^{-1}\,\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1), \quad x \in \ell_c,$$

is the unit matrix I_3. Consequently, all eigenvalues of this matrix equal to one,

$$\lambda_j(x) = 1, \quad j = \overline{1, 3}, \quad x \in \ell_c.$$

Applying a partition of unity, natural local co-ordinate systems and local diffeomorphisms, we can rectify ℓ_c and Σ locally in a standard way. For simplicity, let us denote the local rectified images of ℓ_c and Σ under this diffeomorphisms by the same symbols. Then we identify a one-sided neighbourhood (in Σ) of an arbitrary point $\widetilde{x} \in \ell_c$ as a part of the half-plane $x_2 > 0$. Thus, we assume that $(x_1, 0) \in \ell_c$ and $(x_1, x_{2,+}) \in \Sigma$ for $0 < x_{2,+} < \varepsilon$;. Clearly, $x_{2,+} = \operatorname{dist}(x, \ell_c)$.

Applying the results obtained in Refs. [14] and [37] we can derive the following asymptotic expansion for the solution χ of the strongly elliptic pseudodifferential equation (5.2),

$$\chi(x_1, x_{2,+}) = c_0(x_1)\, x_{2,+}^{\frac{1}{2}} + \sum_{k=1}^{M} c_k(x_1)\, x_{2,+}^{\frac{1}{2}+k} + \chi_{M+1}(x_1, x_{2,+}), \tag{5.4}$$

where M is an arbitrary natural number, $c_k \in \left[C^\infty(\ell_c)\right]^3$, $k = 0, 1, \ldots, M$, and the remainder term satisfies the inclusion

$$\chi_{M+1} \in \left[C^{M+1}(\ell^+_{c,\varepsilon})\right]^3, \quad \ell^+_{c,\varepsilon} = \ell_c \times [0, \varepsilon].$$

Note that, according to [37], the terms in the expansion (5.4) do not contain logarithms, since the principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)}; x, \xi)$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even in ξ.

To derive analogous asymptotic expansion for the solution vector ϕ of Eq. (5.3), we apply the same local technique as above to a one-sided neighbourhood (in S_N) of the curve ℓ_m and preserve the same notation for the local co-ordinates.

Consider a 6×6 matrix $a_0(x_1)$ constructed by the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$,

$$a_0(x_1) := \left[\mathfrak{S}(\mathcal{A}; x_1, 0, +1)\right]^{-1} \mathfrak{S}(\mathcal{A}; x_1, 0, -1), \quad (x_1, 0) \in \ell_m. \tag{5.5}$$

Note that unlike to the above considered case, now (5.5) is not the unit matrix and therefore we proceed as follows.

Denote by $\lambda_1(x_1), \ldots, \lambda_6(x_1)$ the eigenvalues of the matrix a_0. Denote by μ_j, $j = 1, \ldots, l$, $1 \le l \le 6$, the distinct eigenvalues and by m_j their algebraic multiplicities: $m_1 + \cdots + m_l = 6$. It is well known that the matrix $a_0(x_1)$ admits the following decomposition (see, e.g., [38], Chapter 7, Section 7)

$$a_0(x_1) = \mathcal{D}(x_1)\, \mathcal{J}_{a_0}(x_1)\, \mathcal{D}^{-1}(x_1), \quad (x_1, 0) \in \ell_m,$$

where $\mathcal{D}$ is 6×6 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{a_0}$ has a block diagonal structure

$$\mathcal{J}_{a_0}(x_1) := \operatorname{diag}\{\mu_1(x_1) B^{(m_1)}(1), \ldots, \mu_l(x_1) B^{(m_l)}(1)\}.$$

Here $B^{(\nu)}(t)$, $\nu \in \{m_1, \ldots, m_l\}$, are upper triangular matrices:

$$B^{(\nu)}(t) = \|b^{(\nu)}_{jk}(t)\|_{\nu\times\nu}, \qquad b^{(\nu)}_{jk}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}, & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases}$$

i.e.,

$$B^{(\nu)}(t) = \begin{pmatrix} 1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{\nu-2}}{(\nu-2)!} & \dfrac{t^{\nu-1}}{(\nu-1)!} \\ 0 & 1 & t & \cdots & \dfrac{t^{\nu-3}}{(\nu-3)!} & \dfrac{t^{\nu-2}}{(\nu-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{\nu\times\nu}.$$

Denote

$$B_0(t) := \operatorname{diag}\{B^{(m_1)}(t), \ldots, B^{(m_l)}(t)\}.$$

Again, applying the results from Ref. [14] we derive the following asymptotic expansion for the solution ϕ of the strongly elliptic pseudodifferential equation (5.3)

$$\phi(x_1, x_{2,+}) = \mathcal{D}(x_1)\, x_{2,+}^{\frac{1}{2}+\Delta(x_1)}\, B_0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\, \mathcal{D}^{-1}(x_1)\, b_0(x_1)$$
$$+ \sum_{k=1}^{M} \mathcal{D}(x_1)\, x_{2,+}^{\frac{1}{2}+\Delta(x_1)+k}\, B_k\big(x_1, \log x_{2,+}\big) + \phi_{M+1}(x_1, x_{2,+}), \tag{5.6}$$

where $b_0 \in \big[C^\infty(\ell_m)\big]^6$, $\phi_{M+1} \in \big[C^{M+1}(\ell^+_{m,\varepsilon})\big]^6$, $\ell^+_{m,\varepsilon} = \ell_m \times [0, \varepsilon]$, and

$$B_k(x_1, t) = B_0\Big(-\frac{t}{2\pi i}\Big) \sum_{j=1}^{k(2m_0-1)} t^j\, d_{kj}(x_1).$$

Here $m_0 = \max\{m_1, \ldots, m_6\}$, the coefficients $d_{kj} \in \big[C^\infty(\ell_m)\big]^6$ and

$$\Delta := (\Delta_1, \ldots, \Delta_6),$$
$$\Delta_j(x_1) = \frac{1}{2\pi\, i}\log\lambda_j(x_1) = \frac{1}{2\pi}\arg\lambda_j(x_1) + \frac{1}{2\pi\, i}\log|\lambda_j(x_1)|,$$
$$-\pi < \arg\lambda_j(x_1) < \pi, \quad (x_1, 0) \in \ell_m, \quad j = \overline{1, 6}.$$

Furthermore,

$$x_{2,+}^{\frac{1}{2}+\Delta(x_1)} := \operatorname{diag}\Big\{x_{2,+}^{\frac{1}{2}+\Delta_1(x_1)}, \ldots, x_{2,+}^{\frac{1}{2}+\Delta_6(x_1)}\Big\}.$$

Now, having at hand the formulae (5.4) and (5.6) with the help of the asymptotic expansion of potential-type functions obtained in [15] we can write the following spatial asymptotic expansions for the solution vector U of the boundary value problem (2.58)–(2.68) near the crack edge ℓ_c and near the collision curve ℓ_m.

(a) *Asymptotic expansion near the crack edge* ℓ_c:

$$U(x) = \sum_{\mu=\pm 1}\Big[\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j\, z_{s,\mu}^{\frac{1}{2}-j}\, d_{sj}^{(c)}(x_1, \mu)$$
$$+ \sum_{\substack{k,l=0\\ k+l+j+p\ge 1}}^{M+2}\ \sum_{j+p=0}^{M+2-l} x_2^l\, x_3^j\, z_{s,\mu}^{\frac{1}{2}+p+k}\, d_{slkjp}^{(c)}(x_1, \mu)\Big] + U_{M+1}^{(c)}(x) \tag{5.7}$$

with the coefficients

$$d_{sj}^{(c)}(\cdot\,, \mu),\ d_{slkjp}^{(c)}(\cdot\,, \mu) \in \big[C^\infty(\ell_c)\big]^6 \quad\text{and}\quad U_{M+1}^{(c)} \in \big[C^{M+1}(\overline{\Omega}_j)\big]^6, \quad j = 0, 1.$$

Here Ω_j, $j = 0, 1$, are as in Theorem 4.13(iii), and

$$z_{s,+1} = -(x_2 + x_3\zeta_{s,+1}), \quad z_{s,-1} = x_2 - x_3\zeta_{s,-1},$$
$$-\pi < \arg z_{s,\pm 1} < \pi, \qquad \zeta_{s,\pm 1} \in C^\infty(\ell_c), \tag{5.8}$$

where $\{\zeta_{s,\pm 1}\}_{s=1}^{l_0}$ are the different roots in ζ of multiplicity n_s, $s = 1, \ldots, l_0$, of the polynomial $\det A^{(0)}\Big(\big[J_\varkappa^\top(x_1, 0, 0)\big]^{-1}\eta_\pm\Big)$ with $\eta_\pm = (0, \pm 1, \zeta)^\top$, satisfying the condition $\operatorname{Re}\zeta_{s,\pm 1} < 0$. The matrix $J_\varkappa$ stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa$ related to the local co-ordinate system. Under this diffeomorphism ℓ_c and Σ are locally rectified and we assume that $(x_1, 0, 0) \in \ell_c$, $x_2 = \operatorname{dist}(x^{(\Sigma)}, \ell_c)$, $x_3 = \operatorname{dist}(x, \Sigma)$, where $x^{(\Sigma)}$ is the projection of the reference point $x \in \Omega_\Sigma$ onto the plane corresponding to the image of Σ under the diffeomorphism $\varkappa$.

Note that the coefficients $d_{sj}^{(c)}(\cdot\,, \mu)$ can be expressed by the first coefficient c_0 in the asymptotic expansion (5.4) (for details see [15, Theorem 2.3]).

(b) *Asymptotic expansion near the collision curve ℓ_m:*

$$U(x)=\sum_{\mu=\pm1}\Big\{\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1}x_3^j\Big[d_{sj}^{(m)}(x_1,\mu)\,z_{s,\mu}^{\frac12+\Delta(x_1)-j}\,B_0\Big(-\frac{1}{2\pi i}\log z_{s,\mu}\Big)\Big]\widetilde{c}_j(x_1)$$
$$+\sum_{\substack{k,l=0\\ k+l+j+p\geq1}}^{M+2}\sum_{j+p=0}^{M+2-l}x_2^l\,x_3^j\,d_{sljp}^{(m)}(x_1,\mu)\,z_{s,\mu}^{\frac12+\Delta(x_1)+p+k}\,B_{skjp}\big(x_1,\log z_{s,\mu}\big)\Big\}+U_{M+1}^{(m)}(x), \tag{5.9}$$

where $d_{sj}^{(m)}(\cdot\,,\mu)$ and $d_{sljp}^{(m)}(\cdot\,,\mu)$ are matrices with entries belonging to the space $C^\infty(\ell_m)$, $\widetilde{c}_j\in\big[C^\infty(\ell_m)\big]^6$, $U_{M+1}^{(m)}\in\big[C^{M+1}(\overline{\Omega}_1)\big]^6$ and

$$z_{s,\mu}^{\kappa+\Delta(x_1)}:=\mathrm{diag}\big\{z_{s,\mu}^{\kappa+\Delta_1(x_1)},\ \ldots,\ z_{s,\mu}^{\kappa+\Delta_6(x_1)}\big\},\ \kappa\in\mathbb{R},\quad \mu=\pm1,\quad x_1\in\ell_m;$$

$B_{skjp}(x_1,t)$ are polynomials with respect to the variable t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp}=k(2m_0-1)+m_0-1+p+j$, in general, where $m_0=\max\{m_1,\ldots,m_l\}$ and $m_1+\cdots+m_l=6$.

Note that the coefficients $d_{sj}^{(m)}(\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients $\widetilde{c}_j$ can be expressed by means of the first coefficient b_0 in the asymptotic expansion (5.6) (for details see [15, Theorem 2.3]).

Remark 5.1. Note that the above asymptotic expansions hold also true for finitely smooth data. In this case the asymptotic expansions can be obtained as in Ref. [16,14], and [15] with the help of the theory of anisotropic weighted Sobolev and Bessel potential spaces.

6. Analysis of singularities of solutions

Let $x'\in\ell_c$ and $\Pi_{x'}^{(c)}$ be the plane passing through the point x' and orthogonal to the curve ℓ_c. We introduce the polar coordinates (r,α), $r\geq0$, $-\pi\leq\alpha\leq\pi$, in the plane $\Pi_{x'}^{(c)}$ with pole at the point x'. Denote by $\Sigma^\pm$ the two different faces of the crack surface Σ. It is clear that $(r,\pm\pi)\in\Sigma^\pm$.

Denote the similar orthogonal plane to the curve ℓ_m by $\Pi_{x'}^{(m)}$ at the point $x'\in\ell_m$ and introduce there the polar coordinates (r,α), with pole at the point x'. The intersection of the plane $\Pi_{x'}^{(m)}$ and Ω_Σ can be identified with the half-plane $r\geq0$ and $0\leq\alpha\leq\pi$.

In these coordinate systems, the functions $z_{s,\pm1}$ given by (5.8) read as follows

$$z_{s,+1}=-r(\cos\alpha+\zeta_{s,+1}(x')\sin\alpha),$$
$$z_{s,-1}=r(\cos\alpha-\zeta_{s,-1}(x')\sin\alpha),$$

where $x'\in\ell_c\cup\ell_m$, $s=1,\ldots,l_0$. We can rewrite asymptotic expansions (5.7) and (5.9) in more convenient forms, in terms of the variables r and α. Moreover, we establish more refined asymptotic properties.

6.1. *Asymptotic analysis of solutions near the crack edge ℓ_c*

The asymptotic expansion (5.7) yields

$$U=(u,\varphi,\psi,\vartheta)^\top=a_0(x',\alpha)\,r^{1/2}+a_1(x',\alpha)\,r^{3/2}+\cdots,$$

where r is the distance from the reference point $x\in\Pi_{x'}^{(c)}$ to the curve ℓ_c, and $a_j=(a_{j1},\ldots,a_{j6})^\top$, $j=0,1,\ldots,$ are smooth vector functions of $x'\in\ell_c$.

From this representation it follows that in one-sided interior and exterior neighbourhoods of the surface $S_0=\partial\Omega_0$ the vector $U=(u,\varphi,\psi,\vartheta)^\top$ has $C^{\frac12}$-smoothness.

More detailed analysis shows that $a_{06}=0$ and therefore for the temperature function we have the following asymptotic expansion

$$\vartheta=a_{16}(x',\alpha)\,r^{3/2}+a_{26}(x',\alpha)\,r^{5/2}+\cdots.$$

Indeed, we can see that $u_6 = \vartheta$ solves the segregated mixed transmission problem:

$$\eta_{ij}\partial_i\partial_j u_6 = Q^* \text{ in } \quad \Omega\setminus S_0, \tag{6.1}$$

$$\{u_6\}^+ - \{u_6\}^- = \widetilde{f}_6 \text{ on } S_0, \tag{6.2}$$

$$\{[TU]_6\}^+ - \{[TU]_6\}^- = \widetilde{F}_6 \text{ on } S_0, \tag{6.3}$$

$$\{u_6\}^+ = g_6^{(D)} \text{ on } S_D, \tag{6.4}$$

$$\{[TU]_6\}^+ = g_6^{(N)} \text{ on } \quad S_N \tag{6.5}$$

with

$$Q^* = \tau\, T_0\,\lambda_{il}\,\partial_l u_i - \tau\, T_0\, p_i\,\partial_i\varphi - \tau\, T_0\, m_i\,\partial_i\psi + \tau\,\alpha_0\,\vartheta, \quad [TU]_6 = \eta_{il}\, n_i\,\partial_l\vartheta,$$
$$\widetilde{f}_6 \in C^\infty(S_0), \quad \widetilde{F}_6 \in C^\infty(S_0), \quad g_6^{(D)} \in C^\infty(\overline{S}_D), \quad g_6^{(N)} \in C^\infty(\overline{S}_N),$$

where $\widetilde{f}_6$ and $\widetilde{F}_6$ are extensions of the functions f_6 and F_6 from Σ onto the whole of S_0 by zero, and $g_6^{(D)}$ and $g_6^{(N)}$ are the sixth components of the vectors $g^{(D)}$ and $g^{(N)}$, respectively.

The problem (6.1)–(6.5) is a classical transmission problem where transmission conditions are given on the closed interface surface S_0. Regularity of solutions to this problem near the line ℓ_c depends on smoothness of the right hand side function Q^*, since all the other data possess C^∞ smoothness on S_0 (cf. [31], Section 8.2.1).

Let $1 < t < \infty$, $\;1/t - 1/2 + \delta'' < s < 1/t + 1/2 + \delta'$. Then due to Theorem 4.13(i) we deduce

$$U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top \in \left[H_t^{s+1/t}(\Omega_\Sigma)\right]^6.$$

Whence $Q^* \in H_t^{s-1+\frac{1}{t}}(\Omega_\Sigma)$ follows. Using the mapping properties of the volume potential (see [39], Theorem 3.8) we conclude that $u_6 = \vartheta$ belongs to the space $H_t^{s+1+\frac{1}{t}}$ in one-sided neighbourhoods of S_0.

From the embedding theorem (see [26], Theorem 4.6.1) it then follows that for sufficiently large t there holds the inclusion $\vartheta \in C^{1+\varepsilon}$ in a neighbourhood of S_0 with some positive ε. Due to this regularity result, from the expansion

$$\vartheta = a_{06}(x', \alpha)\, r^{1/2} + a_{16}(x', \alpha)\, r^{3/2} + \cdots$$

it follows that $a_{06} = 0$, i.e., actually for ϑ we have

$$\vartheta = a_{16}(x', \alpha)\, r^{3/2} + a_{26}(x', \alpha)\, r^{5/2} + \cdots$$

and, consequently, ϑ possesses $C^{3/2}$-regularity in one-sided closed neighbourhoods of S_0.

6.2. Asymptotic analysis of solutions near the curve ℓ_m

The asymptotic expansion (5.9) yields

$$U(x) = \sum_{\mu=\pm 1}\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} c_{sj\mu}(x', \alpha)\, r^{\gamma + i\,\delta}\, B_0\left(-\frac{1}{2\pi i}\log r\right)\widetilde{c}_{sj\mu}(x', \alpha) + \cdots, \tag{6.6}$$

where

$$r^{\gamma+i\,\delta} := \operatorname{diag}\left\{r^{\gamma_1 + i\,\delta_1}, \ldots, r^{\gamma_6 + i\,\delta_6}\right\},$$
$$\gamma_j = \frac{1}{2} + \frac{1}{2\pi}\arg\lambda_j(x'), \quad \delta_j = \frac{1}{2\pi}\log|\lambda_j(x')|, \quad x' \in \ell_m, \; j = \overline{1,6}, \tag{6.7}$$

and λ_j, $j = \overline{1,6}$, are eigenvalues of the matrix

$$a_0(x') = \left[\mathfrak{S}(\mathcal{A}; x', 0, +1)\right]^{-1}\mathfrak{S}(\mathcal{A}; x', 0, -1), \qquad x' \in \ell_m. \tag{6.8}$$

Here $\mathfrak{S}(\mathcal{A}; x', \xi)$ is the principal homogeneous symbol of the Steklov–Poincaré operator

$$\mathcal{A} = \left(-2^{-1}I_6 + \mathcal{K}\right)\mathcal{H}^{-1}.$$

Moreover, the eigenvalues λ_j, $j=\overline{1,6}$, can be expressed in terms of the eigenvalues β_j, $j=\overline{1,6}$, of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$, where $\mathfrak{S}(\mathcal{K};x',\xi)$ is the principal homogeneous symbol matrix of the singular integral operator $\mathcal{K}$. Indeed, we have the following assertion (see [31, Lemma C.1]).

Lemma 6.1. *The principal homogeneous symbol* $\mathfrak{S}(\mathcal{K};x',\xi)$ $x'\in S$, $\xi=(\xi_1,\xi_2)$, *is an odd matrix-function with respect to* ξ *and*

$$\mathfrak{S}(\mathcal{K};x',\xi)=i\,R(x',\xi),$$

where the entries of the matrix $R(x',\xi)$ *are real-valued functions.*

Proof. Assume, that to every point $x_0\in\Sigma$ there corresponds some orthogonal local coordinate system such that a part of Σ located inside a sphere with a centre at x_0 admits the representation of the form

$$x_3=\gamma(x'),\quad x'=(x_1,x_2),\quad x=(x',\gamma(x'))\in\Sigma, \tag{6.9}$$

where $\gamma\in C^\infty$, $\gamma(0)=\frac{\partial\gamma(0)}{\partial x_1}=\frac{\partial\gamma(0)}{\partial x_2}=0$. The principal homogeneous symbol of the pseudodifferential operator $-\frac{1}{2}I_6+\mathcal{K}$ in the chosen local coordinate system has the form

$$\begin{aligned}
&\mathfrak{S}(-2^{-1}I_6+\mathcal{K};x',\xi)=\|\mathfrak{S}_{pq}(-2^{-1}I_6+\mathcal{K};x',\xi)\|_{6\times6},\quad p,q=1,\dots,6,\\
&\mathfrak{S}_{pq}(-2^{-1}I_6+\mathcal{K};x',\xi)=\frac{1}{2\pi}\int_{l^-}\frac{\mathcal{T}_{pk}(x',\alpha^\top(x')(i\xi,i\zeta))\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta,\\
&\qquad\Delta(\alpha^\top(x')(i\xi,i\zeta))=\det\|A_{kq}\left(\alpha^\top(x')(i\xi,i\zeta)\right)\|_{6\times6},\\
&\qquad\qquad\alpha(x')=\begin{bmatrix}1&0&0\\0&1&0\\ \dfrac{\partial\gamma(x')}{\partial x_1}&\dfrac{\partial\gamma(x')}{\partial x_2}&-1\end{bmatrix},
\end{aligned} \tag{6.10}$$

where $\|A_{kq}(\alpha^\top(x')(i\xi,i\zeta))\|_{6\times6}$ and $\|\mathcal{T}_{pk}(x',\alpha^\top(x')(i\xi,i\zeta))\|_{6\times6}$ are the principal homogeneous symbol matrices of the operators $A(\partial_x,\tau)$ and $\mathcal{T}(\partial_x,n,\tau)$ respectively, written in the local coordinate system (6.9). $\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))$, $q,k=\overline{1,6}$, is a cofactor of $A_{kq}(\alpha^\top(x')(i\xi,i\zeta))$.

Represent the symbols $A(\alpha^\top(x')(i\xi,i\zeta))$ and $\mathcal{T}(x',\alpha^\top(x')(i\xi',i\zeta))$ as

$$\begin{aligned}
&A(\alpha^\top(x')(i\xi,i\zeta))=A^{(2)}(x',i\xi)+A^{(1)}(x',i\xi)(i\zeta)+A^{(0)}(x')(i\zeta)^2,\\
&\mathcal{T}(x',\alpha^\top(x')(i\xi,i\zeta))=\mathcal{T}^{(1)}(x',i\xi)+\mathcal{T}^{(0)}(x')(i\zeta),
\end{aligned}$$

where $A^{(j)}(x',i\xi)=\|A^{(j)}_{kq}(x',i\xi)\|_{6\times6}$, $j=0,1,2$, $\mathcal{T}^{(j)}(x',i\xi)=\|\mathcal{T}^{(j)}_{pk}(x',i\xi)\|_{6\times6}$, $j=0,1$ are homogeneous polynomials in ξ of degree j.

Taking into account (6.10) we get

$$\begin{aligned}
\mathfrak{S}_{pq}(-2^{-1}I_6+\mathcal{K};x',\xi)&=\frac{1}{2\pi}\mathcal{T}^{(0)}_{pk}(x')\int_{l^-}\frac{i\zeta\,\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta\\
&\quad+\frac{1}{2\pi}\mathcal{T}^{(1)}_{pk}(x',i\xi)\int_{l^-}\frac{\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{l^-}\frac{i\zeta\,\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta\\
&\quad=-\frac{i}{12\pi}\frac{\widehat{A}^{(0)}_{qk}(x')}{\det[A^{(0)}(x')]}\int_{l^-}\frac{\partial_\zeta\Delta(\alpha^\top(x')(\xi,\zeta))}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta+\frac{i}{2\pi}\int_{l^-}\frac{\widetilde{\Delta}_{qk}(x',\xi,\zeta)}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta\\
&\quad=\frac{1}{2}\frac{\widehat{A}^{(0)}_{qk}(x')}{\det[A^{(0)}(x')]}+\frac{i}{2\pi}\int_{l^-}\frac{\widetilde{\Delta}_{qk}(x',\xi,\zeta)}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta,
\end{aligned}$$

where $\widehat{A}^{(0)}_{qk}$ is the cofactor of $A^{(0)}_{kq}$ and $\widetilde{\Delta}_{qk}(x',\xi,\zeta)$ is a polynomial of degree 10 in ζ of the form

$$\widetilde{\Delta}_{qk}(x',\xi,\zeta) = \frac{\widehat{A}^{(0)}_{qk}(x')}{6\det[A^{(0)}(x')]}\partial_\zeta \Delta(\alpha^\top(x')(\xi,\zeta)) - \zeta \Delta_{kq}(\alpha^\top(x')(\xi,\zeta)).$$

Therefore

$$\begin{aligned}\mathfrak{S}_{pq}(\mathcal{K};x',\xi) = {} & \frac{i}{2\pi}\mathcal{T}^{(0)}_{pk}(x')\int_{l^-}\frac{\widetilde{\Delta}_{qk}(x',\xi,\zeta)}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta \\ & - \frac{i}{2\pi}\mathcal{T}^{(1)}_{pk}(x',\xi)\int_{l^-}\frac{\Delta_{qk}(\alpha^\top(x')(\xi,\zeta))}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta, \quad p,q=\overline{1,6}. \end{aligned} \tag{6.11}$$

Since $\widetilde{\Delta}_{qk}(x',\xi,\zeta)$ and $\Delta_{qk}(\alpha^\top(x'),\xi,\zeta)$ are polynomials of degree 10 in ζ, from (6.11) we can easily see that

$$\mathfrak{S}_{pq}(\mathcal{K};x',-\xi) = -\mathfrak{S}_{pq}(\mathcal{K};x',\xi)$$

and

$$\mathfrak{S}_{pq}(\mathcal{K};x',\xi) = i R_{pq}(x',\xi), \quad p,q=\overline{1,6},$$

where $R_{pq}(x',\xi)$, $p,q=\overline{1,6}$, are real functions. □

Remark 6.2. It is not difficult to check that the principal homogeneous symbol $\mathfrak{S}(\mathcal{H};x',\xi)$ of the pseudodifferential operator $\mathcal{H}$ is a real even matrix-function with respect to ξ (see Lemma C.2 in [31]).

Theorem 6.3. *Let λ_j, $j=\overline{1,6}$, be the eigenvalues of the matrix (6.8). Then*

$$\lambda_j = \frac{1+2\beta_j}{1-2\beta_j}, \quad j=\overline{1,6},$$

where β_j, $j=\overline{1,6}$, are the eigenvalues of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$.

Proof. The characteristic equation of the matrix a_0 given by (6.8) has the form

$$\det\Big\{\Big[(-2^{-1}I_6+\sigma^+_{\mathcal{K}})\,[\sigma^+_{\mathcal{H}}]^{-1}\Big]^{-1}\Big[(-2^{-1}I_6+\sigma^-_{\mathcal{K}})\,[\sigma^-_{\mathcal{H}}]^{-1}\Big] - \lambda I_6\Big\} = 0, \tag{6.12}$$

where

$$\sigma^\pm_{\mathcal{K}} = \mathfrak{S}(\mathcal{K};x',0,\pm1), \quad \sigma^\pm_{\mathcal{H}} = \mathfrak{S}(\mathcal{H};x',0,\pm1). \tag{6.13}$$

Since the matrix $\mathfrak{S}(\mathcal{K};x',\xi)$ is odd and the matrix $\mathfrak{S}(\mathcal{H};x',\xi)$ is even in ξ (see Lemma 6.1), we have $\sigma^-_{\mathcal{K}} = -\sigma^+_{\mathcal{K}}$ and $\sigma^-_{\mathcal{H}} = \sigma^+_{\mathcal{H}}$. Then the characteristic equation (6.12) can be rewritten as

$$\det\Big\{\sigma^+_{\mathcal{H}}[2^{-1}I_6-\sigma^+_{\mathcal{K}}]^{-1}[2^{-1}I_6+\sigma^+_{\mathcal{K}}]\,[\sigma^+_{\mathcal{H}}]^{-1} - \lambda I_6\Big\} = 0.$$

Since the matrices $\sigma^+_{\mathcal{H}}$ and $2^{-1}I_6 \pm \sigma^+_{\mathcal{K}}$ are non-singular, from the previous equality we derive

$$\det\Big\{[2^{-1}I_6+\sigma^+_{\mathcal{K}}] - \lambda[2^{-1}I_6-\sigma^+_{\mathcal{K}}]\Big\} = 0.$$

Consequently,

$$\det\Big[\sigma^+_{\mathcal{K}} + \frac{1}{2}\Big(\frac{1-\lambda}{1+\lambda}\Big)I_6\Big] = 0. \tag{6.14}$$

Let β_j, $j=\overline{1,6}$, be the eigenvalues of the matrix $\sigma^+_{\mathcal{K}}$. Then it follows from (6.14) that the eigenvalues λ_j of the matrix a_0 and the eigenvalues β_j of $\sigma^+_{\mathcal{K}}$ are related by the equation

$$\frac{\lambda_j-1}{\lambda_j+1} = 2\beta_j, \quad j=\overline{1,6},$$

which completes the proof. □

It can be shown that $\lambda_6 = 1$, i.e., $\beta_6 = 0$ (for details see [31, Section 5.7]). Therefore, $\gamma_6 = 1/2$ and $\delta_6 = 0$ in accordance with (6.7). This implies that one could not expect better smoothness for solutions than $C^{1/2}$, in general.

More detailed analysis leads to the following refined asymptotic behaviour for the temperature function.

Theorem 6.4. *Near the line ℓ_m the function ϑ possesses the following asymptotic:*

$$\vartheta = b_0 r^{1/2} + \mathcal{R}, \tag{6.15}$$

where $\mathcal{R} \in C^{1+\delta'-\varepsilon}$ in a neighbourhood of ℓ_m and $1 + \delta' - \varepsilon > 1/2$ for sufficiently small $\varepsilon > 0$.

Proof. Indeed, $u_6 = \vartheta$ is a solution of the problem (6.1)–(6.5). Since the matrix $[\eta_{ij}]_{3\times 3}$ is positive definite, this problem can be reduced to a system of integral equations, where the principal part is described by the scalar positive-definite Steklov–Poincaré type operator $\mathcal{A} = (-\frac{1}{2}I + \mathcal{K}_{scalar})[H_{scalar}]^{-1}$ on S_N, where $\mathcal{K}_{scalar}$ is compact. This operator possesses an even principal homogeneous symbol $\mathfrak{S}(\mathcal{A}; x, \xi) = -\frac{1}{2}\mathfrak{S}([H_{scalar}]^{-1}; x, \xi) = -\frac{1}{2}[\mathfrak{S}(H_{scalar}; x, \xi)]^{-1}$ which is positive and even in ξ. Hence we can establish refined explicit asymptotic (6.15) for the temperature function $u_6 = \vartheta$ in a neighbourhood of ℓ_m. □

From (6.15) it follows that:

(i) The leading exponent for $u_6 = \vartheta$ in a neighbourhood of line ℓ_m equals $1/2$;
(ii) Logarithmic factors are absent in the first term of the asymptotic expansion of ϑ;
(iii) The temperature function ϑ does not oscillate in a neighbourhood of the collision curve ℓ_m and for the heat flux vector we have no oscillating singularities.

In what follows, we will consider particular type GTEME materials and analyse the exponents $\gamma_j + i\delta_j$ which determine the behaviour of $u = (u_1, u_2, u_3)$, φ, and ψ near the line ℓ_m. Non-zero parameters δ_j lead to the so called oscillating singularities for the first order derivatives of u, φ, and ψ, in general. In turn, this yields oscillating stress singularities which sometimes lead to mechanical contradictions, for example, to overlapping of materials. So, from the practical point of view, it is important to single out classes of solids for which the oscillating effects do not occur.

To this end, we will consider a special class of bodies belonging to the **422** (Tetragonal) or **622** (Hexagonal) class of crystals for which the corresponding system of differential equations reads as follows (see, e.g., [40])

$$\begin{aligned}
&(c_{11}\,\partial_1^2 + c_{66}\,\partial_2^2 + c_{44}\,\partial_3^2)u_1 + (c_{12} + c_{66})\,\partial_1\partial_2 u_2 + (c_{13} + c_{44})\,\partial_1\partial_3 u_3\\
&\quad - e_{14}\,\partial_2\partial_3\varphi - q_{15}\,\partial_2\partial_3\psi - \widetilde{\gamma}_1\,\partial_1\vartheta - \varrho\,\tau^2 u_1 = F_1,\\
&(c_{12} + c_{66})\,\partial_2\partial_1 u_1 + (c_{66}\,\partial_1^2 + c_{11}\,\partial_2^2 + c_{44}\,\partial_3^2)\,u_2 + (c_{13} + c_{44})\,\partial_2\partial_3 u_3\\
&\quad + e_{14}\partial_1\partial_3\varphi - q_{15}\partial_1\partial_3 u_2 - \widetilde{\gamma}_1\partial_2\vartheta - \varrho\,\tau^2 u_2 = F_2,\\
&(c_{13} + c_{44})\,\partial_3\partial_1 u_1 + (c_{13} + c_{44})\,\partial_3\partial_2 u_2 + (c_{44}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{33}\,\partial_3^2)\,u_3\\
&\quad - \widetilde{\gamma}_3\,\partial_3\vartheta - \varrho\,\tau^2 u_3 = F_3,\\
&e_{14}\partial_2\partial_3 u_1 - e_{14}\partial_1\partial_3 u_2 + (\varkappa_{11}\,\partial_1^2 + \varkappa_{11}\,\partial_2^2 + \varkappa_{33}\,\partial_3^2)\,\varphi - (1 + \nu_0\tau)p_3\,\partial_3\vartheta = F_4,\\
&q_{15}\partial_2\partial_3 u_1 - q_{15}\partial_1\partial_3 u_2 + (\mu_{11}\,\partial_1^2 + \mu_{11}\,\partial_2^2 + \mu_{33}\,\partial_3^2)\,\psi - (1 + \nu_0\tau)m_3\,\partial_3\vartheta = F_5,\\
&-\tau\,T_0\,(\widetilde{\gamma}_1\,\partial_1 u_1 + \widetilde{\gamma}_1\,\partial_2 u_2 + \widetilde{\gamma}_3\,\partial_3 u_3) + \tau\,T_0\,p_3\,\partial_3\varphi + \tau\,T_0\,m_3\,\partial_3\psi\\
&\quad + (\eta_{11}\,\partial_1^2 + \eta_{11}\,\partial_2^2 + \eta_{33}\,\partial_3^2)\,\vartheta - (\tau\,d_0 + \tau^2 h_0)\,\vartheta = F_6,
\end{aligned} \tag{6.16}$$

where c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, c_{66}, are the elastic constants, e_{14} is the piezoelectric constant, q_{15} is the piezomagnetic constant, $\varkappa_{11}$ and $\varkappa_{33}$ are the dielectric constants, μ_{11} and μ_{33} are the magnetic permeability constants, $\widetilde{\gamma}_1 = (1+\nu_0\tau)\lambda_{11} = (1+\nu_0\tau)\lambda_{21}$ and $\widetilde{\gamma}_3 = (1+\nu_0\tau)\lambda_{31}$ are the thermal strain constants, η_{11} and η_{33} are the thermal conductivity constants, p_3 is the pyroelectric constant and m_3 is the pyromagnetic constant. Note that in the case of the Hexagonal crystals (**622** class), we have $c_{66} = (c_{11} - c_{12})/2$.

Note that some important polymers and bio-materials are modelled by the above partial differential equations, for example, the *collagen–hydroxyapatite* is one example of such a material. This material is widely used in biology and medicine (see [12]). The other important example is TeO_2 [40].

In this model the thermoelectromechanical stress operator is defined as

$$T(\partial_x, n) = \left\| T_{jk}(\partial_x, n) \right\|_{6\times 6}$$

with

$$\begin{aligned}
&T_{11}(\partial_x, n) = c_{11}n_1\partial_1 + c_{66}n_2\partial_2 + c_{44}n_3\partial_3, && T_{12}(\partial_x, n) = c_{12}n_1\partial_2 + c_{66}n_2\partial_1,\\
&T_{13}(\partial_x, n) = c_{13}n_1\partial_3 + c_{44}n_3\partial_1, && T_{14}(\partial_x, n) = -e_{14}n_3\partial_2,\\
&T_{15}(\partial_x, n) = -q_{15}n_3\partial_2, && T_{16}(\partial_x, n) = -\widetilde{\gamma}_1\, n_1,\\
&T_{21}(\partial_x, n) = c_{66}n_1\partial_2 + c_{12}n_2\partial_1, && T_{22}(\partial_x, n) = c_{66}n_1\partial_1 + c_{11}n_2\partial_2 + c_{44}n_3\partial_3,\\
&T_{23}(\partial_x, n) = c_{13}n_2\partial_3 + c_{44}n_3\partial_2, && T_{24}(\partial_x, n) = e_{14}n_3\partial_1,\\
&T_{25}(\partial_x, n) = q_{15}n_3\partial_1, && T_{26}(\partial_x, n) = -\widetilde{\gamma}_1\, n_2,\\
&T_{31}(\partial_x, n) = c_{44}n_1\partial_3 + c_{13}n_3\partial_1, && T_{32}(\partial_x, n) = c_{44}n_2\,\partial_3 + c_{13}n_3\partial_2,\\
&T_{33}(\partial_x, n) = c_{44}n_1\partial_1 + c_{44}n_2\partial_2 + c_{33}n_3\partial_3, && T_{34}(\partial_x, n) = 0,\\
&T_{35}(\partial_x, n) = 0, && T_{36}(\partial_x, n) = -\widetilde{\gamma}_3\, n_3,
\end{aligned}$$

$$\begin{aligned}
&T_{41}(\partial_x, n) = e_{14}n_2\partial_3, && T_{42}(\partial_x, n) = -e_{14}n_1\partial_3,\\
&T_{43}(\partial_x, n) = e_{14}(n_2\partial_1 - n_1\partial_2), && T_{44}(\partial_x, n) = \varkappa_{11}(n_1\partial_1 + n_2\partial_2) + \varkappa_{33}n_3\partial_3,\\
&T_{45}(\partial_x, n) = 0, && T_{46}(\partial_x, n) = -p_3n_3,\\
&T_{51}(\partial_x, n) = q_{15}n_2\partial_3, && T_{52}(\partial_x, n) = -q_{15}n_1\partial_3,\\
&T_{53}(\partial_x, n) = q_{15}(n_2\partial_1 - n_1\partial_2), && T_{54}(\partial_x, n) = 0,\\
&T_{55}(\partial_x, n) = \mu_{11}(n_1\partial_1 + n_2\partial_2) + \mu_{33}n_3\partial_3, && T_{56}(\partial_x, n) = -m_3n_3,\\
&T_{6j}(\partial_x, n) = 0, \text{ for } j = 1, 2, 3, 4, 5, && T_{66}(\partial_x, n) = \eta_{11}(n_1\partial_1 + n_2\partial_2) + \eta_{33}n_3\partial_3.
\end{aligned}$$

The material constants satisfy the following inequalities which follow from positive definiteness of the internal energy form (see (2.9)–(2.10))

$$\begin{aligned}
&c_{11} > |c_{12}|, \quad c_{44} > 0, \quad c_{66} > 0, \quad c_{33}(c_{11} + c_{12}) > 2c_{13}^2,\\
&\varkappa_{11} > 0, \quad \varkappa_{33} > 0, \quad \eta_{11} > 0, \quad \eta_{33} > 0, \quad \mu_{11} > 0, \quad \mu_{33} > 0.
\end{aligned}$$

From (2.11), (2.14), (2.15), it follows also that

$$\varkappa_{33} > \frac{p_3^2\, T_0}{d_0}, \quad \mu_{33} > \frac{m_3^2\, T_0}{d_0}.$$

Under these conditions the corresponding mixed boundary value problem in question is uniquely solvable.

Furthermore, we assume that mechanical and electric fields are coupled, i.e. $e_{14} \neq 0$, that $\frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} = \alpha$ and the surface S is parallel to the plane of isotropy (i.e., to the plane $x_3 = 0$) in some neighbourhood of ℓ_m.

We will show that under these conditions we can find the exponents involved in the asymptotic expansions of solutions explicitly in terms of the material constants.

In this case the symbol matrix $\sigma_{\mathcal{K}}^+ = \mathfrak{S}(\mathcal{K}; x', 0, +1)$ is calculated explicitly and has the form (see Appendix B):

$$\sigma_{\mathcal{K}}^+ = \begin{bmatrix} 0 & 0 & 0 & A_{14} & A_{15} & 0\\ 0 & 0 & A_{23} & 0 & 0 & 0\\ 0 & A_{32} & 0 & 0 & 0 & 0\\ A_{41} & 0 & 0 & 0 & 0 & 0\\ A_{51} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$A_{14} = -i\,\frac{e_{14}\,c_{66}\,(b_2-b_1)}{2\,b_1\,b_2\sqrt{B}} - i\,\frac{e_{14}q_{15}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2+\varkappa_{11})}{\sqrt{B}}\right], \tag{6.17}$$

$$A_{41} = -i\,\frac{e_{14}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}}, \tag{6.18}$$

$$A_{15} = -i\,\frac{q_{15}\,c_{66}\,(b_2-b_1)}{2\alpha\,b_1\,b_2\sqrt{B}} - i\,\frac{q_{15}e_{14}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2+\varkappa_{11})}{\sqrt{B}}\right], \tag{6.19}$$

$$A_{51} = -i\,\frac{q_{15}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}}, \tag{6.20}$$

$$b_1 = \sqrt{\frac{A-\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}}, \qquad b_2 = \sqrt{\frac{A+\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}},$$

$$\widetilde{e}_{14} = \left(e_{14}^2+\alpha^{-1}q_{15}^2\right)^{1/2}, \quad \alpha = \frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} > 0,$$

$$A = \widetilde{e}_{14}^2 + c_{44}\,\varkappa_{11} + c_{66}\,\varkappa_{33} > 0, \qquad B = A^2 - 4\,c_{44}\,c_{66}\,\varkappa_{11}\,\varkappa_{33} > 0, \quad A > \sqrt{B}.$$

It can be proved that $A_{14}A_{41} + A_{15}A_{51} < 0$ (see Appendix B).

To calculate the entries A_{23} and A_{32}, we have to consider two cases. We set

$$C := c_{11}\,c_{33} - c_{13}^2 - 2\,c_{13}\,c_{44}, \quad D := C^2 - 4\,c_{44}^2\,c_{33}c_{11}.$$

First, let $D > 0$. Then it follows from the positive definiteness of the internal energy that $C > \sqrt{D}$ and we have

$$A_{23} = i\,\frac{c_{44}\,(d_2-d_1)\,(c_{11}-c_{13}\,d_1d_2)}{2\,d_1\,d_2\sqrt{D}}\,, \tag{6.21}$$

$$A_{32} = -i\,\frac{c_{44}\,(d_2-d_1)\,(c_{33}\,d_1\,d_2-c_{13})}{2\,d_1\,d_2\sqrt{D}}\,, \tag{6.22}$$

$$d_1 = \sqrt{\frac{C-\sqrt{D}}{2\,c_{44}\,c_{33}}}\,, \qquad d_2 = \sqrt{\frac{C+\sqrt{D}}{2\,c_{44}\,c_{33}}}\,.$$

These equalities imply $A_{23}\,A_{32} > 0$.

Now, let $D < 0$. We get

$$A_{23} = i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}}-c_{13})}{\sqrt{-D}}, \quad A_{32} = -i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}}-c_{13})}{\sqrt{-D}}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}\,,$$

where

$$a = \frac{1}{2}\sqrt{\frac{-C+2c_{44}\sqrt{c_{11}c_{33}}}{c_{44}c_{33}}} > 0\,.$$

One can easily check that again

$$A_{23}A_{32} = \frac{c_{44}^2\,a^2(\sqrt{c_{11}\,c_{33}}-c_{13})^2}{-D}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}} > 0.$$

The characteristic polynomial of the matrix $\sigma_{\mathcal{K}}^{+}$ can be represented as

$$\det(\sigma_{\mathcal{K}}^{+} - \beta I) = \det \begin{bmatrix} -\beta & A_{14} & 0 & 0 & A_{15} & 0 \\ A_{41} & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & A_{23} & 0 & 0 \\ 0 & 0 & A_{32} & -\beta & 0 & 0 \\ A_{51} & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \end{bmatrix}$$
$$= \beta^2 \left(\beta^2 - A_{23}A_{32}\right) \left(\beta^2 - (A_{14}A_{41} + A_{15}A_{51})\right).$$

Therefore, we have the following expressions for the eigenvalues of the matrix $\sigma_{\mathcal{K}}^{+}$:

$$\beta_{1,2} = \mp i\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}, \quad \beta_{3,4} = \mp\sqrt{A_{23}A_{32}}, \quad \beta_5 = \beta_6 = 0.$$

Then by Theorem 6.3

$$\lambda_1 = \frac{1 - 2i\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}}{1 + 2i\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}}, \ \lambda_2 = \frac{1}{\lambda_1}, \quad \lambda_3 = \frac{1 - 2\sqrt{A_{23}A_{32}}}{1 + 2\sqrt{A_{23}A_{32}}}, \ \lambda_4 = \frac{1}{\lambda_3}, \quad \lambda_5 = \lambda_6 = 1.$$

Note that $|\lambda_1| = |\lambda_2| = 1$. Moreover, since λ_3 and λ_4 are real, they are positive (see Appendix A).

Applying the above results we can explicitly write the exponents of the first terms of the asymptotic expansions of solutions (see (6.7)):

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\arctan 2\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}, \quad \delta_1 = 0,$$
$$\gamma_2 = \frac{1}{2} + \frac{1}{\pi}\arctan 2\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}, \quad \delta_2 = 0,$$
$$\gamma_3 = \gamma_4 = \frac{1}{2}, \qquad \delta_3 = -\delta_4 = \widetilde{\delta} = \frac{1}{2\pi}\log\frac{1 - 2\sqrt{A_{23}A_{32}}}{1 + 2\sqrt{A_{23}A_{32}}},$$
$$\gamma_5 = \gamma_6 = \frac{1}{2}, \qquad \delta_5 = \delta_6 = 0.$$

Note that $B_0(t)$ has the following form

$$B_0(t) = \begin{bmatrix} I_4 & [0]_{4\times 2} \\ [0]_{4\times 2} & B^{(2)}(t) \end{bmatrix}, \qquad \text{where} \quad B^{(2)}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Now we can draw the following conclusions:

1. The solutions of the problem possess the following asymptotic behaviour near the line ℓ_m:

$$(u,\, \varphi,\, \psi)^\top = c_0\, r^{\gamma_1} + c_1\, r^{\frac{1}{2}+i\,\widetilde{\delta}} + c_2\, r^{\frac{1}{2}-i\,\widetilde{\delta}} + c_3 r^{\frac{1}{2}}\ln r + c_4 r^{\frac{1}{2}} + c_5 r^{\gamma_2} + \cdots$$
$$\vartheta = b_3\, r^{1/2} + b_4\, r^{\gamma_2} + \cdots.$$

As we see, the exponent γ_1 characterizing the behaviour of u, φ, and ψ near the line ℓ_m depends on the elastic, piezoelectric, piezomagnetic, dielectric, and permeability constants, and does not depend on the thermal constants. Moreover, γ_1 takes values from the interval $(0,\ 1/2)$.

For the general anisotropic case these exponents also depend on the geometry of the line ℓ_m, in general.

2. Since $\gamma_1 < 1/2$, we have not oscillating singularities for physical fields in some neighbourhood of the curve ℓ_m.

Note that in the classical elasticity theory (for both isotropic and anisotropic solids) for mixed BVPs the dominant exponents are $1/2$, $1/2 \pm i\,\widetilde{\delta}$ with $\widetilde{\delta} \neq 0$ and consequently we have oscillating stress singularities at the collision curve ℓ_m.

The following questions arise naturally:

(a) does there exist a class of GTEME type media for which the real part of the principal exponent defining the dominant stress singularity near the line ℓ_m does not depend on the material constants?

(b) does there exist a class of GTEME type media for which the real part of the principal exponent equals $1/2$?

As we will see below, both question have positive answers.

Indeed, let us consider the class of GTEME type media with cubic anisotropy. Note that such materials as $Bi_{12}GeO_{20}$ and GaAs belong to this class (see, e.g., [40]). The latter material is widely used in the electronic industry.

The corresponding system of differential equations in this case reads as:

$$\begin{aligned}
&(c_{11}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{44}\,\partial_3^2)u_1 + (\,c_{12} + c_{44}\,)\,\partial_1\partial_2 u_2 + (\,c_{12} + c_{44}\,)\,\partial_1\partial_3 u_3\\
&\quad + 2e_{14}\,\partial_2\partial_3\varphi + 2q_{15}\,\partial_2\partial_3\psi - \widetilde{\gamma}_1\,\partial_1\vartheta - \varrho\,\tau^2\,u_1 = F_1,\\
&(\,c_{12} + c_{44}\,)\,\partial_2\partial_1 u_1 + (\,c_{44}\,\partial_1^2 + c_{11}\,\partial_2^2 + c_{44}\,\partial_3^2\,)\,u_2 + (\,c_{12} + c_{44}\,)\,\partial_2\partial_3 u_3\\
&\quad + 2e_{14}\partial_1\partial_3\varphi + 2q_{15}\partial_1\partial_3\psi - \widetilde{\gamma}_1\partial_2\vartheta - \varrho\,\tau^2\,u_2 = F_2,\\
&(\,c_{12} + c_{44}\,)\,\partial_3\partial_1 u_1 + (\,c_{12} + c_{44}\,)\,\partial_3\partial_2 u_2 + (\,c_{44}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{11}\,\partial_3^2\,)\,u_3\\
&\quad + 2e_{14}\partial_1\partial_2\varphi + 2q_{15}\partial_1\partial_2\psi - \widetilde{\gamma}_3\,\partial_3\vartheta - \varrho\,\tau^2\,u_3 = F_3,\\
&\quad -2e_{14}\partial_2\partial_3 u_1 - 2e_{14}\partial_1\partial_3 u_2 - 2e_{14}\partial_1\partial_2 u_3 + (\varkappa_{11}\partial_1^2 + \varkappa_{11}\partial_2^2 + \varkappa_{11}\partial_3^2)\varphi - p_3\partial_3\vartheta = F_4,\\
&\quad -2q_{15}\partial_2\partial_3 u_1 - 2q_{15}\partial_1\partial_3 u_2 - 2q_{15}\partial_1\partial_2 u_3 + (\mu_{11}\partial_1^2 + \mu_{11}\partial_2^2 + \mu_{11}\partial_3^2)\psi - m_3\,\partial_3\vartheta = F_5,\\
&\quad -\tau T_0(\widetilde{\gamma}_1\partial_1 u_1 + \widetilde{\gamma}_1\partial_2 u_2 + \widetilde{\gamma}_3\,\partial_3 u_3) + \tau T_0 p_3\partial_3\varphi + \tau T_0 m_3\partial_3\psi\\
&\quad + (\,\eta_{11}\,\partial_1^2 + \eta_{11}\,\partial_2^2 + \eta_{33}\,\partial_3^2\,)\,\vartheta - \tau\,d_0\,\vartheta = F_6.
\end{aligned} \tag{6.23}$$

The elastic, dielectric, permeability and thermal constants involved in the governing equations satisfy the following conditions:

$$\begin{aligned}
&c_{11} > 0, \quad c_{44} > 0, \quad -1/2 < c_{12}/c_{11} < 1, \quad \varkappa_{11} > 0, \quad \mu_{11} > 0,\\
&\varkappa_{33} > \frac{p_3^2\,T_0}{d_0}, \quad \mu_{33} > \frac{m_3^2\,T_0}{d_0}, \quad \eta_{11} > 0, \quad \eta_{33} > 0.
\end{aligned} \tag{6.24}$$

Introduce the notation,

$$D := C^2 - 4\,c_{11}^2\,c_{44}^2, \quad C := c_{11}^2 - c_{12}^2 - 2\,c_{12}\,c_{44}, \quad a := \frac{1}{2}\sqrt{\frac{-C + 2c_{44}\sqrt{c_{11}}}{c_{44}c_{11}}} > 0.$$

In the case under consideration, the matrix $\sigma_{\mathcal{K}}^{+}$ is self-adjoint and reads as:

$$\sigma_{\mathcal{K}}^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & A_{23} & 0 & 0 & 0\\ 0 & A_{32} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.25}$$

where

$$\begin{aligned}
&A_{23} = \overline{A_{32}} = i\,\frac{c_{44}\,(d_2 - d_1)(c_{11} - c_{12})}{2\sqrt{D}} \quad \text{for} \quad D > 0,\\
&d_1 = \sqrt{\frac{C - \sqrt{D}}{2c_{44}c_{11}}}, \quad d_2 = \sqrt{\frac{C + \sqrt{D}}{2c_{44}c_{11}}},\\
&A_{23} = \overline{A_{32}} = i\,\frac{c_{44}\,a\,(c_{11} - c_{12})}{2\sqrt{-D}} \quad \text{for} \quad D < 0.
\end{aligned}$$

The corresponding eigenvalues read as (see Theorem 6.3)

$$\begin{aligned}
&\beta_j = 0, \;\; j = 1, 2, 5, 6, \quad \beta_{3,4} = \pm|A_{23}|,\\
&\lambda_j = 1, \;\; j = 1, 2, 5, 6, \quad \lambda_3 = \frac{1 + 2|A_{23}|}{1 - 2|A_{23}|} > 0, \quad \lambda_4 = \frac{1}{\lambda_3}\,,
\end{aligned}$$

and

$$\gamma_j = \frac{1}{2},\ \ j = \overline{1,6},\ \ \delta_j = 0,\ \ j = 1,2,5,6,\ \ \delta_3 = -\delta_4 = \widetilde{\delta} = \frac{1}{2\pi}\log\frac{1+2|A_{23}|}{1-2|A_{23}|}.$$

From Lemma 6.1, Remark 6.2 and equalities (6.25) and (6.8) we derive

$$a_0 = \Big[(-2^{-1}I_6 + \sigma_{\mathcal{K}}^{+})\,[\sigma_{\mathcal{H}}^{+}]^{-1}\Big]^{-1}\Big[(-2^{-1}I_6 + \sigma_{\mathcal{K}}^{-})\,[\sigma_{\mathcal{H}}^{-}]^{-1}\Big] = \sigma_{\mathcal{H}}^{+}\,\widetilde{a}_0\,[\sigma_{\mathcal{H}}^{+}]^{-1},$$

where

$$\widetilde{a}_0 = [2^{-1}I_6 - \sigma_{\mathcal{K}}^{+}]^{-1}\,[2^{-1}I_6 + \sigma_{\mathcal{K}}^{+}].$$

This matrix is self-adjoint due to the equality (6.25) and it is similar to a diagonal matrix, i.e., there is a unitary matrix $\mathcal{D}$ such that $\mathcal{D}\,\widetilde{a}_0\,[\mathcal{D}]^{-1}$ is diagonal. Therefore the matrix a_0 can be reduced to a diagonal matrix by the non-degenerate matrix $\sigma_{\mathcal{H}}^{+}\mathcal{D}^{-1}$. In turn, this implies that $B_0(t) = I$ and the leading terms of the asymptotic expansion (6.6) near the curve ℓ_m do not contain logarithmic factors.

As a result we obtain the asymptotic expansion leading to the positive answers to the questions (a) and (b) stated above,

$$\begin{aligned}(u,\varphi,\psi)^\top &= c_0\,r^{1/2} + c_1\,r^{1/2+i\,\widetilde{\delta}} + c_2\,r^{1/2-i\,\widetilde{\delta}} + \mathcal{O}(r^{3/2-\varepsilon}),\\ \vartheta &= b_0\,r^{1/2} + \mathcal{O}(r^{3/2-\varepsilon}),\end{aligned}$$

where ε is an arbitrary positive number. Consequently, u, φ, ψ, and ϑ possess $C^{1/2}$-regularity in a neighbourhood of the collision curve ℓ_m.

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Appendix A. Some results for pseudodifferential equations on manifolds with boundary

Here we collect some results describing the Fredholm properties of strongly elliptic pseudodifferential operators on a compact manifold with boundary. They can be found in [36,16,41,19].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \ldots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\mathfrak{S}(\mathcal{A}; x, 0, \ldots, 0, +1)\,]^{-1}\ \mathfrak{S}(\mathcal{A}; x, 0, \ldots, 0, -1)\,, \quad x \in \partial\overline{\mathcal{M}},$$

and let

$$\delta_j(x) = \mathrm{Re}\,\big[\,(2\pi\,i)^{-1}\ln\lambda_j(x)\,\big],\ \ j = 1, \ldots, N.$$

Here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strict inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$. The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. In particular, if the eigenvalue λ_j is real, then λ_j is positive.

Note that when $\mathfrak{S}(\mathcal{A}, x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ or when it is an even matrix in ξ we have $\delta_j(x) = 0$ for $j = 1, \ldots, N$, since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1, N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators are characterized by the following theorem.

Theorem A. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\mathrm{Re}\left(\sigma_{\mathcal{A}}(x,\xi)\zeta\cdot\zeta\right) \ge c_0\,|\zeta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^N$. Then

$$\mathcal{A} : \widetilde{H}^s_p(\mathcal{M}) \to H^{s-\nu}_p(\mathcal{M}), \qquad \mathcal{A} : \widetilde{B}^s_{p,q}(\mathcal{M}) \to B^{s-\nu}_{p,q}(\mathcal{M}), \tag{A.1}$$

are Fredholm operators with index zero if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x). \tag{A.2}$$

Moreover, the null-spaces and indices of the operators (A.1) are the same (for all values of the parameter $q \in [1,+\infty]$) provided p and s satisfy the inequality (A.2).

We essentially use this theorem in Section 4 to prove the existence and regularity results of solutions to mixed boundary value problems for solids with interior cracks.

Appendix B. Calculation of the symbolic matrices

In this section we calculate the principal homogeneous symbol matrix $\sigma^+_{\mathcal{K}} = \mathfrak{S}(\mathcal{K}; x_1, 0, +1)$ corresponding to the system (6.16) (**422** and **622** classes). To this end we write the fundamental matrix (2.1) in the form (see [31, Section 3])

$$\begin{aligned}\Gamma(x,\tau) &= \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi,\tau)\big] \\ &= \mathcal{F}^{-1}_{\xi'\to x'}\Big[\pm\frac{1}{2\pi}\int_{l^\pm}\big[A(-i\xi',-i\zeta,\tau)\big]^{-1}e^{-i\zeta x_3}\,d\zeta\Big],\end{aligned} \tag{B.1}$$

where the sign "−" corresponds to the case $x_3 > 0$ and the sign "+" to the case $x_3 < 0$. Here $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$, $\xi = (\xi', \xi_3)$, $l^+(l^-)$ is a closed contour with positive counterclockwise orientation enveloping all the roots of the polynomial $\det A(-i\xi', -i\zeta, \tau)$ with respect to the variable ζ in the half-plane $Im\,\zeta > 0$ ($Im\,\zeta < 0$).

First, we write the principal homogeneous symbols $A^{(0)}$ and $\mathcal{T}^{(0)}$ of the operators $A(\partial_x, \tau)$ and $\mathcal{T}(\partial, n)$ at a point $\widetilde{\zeta} = (0, 1, \zeta)$. Choosing a local coordinate system appropriately, we can assume that the exterior unit normal vector at this point reads as $n = (0, 0, 1)$. Then we have

$$A^{(0)}(\widetilde{\zeta}) = -\begin{bmatrix} A^{(0)}_{11} & 0 & 0 & A^{(0)}_{14} & A^{(0)}_{15} & 0 \\ 0 & A^{(0)}_{22} & A^{(0)}_{23} & 0 & 0 & 0 \\ 0 & A^{(0)}_{23} & A^{(0)}_{33} & 0 & 0 & 0 \\ -A^{(0)}_{14} & 0 & 0 & A^{(0)}_{44} & 0 & 0 \\ -A^{(0)}_{15} & 0 & 0 & 0 & A^{(0)}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A^{(0)}_{66} \end{bmatrix}, \tag{B.2}$$

$$\mathcal{T}^{(0)}(\widetilde{\zeta}, n) = -\begin{bmatrix} ic_{44}\zeta & 0 & 0 & -ie_{14} & -iq_{14} & 0 \\ 0 & ic_{44}\zeta & ic_{44} & 0 & 0 & 0 \\ 0 & ic_{13} & ic_{33}\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varkappa_{33}\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & i\mu_{33}\zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & i\eta_{33}\zeta \end{bmatrix}, \tag{B.3}$$

where

$$A^{(0)}_{11} = c_{44}\zeta^2 + c_{66}, \quad A^{(0)}_{14} = -e_{14}, \quad A^{(0)}_{15} = -q_{15},$$
$$A^{(0)}_{22} = c_{44}\zeta^2 + c_{11}, \quad A^{(0)}_{23} = (c_{13} + c_{44})\zeta, \quad A^{(0)}_{33} = c_{33}\zeta^2 + c_{44},$$
$$A^{(0)}_{44} = \varkappa_{33}\zeta^2 + \varkappa_{11}, \quad A^{(0)}_{55} = \mu_{33}\zeta^2 + \mu_{11}, \quad A^{(0)}_{66} = \eta_{33}\zeta^2 + \eta_{11}.$$

Recall, that we assume $\frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} = \alpha$.

From (3.3), (B.1)–(B.3), (6.13), and Theorem 3.3 it follows that

$$-\frac{1}{2}I + \sigma^+_{\mathcal{K}} = \lim_{x_3\to 0}\frac{1}{2\pi}\int_{l^+}\mathcal{T}^{(0)}(\widetilde{\zeta}, n)\big[A^{(0)}(\widetilde{\zeta})\big]^{-1}e^{-i\zeta x_3}d\zeta$$
$$= \frac{1}{2\pi}\int_{l^+}\mathcal{T}^{(0)}(\widetilde{\zeta}, n)\big[A^{(0)}(\widetilde{\zeta})\big]^{-1}d\zeta = \|A_{kj}\|_{6\times 6}, \tag{B.4}$$

where

$$A_{11} = \frac{i}{2\pi}\int_{l^+}\frac{c_{44}\varkappa_{33}\zeta^3 + (c_{44}\varkappa_{11} + \tilde{e}^2_{14})\zeta}{P_1(\zeta)}d\zeta,$$
$$A_{14} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{66}e_{14}}{P_1(\zeta)}d\zeta - \frac{i}{2\pi}\int_{l^+}\frac{e_{14}q^2_{15}\zeta^2}{Q(\zeta)}d\zeta$$
$$A_{15} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{66}q_{15}}{\alpha P_1(\zeta)}d\zeta - \frac{i}{2\pi}\int_{l^+}\frac{e^2_{14}q_{15}\zeta^2}{Q(\zeta)}d\zeta$$
$$A_{1j} = 0, \quad j = 2, 3, 6, \quad A_{22} = \frac{i}{2\pi}\int_{l^+}\frac{c_{33}c_{44}\zeta^3 - c_{13}c_{44}\zeta}{P_2(\zeta)}d\zeta,$$
$$A_{23} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{13}c_{44}\zeta^2 - c_{11}c_{44}}{P_2(\zeta)}d\zeta, \quad A_{2j} = 0, \quad j = 1, 4, 5, 6,$$
$$A_{3j} = 0, \quad j = 1, 4, 5, 6, \quad A_{32} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{33}c_{44}\zeta^2 - c_{13}c_{44}}{P_2(\zeta)}d\zeta,$$
$$A_{33} = \frac{i}{2\pi}\int_{l^+}\frac{c_{33}c_{44}\zeta^3 + (c_{11}c_{33} - c_{13}c_{44} - c^2_{13})\zeta}{P_2(\zeta)}d\zeta,$$
$$A_{41} = -\frac{i}{2\pi}\int_{l^+}\frac{e_{14}\varkappa_{33}\zeta^2}{P_1(\zeta)}d\zeta, \quad A_{4j} = 0, \quad j = 2, 3, 5, 6,$$
$$A_{44} = \frac{i}{2\pi}\int_{l^+}\frac{c_{44}\varkappa_{33}\zeta^3 + c_{66}\varkappa_{33}\zeta}{P_1(\zeta)}d\zeta + \frac{i}{2\pi}\int_{l^+}\frac{\varkappa_{33}q^2_{15}\zeta^3}{Q(\zeta)}d\zeta,$$
$$A_{51} = -\frac{i}{2\pi}\int_{l^+}\frac{q_{15}\varkappa_{33}\zeta^2}{P_1(\zeta)}d\zeta, \quad A_{5j} = 0, \quad j = 2, 3, 4, 6$$
$$A_{55} = \frac{i}{2\pi}\int_{l^+}\frac{c_{44}\varkappa_{33}\zeta^3 + c_{66}\varkappa_{33}\zeta}{P_1(\zeta)}d\zeta + \frac{i}{2\pi}\int_{l^+}\frac{\alpha\varkappa_{33}e^2_{14}\zeta^3}{Q(\zeta)}d\zeta,$$
$$A_{66} = \frac{i}{2\pi}\int_{l^+}\frac{\eta_{33}\zeta}{\eta_{33}\zeta^2 + \eta_{11}}d\zeta, \quad A_{6j} = 0, \quad j = \overline{1, 5},$$
$$P_1(\zeta) = c_{44}\varkappa_{33}\zeta^4 + (c_{44}\varkappa_{11} + c_{66}\varkappa_{33} + \widetilde{e}^2_{14})\zeta^2 + c_{66}\varkappa_{11},$$
$$P_2(\zeta) = c_{33}c_{44}\zeta^4 + (c_{11}c_{33} - 2c_{13}c_{44} - c^2_{13})\zeta^2 + c_{11}c_{44},$$
$$Q(\zeta) = \alpha(\varkappa_{33}\zeta^2 + \varkappa_{11})P_1(\zeta), \quad \widetilde{e}_{14} = \left(e^2_{14} + \alpha^{-1}q^2_{15}\right)^{1/2}.$$

Denote by $\zeta^{(j)}_1$, $\zeta^{(j)}_2$, $j = 1, 2$, the roots of the polynomials P_j with positive imaginary part. Evidently, $\zeta^{(1)}_1$, $\zeta^{(1)}_2$ and $\zeta^{(3)} = i\sqrt{\eta_{11}/\eta_{33}}$ are then the roots of $Q(\zeta)$ with positive imaginary parts.

We have the following explicit formulas,

$$\zeta_1^{(1)} = ib_1 = i\sqrt{\frac{A-\sqrt{B}}{2c_{44}\varkappa_{33}}}, \qquad \zeta_2^{(1)} = ib_2 = i\sqrt{\frac{A+\sqrt{B}}{2c_{44}\varkappa_{33}}},$$
$$\zeta_1^{(2)} = id_1 = i\sqrt{\frac{C-\sqrt{D}}{2c_{44}c_{33}}}, \qquad \zeta_2^{(2)} = id_2 = i\sqrt{\frac{C+\sqrt{D}}{2c_{44}c_{33}}},$$
$$A = \widetilde{e}_{14}^2 + c_{44}\varkappa_{11} + c_{66}\varkappa_{33} > 0, \qquad B = A^2 - 4c_{44}c_{66}\varkappa_{11}\varkappa_{33} > 0,$$
$$C = c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}, \qquad D = C^2 - 4c_{44}^2 c_{33}c_{11}.$$

Note that, if $D > 0$, then the roots $\zeta_1^{(2)}$ and $\zeta_2^{(2)}$ are purely imaginary. For $D < 0$ the roots are complex numbers with opposite real parts and equal imaginary parts:

$$\zeta_1^{(2)} = a + i\,b, \qquad \zeta_2^{(2)} = -a + i\,b, \qquad a > 0, \qquad b > 0.$$

Curvilinear integrals participating in (B.4) can be calculated explicitly by applying theory of residues and Cauchy's theorem

$$\int_{l^+} \frac{d\zeta}{P_1(\zeta)} = \frac{i\pi(\zeta_2^{(1)} - \zeta_1^{(1)})}{\zeta_1^{(1)}\zeta_2^{(1)}\sqrt{B}}, \qquad \int_{l^+} \frac{d\zeta}{P_2(\zeta)} = \frac{i\pi(\zeta_2^{(2)} - \zeta_1^{(2)})}{\zeta_1^{(2)}\zeta_2^{(2)}\sqrt{D}},$$
$$\int_{l^+} \frac{\zeta}{P_1(\zeta)}d\zeta = 0, \qquad \int_{l^+} \frac{\zeta}{P_2(\zeta)}d\zeta = 0,$$
$$\int_{l^+} \frac{\zeta^2}{P_1(\zeta)}d\zeta = -\frac{i\pi}{\sqrt{B}}(\zeta_2^{(1)} - \zeta_1^{(1)}), \qquad \int_{l^+} \frac{\zeta^2}{P_2(\zeta)}d\zeta = -\frac{i\pi}{\sqrt{D}}(\zeta_2^{(2)} - \zeta_1^{(2)}),$$
$$\int_{l^+} \frac{\zeta^3}{P_1(\zeta)}d\zeta = \frac{i\pi}{c_{44}\varkappa_{33}}, \qquad \int_{l^+} \frac{\zeta^3}{P_2(\zeta)}d\zeta = \frac{i\pi}{c_{44}c_{33}},$$
$$\int_{l^+} \frac{\zeta^2}{Q(\zeta)}d\zeta = \frac{\pi}{\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2 + \varkappa_{11})}{\sqrt{B}}\right],$$

$$\int_{l^+} \frac{\zeta^3}{Q(\zeta)}d\zeta = 2\pi i\sum_{k=1}^{3}\frac{\zeta^3}{Q'(\zeta)}\bigg|_{\zeta=\zeta_k}$$
$$= -\frac{\pi i}{\alpha}\bigg(\frac{b_1^2}{(\varkappa_{11} - \varkappa_{33}b_1^2)(c_{44}\varkappa_{11} + c_{66}\varkappa_{33} + \widetilde{e}_{14}^2 - 2c_{44}\varkappa_{33}b_1^2)}$$
$$+ \frac{b_2^2}{(\varkappa_{11} - \varkappa_{33}b_2^2)(c_{44}\varkappa_{11} + c_{66}\varkappa_{33} + \widetilde{e}_{14}^2 - 2c_{44}\varkappa_{33}b_2^2)} + \frac{b_3^2}{\varkappa_{33}P_1(ib_3)}\bigg).$$

Note, that the last equality implies that the integrals

$$\frac{i}{2\pi}\int_{l^+} \frac{\varkappa_{33}q_{15}^2\zeta^3}{Q(\zeta)}d\zeta \quad \text{and} \quad \frac{i}{2\pi}\int_{l^+} \frac{\alpha\varkappa_{33}e_{14}^2\zeta^3}{Q(\zeta)}d\zeta$$

which are involved in A_{44} and A_{55} are real, therefore due to Lemma 6.1 they must be zero.

As a result we obtain

$$A_{jj} = -\frac{1}{2}, \quad j = \overline{1,6}, \quad A_{1j} = 0, \quad j = 2,3,6,$$
$$A_{14} = \frac{e_{14}c_{66}(\zeta_2^{(1)} - \zeta_1^{(1)})}{2\zeta_1^{(1)}\zeta_2^{(1)}\sqrt{B}} - \frac{e_{14}q_{15}^2}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[i\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(\zeta_2 - \zeta_1)(-\varkappa_{33}\,\zeta_1\zeta_2 + \varkappa_{11})}{\sqrt{B}}\right],$$
$$A_{15} = \frac{q_{15}c_{66}(\zeta_2^{(1)} - \zeta_1^{(1)})}{2\alpha\zeta_1^{(1)}\zeta_2^{(1)}\sqrt{B}} - \frac{e_{14}^2q_{15}}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[i\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(\zeta_2 - \zeta_1)(-\varkappa_{33}\,\zeta_1\zeta_2 + \varkappa_{11})}{\sqrt{B}}\right],$$
$$A_{2j} = 0, \quad j = 1,4,5,6, \quad A_{23} = -\frac{c_{44}(\zeta_2^{(2)} - \zeta_1^{(2)})(c_{11} + c_{13}\zeta_1^{(2)}\zeta_2^{(2)})}{2\zeta_1^{(2)}\zeta_2^{(2)}\sqrt{D}},$$

$$A_{3j}=0,\quad j=1,4,5,6,\quad A_{32}=-\frac{c_{44}(\zeta_2^{(2)}-\zeta_1^{(2)})(c_{33}\zeta_1^{(2)}\zeta_2^{(2)}+c_{13})}{2\zeta_1^{(2)}\zeta_2^{(2)}\sqrt{D}},$$
$$A_{41}=-\frac{e_{14}\varkappa_{33}(\zeta_2^{(1)}-\zeta_1^{(1)})}{2\sqrt{B}},\quad A_{4j}=0,\quad j=2,3,5,6,$$
$$A_{51}=-\frac{q_{15}\varkappa_{33}(\zeta_2^{(1)}-\zeta_1^{(1)})}{2\sqrt{B}},\quad A_{5j}=0,\quad j=2,3,4,6,$$
$$A_{6j}=0,\quad j=\overline{1,5}.$$

Now, taking into account that

$$\zeta_j^{(1)}=i\,b_j,\quad b_j>0,\quad j=1,2,$$
$$\zeta_j^{(2)}=i\,d_j,\quad d_j>0,\quad j=1,2,\quad \text{if } D>0,$$
$$\zeta_1^{(2)}=a+i\,b,\quad \zeta_2^{(2)}=-a+i\,b,\quad a>0,\quad b>0,\quad \zeta_1^{(2)}\zeta_2^{(2)}=-\sqrt{\frac{c_{11}}{c_{33}}},\quad \text{if } D<0,$$

we obtain (6.17)–(6.22).

One can calculate the homogeneous symbol matrix $\sigma_{\mathcal{K}}^{+}=\sigma_{\mathcal{K}}(x_1,0,+1)$ corresponding to the system (6.23) quite similarly.

Now we prove that

$$A_{14}A_{41}+A_{15}A_{51}<0. \tag{B.5}$$

In view of the inequalities (6.24) and the relation

$$A_{14}A_{41}+A_{15}A_{51}=-\frac{c_{66}\varkappa_{33}(b_2-b_1)^2\widetilde{e}_{14}^{\,2}}{4Bb_1b_2}-\frac{1}{\alpha}\frac{e_{14}^2q_{15}^2\varkappa_{33}(b_2-b_1)}{2\sqrt{B}\varkappa_{11}\widetilde{e}_{14}^{\,2}}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}b_1b_2+\varkappa_{11})}{\sqrt{B}}\right],$$

and since

$$b_2-b_1>0,\quad b_1>0,\quad B>0,$$

it is sufficient to show that

$$\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}b_1b_2+\varkappa_{11})}{\sqrt{B}}>0. \tag{B.6}$$

Rewrite this inequality as

$$\varkappa_{11}B>c_{44}^2\varkappa_{33}(b_2-b_1)^2(\varkappa_{33}b_1b_2+\varkappa_{11})^2. \tag{B.7}$$

Taking into account the equalities

$$(b_2-b_1)^2=\tfrac{A}{c_{44}\varkappa_{33}}-2\sqrt{\tfrac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}},\quad b_1b_2=\sqrt{\tfrac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}},\quad B=A^2-4c_{66}c_{44}\varkappa_{11}\varkappa_{33},$$

we find that (B.7) is equivalent to the relation

$$\varkappa_{11}\Big(A^2-4c_{44}c_{66}\varkappa_{11}\varkappa_{33}\Big)>c_{44}\Big(A-2\sqrt{c_{44}c_{66}\varkappa_{11}\varkappa_{33}}\Big)\left(\varkappa_{33}\sqrt{\frac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}}+\varkappa_{11}\right)^2.$$

In turn the last inequality is equivalent to the following one

$$\varkappa_{11}\Big(A+2\sqrt{c_{44}c_{66}\varkappa_{11}\varkappa_{33}}\Big)>c_{44}\left(\varkappa_{33}\sqrt{\frac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}}+\varkappa_{11}\right)^2.$$

At last, substituting here $A = \widetilde{e}_{14}^2 + c_{44}\varkappa_{11} + c_{66}\varkappa_{33}$ we arrive at the evident inequality

$$\widetilde{e}_{14}^2 + \left(\sqrt{c_{44}\varkappa_{11}} + \sqrt{c_{44}\varkappa_{11}}\right)^2 > \left(\sqrt{c_{44}\varkappa_{11}} + \sqrt{c_{44}\varkappa_{11}}\right)^2.$$

Thus (B.6) is valid and consequently (B.5) holds as well.

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