


Singular localised boundary-domain integral equations of acoustic scattering by inhomogeneous anisotropic obstacle

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We consider the time-harmonic acoustic wave scattering by a bounded *anisotropic inhomogeneity* embedded in an unbounded *anisotropic* homogeneous medium. The material parameters may have discontinuities across the interface between the inhomogeneous interior and homogeneous exterior regions. The corresponding mathematical problem is formulated as a transmission problems for a second-order elliptic partial differential equation of Helmholtz type with discontinuous variable coefficients. Using a localised quasi-parametrix based on the harmonic fundamental solution, the transmission problem for arbitrary values of the frequency parameter is reduced equivalently to a system of *singular localised boundary-domain integral equations*. Fredholm properties of the corresponding *localised boundary-domain integral operator* are studied and its invertibility is established in appropriate Sobolev-Slobodetskii (Bessel potential) spaces, which implies existence and uniqueness results for the localised boundary-domain integral equations system and the corresponding acoustic scattering transmission problem.

KEYWORDS

acoustic scattering, localised parametrix, localised boundary-domain integral equations, partial differential equations, pseudodifferential equations, transmission problems

1 | INTRODUCTION

We consider the time-harmonic acoustic wave scattering by a bounded *anisotropic inhomogeneous obstacle* embedded in an unbounded *anisotropic* homogeneous medium. We assume that the material parameters and speed of sound are functions of position within the inhomogeneous bounded obstacle. The physical model problem with a frequency parameter $\omega \in \mathbb{R}$ is formulated mathematically as a transmission problem for a second-order elliptic partial differential equation with variable coefficients $A_2(x, \partial_x)u(x) \equiv \partial_{x_k}(a_{kj}^{(2)}(x)\partial_{x_j}u(x)) + \omega^2 \kappa_2(x)u(x) = f_2$ in the *inhomogeneous anisotropic* bounded region $\Omega^+ \subset \mathbb{R}^3$ and for a Helmholtz type equation with constant coefficients $A_1(\partial_x)u(x) \equiv a_{kj}^{(1)}\partial_{x_k}\partial_{x_j}u(x) + \omega^2 \kappa_1 u(x) = f_1$ in the *homogeneous anisotropic* unbounded region $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. The material parameters $a_{kj}^{(q)}$ and κ_q are not assumed

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to be continuous across the interface $S = \partial\Omega^- = \partial\Omega^+$ between the inhomogeneous interior and homogeneous exterior regions. The transmission conditions are assumed on the interface, relating the interior and exterior traces of the wave amplitude u and its co-normal derivative on S .

The transmission problems for the Helmholtz equation, ie, when $A_2(x, \partial) = A_1(\partial) = \Delta + \omega^2$, which corresponds to a *homogeneous isotropic media*, are well studied in the case of smooth and Lipschitz interface (see Costabel and Stephan,¹ Kleinman and Martin,² Kress and Roach,³ Torres and Welland,⁴ and the references therein).

The *special isotropic transmission problems* when $A_2(x, \partial_x) = \Delta + \omega^2 \kappa_2(x)$ and $A_1(\partial_x) = \Delta + \omega^2$ is the Helmholtz operator are also well presented in the literature (see Colton and Kress,⁵ Nédélec,⁶ and the references therein). The acoustic scattering problem in the whole space corresponding to a more general isotropic case, when $a_{kj}^{(2)}(x) = a(x) \delta_{kj}$, where δ_{kj} is Kronecker delta and $A_1(\partial_x) = \Delta + \omega^2$, was analysed by the indirect boundary-domain integral equation method by Werner.^{7,8} Applying the potential method based on the Helmholtz fundamental solution, Werner reduced the problem to the *Fredholm-Riesz type integral equations* system and proved its unique solvability. The same problem by the direct method was considered by Martin,⁹ where the problem was reduced to a singular integro-differential equation in the inhomogeneous bounded region Ω^+ . Using the uniqueness and existence results obtained by Werner,^{7,8} the equivalence of the integro-differential equation to the initial transmission problem and its unique solvability were shown for special type right-hand side functions associated with Green's third formula.

Note that the wave scattering problems for the general inhomogeneous anisotropic case described above can be studied by the variational method incorporated with the nonlocal approach and also by the classical potential method when the corresponding fundamental solution is available in an explicit form. However, fundamental solutions for second-order elliptic partial differential equations with variable coefficients are not available in explicit form, in general. Application of the potential method based on the corresponding Levi function, which always can be constructed explicitly, leads to Fredholm-Riesz type integral equations but invertibility of the corresponding integral operators can be proved only for particular cases (see Miranda¹⁰).

Our goal here is to show that the acoustic transmission problems for anisotropic heterogeneous structures can be equivalently reformulated as systems of singular *localised boundary-domain integral equations* (LBDIEs) with the help of a *localised harmonic parametrix* based on the harmonic fundamental solution, which is a *quasi-parametrix* for the considered PDEs of acoustics, and to prove that the corresponding singular *localised boundary-domain integral operators* (LBDIO) are invertible for an arbitrary value of the frequency parameter. Beside a pure mathematical interest, these results seem to be important from the point of view of applications, since LBDIE system can be applied in constructing convenient numerical algorithms (cf Mikhailov¹¹, Zhu et al^{12,13} and Sladek et al¹⁴). The main novelty of the paper is in application of the singular localised boundary-domain integral equations method to the problem of acoustic transmission through a penetrable, anisotropic, inhomogeneous obstacle.

The paper is organised as follows. First, after mathematical formulation of the problem, we introduce layer and volume potentials based on a localised harmonic parametrix and derive basic integral relations in bounded inhomogeneous and unbounded homogeneous anisotropic regions. Then we reduce the transmission problem under consideration to the localised boundary-domain singular integral equations system and prove the equivalence theorem for arbitrary values of the frequency parameter, which plays a crucial role in our analysis. Afterwards, applying the Vishik-Eskin approach, we investigate Fredholm properties of the corresponding matrix LBDIO, containing singular integral operators over the interface surface and the bounded region occupied by the inhomogeneous obstacle, and prove invertibility of the LBDIO in appropriate Sobolev-Slobodetskii (Bessel potential spaces). This invertibility property implies then, in particular, existence and uniqueness results for the LBDIE system and the corresponding original transmission problem.

Next, we analyse also an alternative nonlocal approach based on coupling of variational and boundary integral equation methods, which reduces the transmission problem for unbounded composite structure to the variational equation containing a coercive sesquilinear form, which lives on the bounded inhomogeneous region and the interface manifold. Both approaches presented in the paper can be applied in the study of similar wave scattering problems for multilayer piecewise inhomogeneous anisotropic structures.

Finally, for the readers convenience, we collected necessary auxiliary material related to classes of localising functions, properties of localised potentials and anisotropic radiating potentials in three brief appendices.

2 | FORMULATION OF THE TRANSMISSION PROBLEM

Let $\Omega^+ = \Omega_2$ be a bounded domain in \mathbb{R}^3 with a simply connected boundary $\partial\Omega_2 = S$, and $\Omega^- = \Omega_1 := \mathbb{R}^3 \setminus \overline{\Omega_2}$. For simplicity, we assume that $S \in C^\infty$ if not otherwise stated. Throughout the paper, $n = (n_1, n_2, n_3)$ denotes the unit normal vector to S directed outward the domain Ω_2 .

We assume that the propagation region of a time harmonic acoustic wave u^{tot} is the whole space \mathbb{R}^3 that consists of an inhomogeneous part Ω_2 and a homogeneous part Ω_1 . Acoustic wave propagation is governed by the uniformly elliptic second-order scalar partial differential equation

$$A u^{tot}(x) \equiv \partial_k (a_{kj}(x) \partial_j u^{tot}(x)) + \omega^2 \kappa(x) u^{tot}(x) = f(x), \quad x \in \Omega_2 \cup \Omega_1, \quad (1)$$

where $\partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$, $a_{kj}(x) = a_{jk}(x)$, and $\kappa(x)$ are real-valued functions, $\omega \in \mathbb{R}$ is a frequency parameter, while $f \in L_{2,comp}(\mathbb{R}^3)$ is the volume force amplitude. Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed.

Note that in the mathematical model of an inhomogeneous absorbing medium, the function κ is complex valued, with nonzero real and imaginary parts, in general (see, eg, Colton and Kress,⁵ chapter 8). Here, we treat only the case when the κ is a real-valued function, but it should be mentioned that the complex-valued case can be also considered by the approach developed here.

In our further analysis, it is assumed that the real-valued variable coefficients a_{kj} and κ are constant in the homogeneous unbounded region Ω_1 and the following relations hold:

$$a_{kj}(x) = a_{jk}(x) = \begin{cases} a_{kj}^{(1)} & \text{for } x \in \Omega_1, \\ a_{kj}^{(2)}(x) & \text{for } x \in \Omega_2, \end{cases} \quad \kappa(x) = \begin{cases} \kappa_1 > 0 & \text{for } x \in \Omega_1, \\ \kappa_2(x) > 0 & \text{for } x \in \Omega_2, \end{cases} \quad (2)$$

where $a_{kj}^{(1)}$ and κ_1 are constants, while $a_{kj}^{(2)}$ and κ_2 are smooth function in $\overline{\Omega_2}$,

$$a_{kj}^{(2)}, \kappa_2 \in C^2(\overline{\Omega_2}), \quad j, k = 1, 2, 3. \quad (3)$$

Moreover, the matrices $\mathbf{a}_q = [a_{kj}^{(q)}]_{k,j=1}^3$ are uniformly positive definite, ie, there are positive constants c_1 and c_2 such that

$$c_1 |\xi|^2 \leq a_{kj}^{(q)}(x) \xi_k \xi_j \leq c_2 |\xi|^2 \quad \forall x \in \overline{\Omega_q}, \quad \forall \xi \in \mathbb{R}^3, \quad q = 1, 2. \quad (4)$$

We do not assume that the coefficients a_{kj} and κ are continuous across S in general, ie, the case $a_{kj}^{(2)}(x) \neq a_{kj}^{(1)}$ and $\kappa_2(x) \neq \kappa_1$ for $x \in S$ is covered by our analysis. Further, let us denote

$$A_1 v(x) := a_{kj}^{(1)} \partial_{x_k} \partial_{x_j} v(x) + \omega^2 \kappa_1 v(x) \quad \text{for } x \in \Omega_1, \quad (5)$$

$$A_2 v(x) := \partial_{x_k} (a_{kj}^{(2)}(x) \partial_{x_j} v(x)) + \omega^2 \kappa_2(x) v(x) \quad \text{for } x \in \Omega_2.$$

For a function v sufficiently smooth in Ω_1 and Ω_2 , the classical co-normal derivative operators, T_{cq}^\pm are well defined as

$$T_{cq}^\pm v(x) := a_{kj}^{(q)} n_k(x) \gamma^\pm(\partial_{x_j} v(x)), \quad x \in S, \quad q = 1, 2; \quad (6)$$

here, the symbols γ^+ and γ^- denote one-sided boundary trace operators on S from the interior and exterior domains, respectively. Their continuous right inverse operators, which are nonuniquely defined, are denoted by symbols $(\gamma^\pm)^{-1}$.

By $H^s(\Omega) = H_2^s(\Omega)$, $H_{loc}^s(\Omega) = H_{2,loc}^s(\Omega)$, $H_{comp}^s(\Omega) = H_{2,comp}^s(\Omega)$, and $H^s(S) = H_2^s(S)$, $s \in \mathbb{R}$, we denote the L_2 -based Bessel potential spaces on an open domain $\Omega \subset \mathbb{R}^3$ and on a closed manifold S without boundary, while $\mathcal{D}(\Omega)$ stands for the space of infinitely differentiable test functions with support in Ω . Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω . Let the symbol r_Ω denote the restriction operator onto Ω .

Since the boundary traces of gradients, $\gamma^\pm(\partial_{x_j} v(x))$ are generally not well defined on functions from $H^1(\Omega_q)$, the classical co-normal derivatives (6) are not well defined on such functions either, cf Mikhailov,¹⁴ Appendix A, where an example of such function, for which the classical co-normal derivative exists at no boundary point. Let us introduce the following subspaces of $H^1(\Omega_2)$ and $H_{loc}^1(\Omega_1)$ to which the classical co-normal derivatives can be continuously extended, cf, eg, Grisvard,¹⁵ Costabel,¹⁶ and Mikhailov¹⁷:

$$H^{1,0}(\Omega_2; A_2) := \{v \in H^1(\Omega_2) : A_2 v \in H^0(\Omega_2)\}, \quad H_{loc}^{1,0}(\Omega_1; A_1) := \{v \in H_{loc}^1(\Omega_1) : A_1 v \in H_{loc}^0(\Omega_1)\}.$$

We will also use the corresponding spaces with the Laplace operator Δ instead of A_q .

Motivated by the first Green identity well known for smooth functions, the classical co-normal derivative operators (6) can be extended by continuity to functions from the spaces $H_{loc}^{1,0}(\Omega_1; A_1)$ and $H^{1,0}(\Omega_2; A_2)$ giving the *canonical* co-normal derivative operators, T_1^\pm and T_2^\pm , defined in the weak form as

$$\begin{aligned} \langle T_q^+ u, g \rangle_S &:= \int_{\Omega_2} [a_{kj}^{(q)}(x) [\partial_j u(x)] \partial_k (\gamma^+)^{-1} g(x) - \omega^2 \kappa_q(x) u(x) (\gamma^+)^{-1} g(x)] dx \\ &\quad + \int_{\Omega_q} [A_q u(x)] (\gamma^+)^{-1} g(x) dx, \quad u \in H^{1,0}(\Omega_2; A_q), \quad \forall g \in H^{\frac{1}{2}}(S), \end{aligned} \tag{7}$$

$$\begin{aligned} \langle T_1^- u, g \rangle_S &:= - \int_{\Omega_1} [a_{kj}^{(1)} [\partial_j u(x)] \partial_k (\gamma^-)^{-1} g(x) - \omega^2 \kappa_1 u(x) (\gamma^-)^{-1} g(x)] dx \\ &\quad - \int_{\Omega_1} [A_1 u(x)] (\gamma^-)^{-1} g(x) dx, \quad u \in H_{loc}^{1,0}(\Omega_1; A_1), \quad \forall g \in H^{\frac{1}{2}}(S), \end{aligned} \tag{8}$$

where $(\gamma^+)^{-1} : H^{\frac{1}{2}}(S) \rightarrow H^1(\Omega_2)$ and $(\gamma^-)^{-1} : H^{\frac{1}{2}}(S) \rightarrow H_{comp}^1(\Omega_1)$ are the right inverse operators to the trace operators γ^\pm , and the angular brackets $\langle \cdot, \cdot \rangle_S$ should be understood as duality pairing of $H^{-\frac{1}{2}}(S)$ with $H^{\frac{1}{2}}(S)$, which extends the usual bilinear $L_2(S)$ inner product.

The canonical co-normal derivatives $T_2^- u$ and $T_1^+ u$ can be defined analogously for functions from the spaces $H_{loc}^{1,0}(\Omega_1; A_2)$ and $H^{1,0}(\Omega_2; A_1)$, respectively, provided that the variable coefficients $a_{kj}^{(2)}(x)$ and $\kappa_2(x)$ are continuously extended from Ω_2 to the whole space \mathbb{R}^3 preserving the smoothness. It is evident that for functions from the space $H^2(\Omega_2)$ and $H_{loc}^2(\Omega_1)$, the classical and canonical co-normal derivative operators coincide. Concerning the *canonical* and *generalised* co-normal derivatives in wider functional spaces, see Mikhailov.¹⁷

For two times continuously differentiable function w in a neighbourhood of S , we employ also the notation $T_q(x, \partial_x)w := a_{kj}^{(q)} n_k(x) (\partial_{x_j} w(x))$, $x \in S$, to denote both the classical and the canonical co-normal derivatives.

Recall that the definitions of the co-normal derivatives T_q^\pm do not depend on the choice of the right inverse operators $(\gamma^\pm)^{-1}$, and the following Green's first and second identities hold (cf Mikhailov,¹⁷ Theorem 3.9),

$$\langle T_q^+ u, \gamma^+ v \rangle_S = \int_{\Omega_2} [a_{kj}^{(q)} \partial_j u \partial_k v - \omega^2 \kappa_q u v] dx + \int_{\Omega_2} v A_q u dx, \quad u \in H^{1,0}(\Omega_2; A_q), \quad v \in H^1(\Omega_2), \quad q = 1, 2, \tag{9}$$

$$\langle T_2^+ u, \gamma^+ v \rangle_S - \langle T_2^+ v, \gamma^+ u \rangle_S = \int_{\Omega_2} [v A_2 u - u A_2 v] dx, \quad u, v \in H^{1,0}(\Omega_2; A_2),$$

$$\langle T_1^- u, \gamma^- v \rangle_S = - \int_{\Omega_1} [a_{kj}^{(1)} \partial_j u \partial_k v - \omega^2 \kappa_1 u v] dx - \int_{\Omega_1} v A_1 u dx, \quad u \in H_{loc}^{1,0}(\Omega_1; A_1), \quad v \in H_{comp}^1(\Omega_1). \tag{10}$$

By $Z(\Omega_1)$, we denote a subclass of complex-valued functions from $H_{loc}^1(\Omega_1)$ satisfying the Sommerfeld radiation conditions at infinity (see Vekua¹⁸ and Colton and Kress⁵ for the Helmholtz operator and Vainberg¹⁹ and Jentsch et al²⁰ for the ‘anisotropic’ operator A_1 defined by (5)). Denote by S_ω the characteristic surface (ellipsoid) associated with the operator A_1 ,

$$a_{kj}^{(1)} \xi_k \xi_j - \omega^2 \kappa_1 = 0, \quad \xi \in \mathbb{R}^3.$$

For an arbitrary vector $\eta \in \mathbb{R}^3$ with $|\eta| = 1$, there exists only one point $\xi(\eta) \in S_\omega$ such that the outward unit normal vector $n(\xi(\eta))$ to S_ω at the point $\xi(\eta)$ has the same direction as η , ie, $n(\xi(\eta)) = \eta$. Note that $\xi(-\eta) = -\xi(\eta) \in S_\omega$ and $n(-\xi(\eta)) = -\eta$. It can easily be verified that

$$\xi(\eta) = \omega \kappa_1^{1/2} (\mathbf{a}_1^{-1} \eta \cdot \eta)^{-1/2} \mathbf{a}_1^{-1} \eta, \tag{11}$$

where \mathbf{a}_1^{-1} is the matrix inverse to $\mathbf{a}_1 := [a_{kj}^{(1)}]_{k,j=1}^3$.

Definition 1. A complex-valued function v belongs to the class $Z(\Omega_1)$ if there exists a ball $B(R)$ of radius R centred at the origin such that $v \in C^1(\Omega_1 \setminus B(R))$ and v satisfies the Sommerfeld radiation conditions associated with the operator $A_1(\partial)$ for sufficiently large $|x|$,

$$v(x) = \mathcal{O}(|x|^{-1}), \quad \partial_k v(x) - i\xi_k(\eta)v(x) = \mathcal{O}(|x|^{-2}), \quad k = 1, 2, 3, \tag{12}$$

where $\xi(\eta) \in S_\omega$ corresponds to the vector $\eta = x/|x|$ (ie, $\xi(\eta)$ is given by (11) with $\eta = x/|x|$).

Note that due to the ellipticity of the operator $A_1(\partial_x)$, any solution to the constant coefficient homogeneous equation $A_1(\partial_x)v(x) = 0$ in an open region $\Omega \subset \mathbb{R}^3$ is a real analytic function of x in Ω .

Conditions (12) are equivalent to the classical Sommerfeld radiation conditions for the Helmholtz equation if $A_1(\partial) = \Delta(\partial) + \omega^2$, ie, if $\kappa_1 = 1$ and $a_{kj}^{(1)} = \delta_{kj}$, where δ_{kj} is the Kronecker delta. There holds the following analogue of the classical Rellich-Vekua lemma (for details, see Jentsch et al²⁰ and Natroshvili et al²¹).

Lemma 1. Let $v \in Z(\Omega_1)$ be a solution of the equation $A_1(\partial_x)v = 0$ in Ω_1 and let

$$\lim_{R \rightarrow +\infty} \operatorname{Im} \left\{ \int_{\Sigma_R} \overline{v(x)} T_1 v(x) d\Sigma_R \right\} = 0, \tag{13}$$

where Σ_R is the sphere with radius R centred at the origin. Then $v = 0$ in Ω_1 .

Remark 1. For $x \in \Sigma_R$ and $\eta = x/|x|$, we have $n(x) = \eta$, and in view of (6) and (12) for a function $v \in Z(\Omega_1)$, we get

$$T_1(x, \partial_x)v(x) = a_{kj}^{(1)} n_k(x) [i\xi_j(\eta)v(x)] + \mathcal{O}(|x|^{-2}) = ia_{kj}^{(1)} \eta_k \xi_j(\eta)v(x) + \mathcal{O}(|x|^{-2}).$$

Therefore, by (11) and the symmetry condition $a_{kj} = a_{jk}$, we arrive at the relation

$$\overline{v(x)} T_1 v(x) = i\omega\kappa_1^{1/2} |v(x)|^2 (\mathbf{a}_1^{-1} \eta \cdot \eta)^{-1/2} \mathbf{a}_1 \eta \cdot \mathbf{a}_1^{-1} \eta + \mathcal{O}(|x|^{-3}) = i\omega\kappa_1^{1/2} (\mathbf{a}_1^{-1} \eta \cdot \eta)^{-1/2} |v(x)|^2 + \mathcal{O}(|x|^{-3}),$$

On the other hand, matrix \mathbf{a}_1 is positive definite, cf (4), which implies positive definiteness of the inverse matrix \mathbf{a}_1^{-1} . Hence, there are positive constants δ_0 and δ_1 such that the inequality $0 < \delta_0 \leq (\mathbf{a}_1^{-1} \eta \cdot \eta)^{-1/2} \leq \delta_1 < \infty$ holds for all $\eta \in \Sigma_1$. Consequently, (13) for $\omega \neq 0$ is equivalent to the condition in the well-known Rellich-Vekua lemma in the theory of the Helmholtz equation, Vekua,¹⁸ Rellich,²² and Colton and Kress,⁵

$$\lim_{R \rightarrow +\infty} \int_{\Sigma_R} |v(x)|^2 d\Sigma_R = 0.$$

In the unbounded region Ω_1 , we have a total wave field $u^{tot} = u^{inc} + u^{sc}$, where u^{inc} is a wave motion initiating known incident field and u^{sc} is a radiating unknown scattered field. It is often assumed that the incident field is defined in the whole of \mathbb{R}^3 , being, for example, a corresponding plane wave that solves the homogeneous equation $A_1 u^{inc} = 0$ in \mathbb{R}^3 but does not satisfy the Sommerfeld radiation conditions at infinity. Motivated by relations (2), let us set $u_1(x) := u^{sc}(x)$ for $x \in \Omega_1$ and $u_2(x) := u^{tot}(x)$ for $x \in \Omega_2$.

Now we formulate the transmission problem associated with the time-harmonic acoustic wave scattering by a bounded anisotropic inhomogeneity embedded in an unbounded anisotropic homogeneous medium:

Find complex-valued functions $u_1 \in H_{loc}^{1,0}(\Omega_1, A_1) \cap Z(\Omega_1)$ and $u_2 \in H^{1,0}(\Omega_2, A_2)$ satisfying the differential equations

$$A_1 u_1(x) = f_1(x) \quad \text{for } x \in \Omega_1, \tag{14}$$

$$A_2 u_2(x) = f_2(x) \quad \text{for } x \in \Omega_2, \tag{15}$$

and the transmission conditions on the interface S ,

$$\gamma^+ u_2 - \gamma^- u_1 = \varphi_0 \quad \text{on } S, \tag{16}$$

$$T_2^+ u_2 - T_1^- u_1 = \psi_0 \quad \text{on } S, \tag{17}$$

where

$$f_2 := r_{\Omega_2} f \in H^0(\Omega_2), \quad f_1 := r_{\Omega_1} f \in H_{comp}^0(\Omega_1), \quad f \in H_{comp}^0(\mathbb{R}^3), \quad \varphi_0 \in H^{\frac{1}{2}}(S), \quad \psi_0 \in H^{-\frac{1}{2}}(S). \tag{18}$$

In the above setting, Equations (14) and (15) are understood in the distributional sense, the Dirichlet type transmission condition (16) is understood in the usual trace sense, while the Neumann type transmission condition (16) is understood in the canonical co-normal derivative sense defined by the relations (7) and (8).

If the interface continuity of u^{tot} and its co-normal derivatives is assumed, then $\varphi_0 = \gamma^- u^{inc}$, $\psi_0 = T_1^- u^{inc}$.

Remark 2. If the variable coefficients a_{kj} and the function κ in (1) and (2) belong to $C^2(\mathbb{R}^3)$ and $u^{inc} \in H^2_{loc}(\mathbb{R}^3)$, then conditions (16) and (17) can be reduced to the homogeneous ones by introducing a new unknown function $\tilde{u} := u^{tot} - u^{inc}$ in \mathbb{R}^3 , since $T_1^- u^{inc} = T_2^+ u^{inc}$ on S . For the function \tilde{u} , the above formulated transmission problem is reduced then to the following one:

Find a solution $\tilde{u} \in H^2_{loc}(\mathbb{R}^3) \cap Z(\mathbb{R}^3)$ to the differential equation

$$A\tilde{u}(x) \equiv \partial_{x_k} (a_{kj}(x)\partial_{x_j}\tilde{u}(x)) + \omega^2 \kappa(x)\tilde{u}(x) = \tilde{f}(x), \quad x \in \mathbb{R}^3, \tag{19}$$

where $\tilde{f} := f - Au^{inc} \in H^0_{comp}(\mathbb{R}^3)$ due to the inclusions $f \in H^0_{comp}(\mathbb{R}^3)$ and $Au^{inc} = A_1 u^{inc} = 0$ in Ω_1 .

If $A \equiv \Delta + \omega^2 \kappa(x)$ in \mathbb{R}^3 with κ as in (2), then Equation (19) can be equivalently reduced to the Lippmann-Schwinger type integral equation (see, eg, Colton and Kress,⁵ chapter 8).

In our analysis, even for $C^2(\mathbb{R}^3)$ -smooth coefficients, we always will keep the transmission conditions (16) and (17), which allow us to reduce the problem under consideration to the system of localised boundary-domain integral equations that live on the bounded domain Ω_2 and its boundary S (cf Nédélec,⁶ chapter 2).

Let us prove the uniqueness theorem for the transmission problem.

Theorem 1. *The homogeneous transmission problem (14) - (17) (with $f_1 = 0, f_2 = 0, \varphi_0 = \psi_0 = 0$) possesses only the trivial solution.*

Proof. Denote by $B(R)$ a ball centred at the origin and having radius R , $\Sigma_R := \partial B(R)$. We assume that R is a sufficiently large such that $\bar{\Omega}_2 \subset B(R)$. Let a pair (u_1, u_2) be a solution to the homogeneous transmission problem (14) - (17). Note that $u_1 \in C^\infty(\Omega_1)$ due to ellipticity of the constant coefficient operator A_1 . We can write the first Green identities for the domains Ω_2 and $\Omega_1(R) := \Omega_1 \cap B(R)$ (see (9) and (10)),

$$\int_{\Omega_2} [a_{kj}^{(2)}(x)\partial_j u_2(x)\overline{\partial_k u_2(x)} - \omega^2 \kappa_2(x)|u_2(x)|^2] dx = \langle T_2^+ u_2, \overline{\gamma^+ u_2} \rangle_S, \tag{20}$$

$$\int_{\Omega_1(R)} [a_{kj}^{(1)}\partial_j u_1(x)\overline{\partial_k u_1(x)} - \omega^2 \kappa_1|u_1(x)|^2] dx = -\langle T_1^- u_1, \overline{\gamma^- u_1} \rangle_S + \langle T_1^+ u_1, \overline{\gamma^- u_1} \rangle_{\Sigma_R}. \tag{21}$$

Since the matrices $\mathbf{a}^q = [a_{kj}^{(q)}]_{k,j=1}^3$ are symmetric and positive definite, in view of the homogeneous transmission conditions (16) and (17), after adding (20) and (21) and taking the imaginary part, we get

$$\text{Im} \left\{ \int_{\Sigma_R} \overline{u_1(x)} T_1 u_1(x) d\Sigma_R \right\} = 0.$$

Whence by Lemma 1 we deduce that $u_1 = 0$ in Ω_1 . In view of (16) and (17) then we see that the function u_2 solves the homogeneous Cauchy problem in Ω_2 for the elliptic partial differential equation $A_2 u_2 = 0$ with variable coefficients $a_{kj}^{(2)}$ and κ_2 being $C^2(\bar{\Omega}_2)$ -smooth functions, see (3). By the interior and boundary regularity properties of solutions to elliptic problems, we have $u_2 \in C^2(\bar{\Omega}_2)$ and therefore $u_2 = 0$ in Ω_2 due to the well-known uniqueness theorem for the Cauchy problem (see, eg, Landis,²³ Theorem 3; Calderon,²⁴ Theorem 6). \square

Remark 3. Due to the recent results concerning the Cauchy problem for scalar elliptic operators, one can reduce the smoothness of coefficients $a_{kj}^{(2)}$ and κ_2 to the Lipschitz continuity and require that Ω_2 is a Dini domain, see, eg, Theorem 2.9 in Tao et al.²⁵

3 | REDUCTION TO LBDIE SYSTEM AND EQUIVALENCE THEOREM

3.1 | Integral relations in the nonhomogeneous bounded domain

As it has already been mentioned, our goal is to reduce the above-stated transmission problem to the corresponding system of localised boundary-domain integral equations. To this end, let us define a *localised parametrix* associated with

the fundamental solution $-(4\pi|x|)^{-1}$ of the Laplace operator,

$$P_\chi(x) := -\frac{\chi(x)}{4\pi|x|},$$

where χ is a cut-off function $\chi \in X_+^4$, see Appendix A. Throughout the paper, we assume that this condition is satisfied and χ has a compact support if not otherwise stated.

Let us consider Green's second identity for functions $u_2, v_2 \in H^{1,0}(\Omega_2; A_2)$,

$$\int_{\Omega_2(y,\epsilon)} (v_2 A_2 u_2 - u_2 A_2 v_2) dx = \langle T_2^+ u_2, \gamma^+ v_2 \rangle_{\partial\Omega_2(y,\epsilon)} - \langle \gamma^+ u_2, T_2^+ v_2 \rangle_{\partial\Omega_2(y,\epsilon)},$$

where $\Omega_2(y, \epsilon) := \Omega_2 \setminus B(y, \epsilon)$ with $B(y, \epsilon)$ being a ball centred at the point $y \in \Omega_2$ with radius $\epsilon > 0$. Substituting for $v_2(x)$ the parametrix $P_\chi(x - y)$, by standard limiting arguments as $\epsilon \rightarrow 0$, one can derive Green's third identity for $u \in H^{1,0}(\Omega_2, A_2)$ (cf Chkadua et al²⁶),

$$\beta u_2 + \mathcal{N}_\chi u_2 - V_\chi T_2^+ u_2 + W_\chi \gamma^+ u_2 = \mathcal{P}_\chi A_2 u_2 \quad \text{in } \Omega_2, \tag{22}$$

where

$$\beta(y) = \frac{1}{3} \left[a_{11}^{(2)}(y) + a_{22}^{(2)}(y) + a_{33}^{(2)}(y) \right], \tag{23}$$

\mathcal{N}_χ is a singular localised integral operator that is understood in the Cauchy principal value sense,

$$\mathcal{N}_\chi u_2(y) := \text{v.p.} \int_{\Omega_2} [A_2(x, \partial_x) P_\chi(x - y)] u_2(x) dx = \lim_{\epsilon \rightarrow 0} \int_{\Omega_2(y,\epsilon)} [A_2(x, \partial_x) P_\chi(x - y)] u_2(x) dx, \quad y \in \mathbb{R}^3, \tag{24}$$

V_χ, W_χ , and \mathcal{P}_χ are the localised single layer, double layer, and Newtonian volume potentials, respectively,

$$V_\chi g(y) := - \int_S P_\chi(x - y) g(x) dS_x, \quad W_\chi g(y) := - \int_S [T_2(x, \partial_x) P_\chi(x - y)] g(x) dS_x, \quad y \in \mathbb{R}^3 \setminus S, \tag{25}$$

$$\mathcal{P}_\chi h(y) := \int_{\Omega_2} P_\chi(x - y) h(x) dx, \quad y \in \mathbb{R}^3. \tag{26}$$

Note that if P_χ is replaced with the corresponding fundamental solution, then $\mathcal{N}_\chi u_2 = 0$, $\beta = 1$, and the third Green identity reduces to the familiar integral representation formula.

If the domain of integration in (24) and (26) is the whole space \mathbb{R}^3 , we employ the notation

$$\mathbf{N}_\chi h(y) := \text{v.p.} \int_{\mathbb{R}^3} [A_2(x, \partial_x) P_\chi(x - y)] h(x) dx, \quad \mathbf{P}_\chi h(y) := \int_{\mathbb{R}^3} P_\chi(x - y) h(x) dx, \tag{27}$$

where the operator $A_2(x, \partial_x)$ in the first integral in (27) is assumed to be extended to the whole \mathbb{R}^3 . Some mapping properties of the above potentials needed in our analysis are collected in Appendix B.

In view of the following distributional equality,

$$\frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|} = -\frac{4\pi \delta_{kj}}{3} \delta(x - y) + \text{v.p.} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|},$$

where δ_{kj} is the Kronecker delta and $\delta(\cdot)$ is the Dirac distribution, we have (again in the distributional sense)

$$\begin{aligned}
 A_2(x, \partial_x)P_\chi(x - y) &= a_{kj}^{(2)}(x) \frac{\partial^2 P_\chi(x - y)}{\partial x_k \partial x_j} + \frac{a_{kj}^{(2)}(x)}{\partial x_k} \frac{\partial P_\chi(x - y)}{\partial x_j} + \omega^2 \kappa_2(x) P_\chi(x - y) \\
 &= \beta(x) \delta(x - y) + \text{v.p. } A_2(x, \partial_x)P_\chi(x - y),
 \end{aligned}
 \tag{28}$$

where

$$\begin{aligned}
 \text{v.p. } A_2(x, \partial_x)P_\chi(x - y) &= \text{v.p.} \left[-\frac{a_{kj}^{(2)}(x)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|} \right] + R(x, y) = \text{v.p.} \left[-\frac{a_{kj}^{(2)}(y)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|} \right] + \tilde{R}(x, y), \\
 R(x, y) &:= -\frac{1}{4\pi} \left\{ \frac{\partial}{\partial x_k} \left[\frac{\partial \chi(x - y)}{\partial x_j} \frac{a_{kj}^{(2)}(x)}{|x - y|} \right] + \frac{\partial \left[a_{kj}^{(2)}(x) \chi(x - y) \right]}{\partial x_k} \frac{\partial}{\partial x_j} \frac{1}{|x - y|} \right. \\
 &\quad \left. + a_{kj}^{(2)}(x) [\chi(x - y) - 1] \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|} \right\} + \omega^2 \kappa_2(x) P_\chi(x - y), \\
 \tilde{R}(x, y) &:= R(x, y) - \frac{a_{kj}^{(2)}(x) - a_{kj}^{(2)}(y)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|}.
 \end{aligned}
 \tag{29}$$

Since $\chi(0) = 1$, the functions $R(x, y)$ and $\tilde{R}(x, y)$ possess weak singularities of type $\mathcal{O}(|x - y|^{-2})$ as $x \rightarrow y$. However, the whole term $\text{v.p. } A_2(x, \partial_x)P_\chi(x - y)$ possesses the strong Cauchy singularity as $x \rightarrow y$. Thus, although P_χ is a parametrix for the Laplace operator, it is not a parametrix for the operator A_2 , and we will call it instead a *quasi-parametrix* for A_2 .

It is evident that if $a_{kj}^{(2)}(x) = a_2(x) \delta_{kj}$, then the terms in square brackets in formula (29) vanish and $\text{v.p. } A_2(x, \partial_x)P_\chi(x - y)$ becomes a weakly singular kernel.

Using the integration by parts formula in (24), one can easily derive the following relation for $u_2 \in H^1(\Omega_2)$

$$\mathcal{N}_\chi u_2 = -\beta u_2 - W_\chi \gamma^+ u_2 + \mathcal{Q}_\chi u_2 \quad \text{in } \Omega_2,
 \tag{30}$$

where

$$\mathcal{Q}_\chi u_2(y) := - \int_{\Omega_2} a_{kl}^{(2)}(x) \frac{\partial P_\chi(x - y)}{\partial x_l} \frac{\partial u_2(x)}{\partial x_k} dx = \partial_{y_l} P_\chi (a_{kl}^{(2)} \partial_k u_2)(y), \quad \forall y \in \Omega_2.
 \tag{31}$$

From Green's third identity (22) and Theorem 8, we deduce

$$\beta u_2 + \mathcal{N}_\chi u_2 \in H^{1,0}(\Omega_2, \Delta) \quad \text{for } u_2 \in H^{1,0}(\Omega_2, A_2),
 \tag{32}$$

which, in turn, along with relations (30) and (31) implies

$$\mathcal{Q}_\chi u_2 = \partial_{y_l} P_\chi (a_{kl}^{(2)} \partial_k u_2) \in H^{1,0}(\Omega_2, \Delta) \quad \text{for } u \in H^{1,0}(\Omega_2, A_2).$$

In what follows, in our analysis, we need the explicit expression of the principal homogeneous symbol $\mathfrak{S}_0(\mathbf{N}_\chi; y, \xi)$ of the singular integral operator \mathbf{N}_χ , which due to (28) and (29) reads as

$$\begin{aligned}
 \mathfrak{S}_0(\mathbf{N}_\chi; y, \xi) &= \mathcal{F}_{z \rightarrow \xi} \left(-\text{v.p.} \left[\frac{a_{kl}^{(2)}(y)}{4\pi} \frac{\partial^2}{\partial z_k \partial z_l} \frac{1}{|z|} \right] \right) = -\frac{a_{kl}^{(2)}(y)}{4\pi} \mathcal{F}_{z \rightarrow \xi} \left(\text{v.p.} \left[\frac{\partial^2}{\partial z_k \partial z_l} \frac{1}{|z|} \right] \right) \\
 &= -\frac{a_{kl}^{(2)}(y)}{4\pi} \mathcal{F}_{z \rightarrow \xi} \left[\frac{4\pi \delta_{kl}}{3} \delta(z) + \frac{\partial^2}{\partial z_k \partial z_l} \frac{1}{|z|} \right] = -\beta(y) - a_{kl}^{(2)}(y) (-i \xi_k) (-i \xi_l) \mathcal{F}_{z \rightarrow \xi} \left[\frac{1}{4\pi |z|} \right] \\
 &= -\beta(y) + \frac{a_{kl}^{(2)}(y) \xi_k \xi_l}{|\xi|^2} = \frac{A_2(y, \xi)}{|\xi|^2} - \beta(y), \quad y \in \overline{\Omega_2}, \quad \xi \in \mathbb{R}^3,
 \end{aligned}
 \tag{33}$$

where $A_2(y, \xi) = a_{kl}^{(2)}(y) \xi_k \xi_l$. Here and in what follows, \mathcal{F} and \mathcal{F}^{-1} denote the distributional direct and inverse Fourier transform operators that for a summable function g read as

$$\mathcal{F}_{z \rightarrow \xi}[g] = \int_{\mathbb{R}^n} g(z) e^{iz \cdot \xi} dz, \quad \mathcal{F}_{\xi \rightarrow z}[g] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(\xi) e^{-iz \cdot \xi} d\xi.$$

In derivation of formula (33), we employed that $\mathcal{F}_{z \rightarrow \xi}[(4\pi |z|)^{-1}] = |\xi|^{-2}$ and $\mathcal{F}_{z \rightarrow \xi}[\partial_j g] = -i \xi_j \mathcal{F}_{z \rightarrow \xi}[g]$ for $n = 3$.

Note that the principal homogeneous symbol $\mathfrak{S}_0(\mathbf{N}_\chi; y, \xi)$ is a rational homogeneous even function of order zero in ξ . In view of Theorem 9 in the Appendix, the interior trace of equality (22) on S reads as

$$\mathcal{N}_\chi^+ u_2 - \mathcal{V}_\chi T_2^+ u_2 + [(\beta - \mu)I + \mathcal{W}_\chi] \gamma^+ u_2 = \mathcal{P}_\chi^+ A_2 u_2 \quad \text{on } S, \tag{34}$$

where the functions β and μ are defined by (23) and (B2), $\mathcal{N}_\chi^+ = \gamma^+ \mathcal{N}_\chi$, $\mathcal{P}_\chi^+ = \gamma^+ \mathcal{P}_\chi$, while the operators \mathcal{V}_χ and \mathcal{W}_χ , generated by the direct values of the single and double layer potentials, are given by formulas (B1).

Finally, we formulate a technical lemma that follows from formulas (30), (31), and Theorem 8.

Lemma 2. *Let $\Phi \in H^{1,0}(\Omega_2; \Delta)$, $\psi \in H^{-\frac{1}{2}}(S)$, $\varphi \in H^{\frac{1}{2}}(S)$, $\chi \in X^3$, and the function β be defined by (23). Moreover, let $u_2 \in H^1(\Omega_2)$ and the following equation hold,*

$$\beta u_2 + \mathcal{N}_\chi u_2 - V_\chi \psi + W_\chi \varphi = \Phi \quad \text{in } \Omega_2.$$

Then $u_2 \in H^{1,0}(\Omega_2; A_2)$ and the following estimate holds for some constant $C > 0$,

$$\|u_2\|_{H^{1,0}(\Omega_2; A_2)} \leq C(\|u_2\|_{H^1(\Omega_2)} + \|\psi\|_{H^{-\frac{1}{2}}(S)} + \|\varphi\|_{H^{\frac{1}{2}}(S)} + \|\Phi\|_{H^{1,0}(\Omega_2; \Delta)}).$$

3.2 | Integral relations in the homogeneous unbounded domain

For any radiating solution $u_1 \in H_{loc}^{1,0}(\Omega_1, A_1) \cap Z(\Omega_1)$ with $A_1 u_1 \in H_{comp}^0(\Omega_1)$, there holds Green's third identity (for details, see the references Colton and Kress,⁵ Vekua,¹⁸ Jentsch et al,²⁰ and Natroshvili et al²¹)

$$u_1 + V_\omega T_1^- u_1 - W_\omega \gamma^- u_1 = \mathcal{P}_\omega A_1 u_1 \quad \text{in } \Omega_1, \tag{35}$$

where

$$V_\omega g(y) := - \int_S \Gamma(x - y, \omega) g(x) dS_x, \quad W_\omega g(y) := - \int_S [T_1(x, \partial_x) \Gamma(x - y, \omega)] g(x) dS_x, \quad y \in \mathbb{R}^3 \setminus S, \tag{36}$$

$$\mathcal{P}_\omega f(y) := \int_{\Omega_1} \Gamma(x - y, \omega) f(x) dx, \quad y \in \mathbb{R}^3. \tag{37}$$

Here, $T_1(x, \partial_x) = a_{kj}^{(1)} n_k(x) \partial_{x_j}$, $n(x)$ is the outward unit normal vector to S at the point $x \in S$, and

$$\Gamma(x, \omega) = - \frac{\exp \left\{ i\omega \kappa_1^{1/2} (\mathbf{a}_1^{-1} x \cdot x)^{1/2} \right\}}{4\pi (\det \mathbf{a}_1)^{1/2} (\mathbf{a}_1^{-1} x \cdot x)^{1/2}} \tag{38}$$

is a radiating fundamental solution of the operator A_1 (see, eg, Lemma 1.1 in Jentsch et al²⁰). If x belongs to a bounded subset of \mathbb{R}^3 , then for sufficiently large $|y|$, we have the following asymptotic formula

$$\Gamma(y - x, \omega) = c(\xi) \frac{\exp \{ i\xi \cdot (y - x) \}}{|y|} + \mathcal{O}(|y|^{-2}), \quad c(\xi) = - \frac{|\mathbf{a}_1 \xi|}{4\pi \omega \kappa_1^{1/2} (\det \mathbf{a}_1)^{1/2}}, \tag{39}$$

where $\xi = \xi(\eta) \in S_\omega$ corresponds to the direction $\eta = y/|y|$ and is given by (11). The asymptotic formula (39) can be differentiated arbitrarily many times with respect to x and y .

The mapping properties of these potentials and the boundary operators generated by them are collected in Appendix C.

Evidently, the layer potentials $V_\omega g$ and $W_\omega g$ solve the homogeneous differential Equation (14), ie,

$$A_1 V_\omega g = A_1 W_\omega g = 0 \quad \text{in } \mathbb{R}^3 \setminus S, \tag{40}$$

while for $f_1 \in H_{comp}^0(\Omega_1)$, the volume potential $\mathcal{P}_\omega f_1 \in H_{loc}^2(\mathbb{R}^3)$ solves the following nonhomogeneous equation (see Lemma 5(i))

$$A_1 \mathcal{P}_\omega f_1 = \begin{cases} f_1 & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2. \end{cases} \tag{41}$$

The exterior trace and co-normal derivative of the third Green identity (35) on S read as (see Lemma 5(ii))

$$\mathcal{V}_\omega T_1^- u_1 + \left(\frac{1}{2} I - \mathcal{W}_\omega \right) \gamma^- u_1 = \gamma^- \mathcal{P}_\omega A_1 u_1 \quad \text{on } S, \tag{42}$$

$$\left(\frac{1}{2} I + \mathcal{W}'_\omega \right) T_1^- u_1 - \mathcal{L}_\omega \gamma^- u_1 = T_1^- \mathcal{P}_\omega A_1 u_1 \quad \text{on } S, \tag{43}$$

where the integral operators \mathcal{V}_ω , \mathcal{W}_ω , \mathcal{W}'_ω , and \mathcal{L}_ω are defined in Appendix C by formulas (C1) - (C4). Note that the operators \mathcal{V}_ω , $2^{-1}I - \mathcal{W}_\omega$, $2^{-1}I + \mathcal{W}'_\omega$, and \mathcal{L}_ω involved in (42) and (43) are not invertible for resonant values of the frequency parameter ω . The set of these resonant values is countable and consists of eigenfrequencies of the interior Dirichlet and Neumann boundary value problems for the operator A_1 in the bounded domain Ω_2 (see Vekua¹⁸ section 4; Colton and Kress²⁷ chapter 3 Chen and Zhou²⁸ section 7.7). Therefore, to obtain Dirichlet-to-Neumann or Neumann-to-Dirichlet mappings for arbitrary values of the frequency parameter ω , we apply the ideas of the so-called *combined-field integral equations*, cf Burton and Miller,²⁹ Brakhage and Werner,³⁰ Colton and Kress,^{5,27} Leis,³¹ and Panich.³²

Multiply Equation (42) by $-i\alpha$ with some fixed positive α and add to Equation (43) to obtain

$$\mathcal{K}_\omega T_1^- u_1 - \mathcal{M}_\omega \gamma^- u_1 = \Psi_\omega A_1 u_1 \quad \text{on } S, \quad (44)$$

where

$$\mathcal{K}_\omega g := \left(\frac{1}{2}I + \mathcal{W}'_\omega - i\alpha \mathcal{V}_\omega\right)g = (T_1^+ - i\alpha \gamma^+)V_\omega g \quad \text{on } S, \quad (45)$$

$$\mathcal{M}_\omega h := \left[\mathcal{L}_\omega - i\alpha \left(-\frac{1}{2}I + \mathcal{W}_\omega\right)\right]h = (T_1^+ - i\alpha \gamma^+)W_\omega h \quad \text{on } S, \quad (46)$$

$$\Psi_\omega f_1 := (T_1^- - i\alpha \gamma^-)\mathcal{P}_\omega f_1 = (T_1^+ - i\alpha \gamma^+)\mathcal{P}_\omega f_1 \quad \text{on } S, \quad (47)$$

for $f_1 \in H_{comp}^0(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S)$, and $h \in H^{\frac{1}{2}}(S)$.

In view of Lemma 6, from (44) we derive the following analogue of the Steklov-Poincaré type relation for arbitrary $u_1 \in H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1)$

$$T_1^- u_1 = \mathcal{K}_\omega^{-1}(\mathcal{M}_\omega \gamma^- u_1 + \Psi_\omega A_1 u_1) \quad \text{on } S, \quad (48)$$

where $\mathcal{K}_\omega^{-1} : H^{-\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S)$ is the inverse to the operator $\mathcal{K}_\omega : H^{-\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S)$.

3.3 | Equivalent reduction to a system of integral equations

Let us set

$$\varphi_1 = \gamma^- u_1, \quad \varphi_2 := \gamma^+ u_2, \quad \psi_1 = T_1^- u_1, \quad \psi_2 := T_2^+ u_2. \quad (49)$$

If a pair (u_1, u_2) solves the transmission problem (14) - (17), then by notation (49) and relations (22), (34), (44), and (35), the following equations hold true:

$$\beta u_2 + \mathcal{N}_\chi u_2 - V_\chi \psi_2 + W_\chi \varphi_2 = \mathcal{P}_\chi f_2 \quad \text{in } \Omega_2, \quad (50)$$

$$\mathcal{N}_\chi^+ u_2 - \mathcal{V}_\chi \psi_2 + [(\beta - \mu)I + \mathcal{W}_\chi]\varphi_2 = \mathcal{P}_\chi^+ f_2 \quad \text{on } S, \quad (51)$$

$$\mathcal{K}_\omega \psi_2 - \mathcal{M}_\omega \varphi_2 = \Psi_\omega f_1 + \mathcal{K}_\omega \psi_0 - \mathcal{M}_\omega \varphi_0 \quad \text{on } S, \quad (52)$$

$$\psi_2 - \psi_1 = \psi_0 \quad \text{on } S, \quad (53)$$

$$\varphi_2 - \varphi_1 = \varphi_0 \quad \text{on } S, \quad (54)$$

$$u_1 + V_\omega \psi_1 - W_\omega \varphi_1 = \mathcal{P}_\omega f_1 \quad \text{in } \Omega_1. \quad (55)$$

Let us consider relations (50) - (55) as a LBDIE system with respect to the unknowns $(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbf{H}$, where

$$\mathbf{H} := H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times (H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1)). \quad (56)$$

Note that if \mathcal{P}_χ would be replaced with the corresponding fundamental solution, then we would have $\mathcal{N}_\chi u_2 = 0$, $\mathcal{N}_\chi^+ u_2 = 0$, $\beta = 1$, and $\mu = 1/2$ in (50) and (51). Thus, the system could be split to the boundary integral equation system (51) - (54) and the representation formulas (50), (55) for the functions u_1 and u_2 in the domains Ω_1 and Ω_2 , respectively.

Let us prove the following equivalence theorem.

Theorem 2. *Let conditions (18) hold.*

- (i) *If a pair $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times (H^{1,0}_{loc}(\Omega_1; A_1) \cap Z(\Omega_1))$ solves transmission problem (14) - (17), then the vector $(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbf{H}$ with ψ_q and $\varphi_q, q = 1, 2$, defined by (49), solves LBDIE system (50) - (55).*
- (ii) *Vice versa, if a vector $(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbf{H}$ solves LBDIE system (50) - (55), then the pair $(u_2, u_1) \in H^{1,0}(\Omega_1; A_1) \times (H^{1,0}_{loc}(\Omega_1; A_1) \cap Z(\Omega_1))$ solves transmission problem (14) - (17) and relations (49) hold true.*

Proof.

- (i) The first part of the theorem directly follows from the formulation of the transmission problem (14) - (17) and relations (22), (34), (35), and (44).
- (ii) Now let a vector $(u_2, \psi_2, \varphi_2, \psi_1, \varphi_1, u_1) \in \mathbf{H}$ solve system (50) - (55). Taking the trace of (50) on S and comparing with (51) lead to the equation

$$\gamma^+ u_2 = \varphi_2 \text{ on } S. \tag{57}$$

Further, since $u_2 \in H^{1,0}(\Omega_2; A_2)$, we can write Green's third identity (22), which in view of (57) can be rewritten as

$$\beta u_2 + \mathcal{N}_\chi u_2 - V_\chi T^+ u_2 + W_\chi \varphi_2 = \mathcal{P}_\chi A_2 u_2 \text{ in } \Omega_2. \tag{58}$$

From (50) and (58), it follows that

$$V_\chi(T^+ u_2 - \psi_2) + \mathcal{P}_\chi(A_2 u_2 - f_2) = 0 \text{ in } \Omega_2.$$

Whence by Lemma 6.3 in Chkadua et al,³³ we deduce

$$A_2 u_2 = f_2 \text{ in } \Omega_2, \quad T^+ u_2 = \psi_2 \text{ on } S. \tag{59}$$

From Equation (55), it follows that

$$A_1 u_1 = f_1 \text{ in } \Omega_1. \tag{60}$$

From (52), (54), and (53), we derive

$$\mathcal{K}_\omega \psi_1 - \mathcal{M}_\omega \varphi_2 - \Psi_\omega f_1 = 0 \text{ on } S. \tag{61}$$

Now, let us consider the function

$$w := V_\omega \psi_1 - W_\omega \varphi_1 - \mathcal{P}_\omega f_1 \text{ in } \Omega_2. \tag{62}$$

In view of the inclusion $\mathcal{P}_\omega f_1 \in H^2_{loc}(\mathbb{R}^3)$ it follows that $\gamma^+ \mathcal{P}_\omega f_1 = \gamma^- \mathcal{P}_\omega f_1$ and $T^+_1 \mathcal{P}_\omega f_1 = T^-_1 \mathcal{P}_\omega f_1$ on S . Whence due to (45) - (47), (61), and Lemma 5, we have $w \in H^{1,0}(\Omega_2; A_1)$ and

$$\begin{aligned} (T^+_1 - i\alpha \gamma^+)w &= \left(\frac{1}{2}I + \mathcal{W}'_\omega - i\alpha \mathcal{V}_\omega\right)\psi_1 - \left[\mathcal{L}_\omega - i\alpha \left(-\frac{1}{2}I + \mathcal{W}_\omega\right)\right]\varphi_1 - (T^- - i\alpha \gamma^-)\mathcal{P}_\omega f_1 \\ &= \mathcal{K}_\omega \psi_1 - \mathcal{M}_\omega \varphi_1 - \Psi_\omega f_1 = 0 \text{ on } S. \end{aligned}$$

Consequently, in view of (40) and (41), we see that the function w solves the homogeneous Robin type interior boundary value problem,

$$A_1 w = 0 \text{ in } \Omega_2, \quad T^+_1 w - i\alpha \gamma^+ w = 0 \text{ on } S.$$

By Green's first identity (9) for the operator A_1 , we have

$$\int_{\Omega_2} \overline{w(x)} A_1 w(x) \, dx = - \int_{\Omega_2} \left[a^{(1)}_{kj} \partial_j w(x) \overline{\partial_k w(x)} - \omega^2 \kappa_1 |w(x)|^2 \right] dx + \langle T^+_1 w, \overline{\gamma^+ w} \rangle_S,$$

and since for the real symmetric matrix $a^{(1)}_{kj}$ the function $a^{(1)}_{kj} \partial_j w(x) \overline{\partial_k w(x)}$ is also real-valued, it follows that $\gamma^+ w = 0$ and $T^+_1 w = 0$ on S for real $\alpha \neq 0$. Consequently, the function w defined in (62) vanishes identically

in Ω_2 in view of the corresponding Green's third identity. Due to the jump relations for the layer potentials presented in Lemma 5(ii) and since $\mathcal{P}_\omega f_1 \in H_{loc}^2(\mathbb{R}^3)$, we have from (55) and (62) the following relations:

$$\gamma^- u_1 = \gamma^- u_1 + \gamma^+ w = \varphi_1, \quad T_1^- u_1 = T_1^- u_1 + T_1^+ w = \psi_1. \quad (63)$$

From Equations (53) and (54) and relations (57), (59), (60), and (63), it follows that the pair (u_2, u_1) solves the transmission problem (14) and relations (49) hold true. \square

From uniqueness Theorem 1 and the equivalence Theorem 2, the following assertion follows directly.

Corollary 1. *Let conditions (18) be fulfilled. Then the LBDIE system (50) - (55) possesses at most one solution in the space \mathbf{H} defined in (56).*

4 | ANALYSIS OF THE LBDIO

Let us rewrite the LBDIE system (50) - (55) in a more convenient form for our further purposes

$$(\beta I + \mathbf{N}_\chi) \mathring{E} u_2 - V_\chi \psi_2 + W_\chi \varphi_2 = \mathcal{P}_\chi f_2 \quad \text{in } \Omega_2, \quad (64)$$

$$\mathbf{N}_\chi^+ \mathring{E} u_2 - \mathcal{V}_\chi \psi_2 + [(\beta - \mu)I + \mathcal{W}_\chi] \varphi_2 = \mathcal{P}_\chi^+ f_2 \quad \text{on } S, \quad (65)$$

$$\mathcal{K}_\omega \psi_2 - \mathcal{M}_\omega \varphi_2 = \Psi_\omega f_1 + \mathcal{K}_\omega \psi_0 - \mathcal{M}_\omega \varphi_0 \quad \text{on } S, \quad (66)$$

$$\psi_2 - \psi_1 = \psi_0 \quad \text{on } S, \quad (67)$$

$$\varphi_2 - \varphi_1 = \varphi_0 \quad \text{on } S, \quad (68)$$

$$u_1 + V_\omega \psi_1 - W_\omega \varphi_1 = \mathcal{P}_\omega f_1 \quad \text{in } \Omega_1, \quad (69)$$

where $\mathring{E} = \mathring{E}_{\Omega_2}$ denotes the extension operator by zero from Ω_2 onto Ω_1 , \mathbf{N}_χ is a pseudodifferential operator given in (27), $\mathbf{N}_\chi^+ = \gamma^+ \mathbf{N}_\chi$, and $\mathcal{P}_\chi^+ = \gamma^+ \mathcal{P}_\chi$. Note that for a function $u_2 \in H^1(\Omega_2)$, we have $\beta u_2 + \mathcal{N}_\chi u_2 = (\beta I + \mathbf{N}_\chi) \mathring{E} u_2$ in Ω_2 .

It can easily be seen that if the unknowns (u_2, ψ_2, φ_2) are determined from the first three equations of system (64) - (69), then the unknowns (ψ_1, φ_1, u_1) are determined explicitly from the last three equations of the same system. Therefore, the main task is to investigate the matrix integral operator generated by the left hand side expressions in (64) - (66).

Let us rewrite the first three equations of the LBDIE system (64) - (69) in matrix form

$$\mathbf{M}U = F,$$

where $U := (u_2, \psi_2, \varphi_2)^\top$, $F := (F_1, F_2, F_3)^\top$, $F_1 := \mathcal{P}_\chi f_2$, $F_2 := \mathcal{P}_\chi^+ f_2$, $F_3 := \Psi_\omega f_1 + \mathcal{K}_\omega \psi_0 - \mathcal{M}_\omega \varphi_0$,

$$\mathbf{M} := \begin{bmatrix} r_{\Omega_2}(\beta I + \mathbf{N}_\chi) \mathring{E} & -r_{\Omega_2} V_\chi & r_{\Omega_2} W_\chi \\ \mathbf{N}_\chi^+ \mathring{E} & -\mathcal{V}_\chi & (\beta - \mu)I + \mathcal{W}_\chi \\ 0 & \mathcal{K}_\omega & -\mathcal{M}_\omega \end{bmatrix}. \quad (70)$$

Let us introduce the spaces

$$\mathbb{H} := H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S), \quad \mathbb{F} := H^{1,0}(\Omega_2; \Delta) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S),$$

$$\mathbb{X} := H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S), \quad \mathbb{Y} := H^1(\Omega_2) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S).$$

Recall that for $\chi \in X_+^4$, the principal homogeneous symbol $\mathfrak{S}_0(\mathbf{N}_\chi; y, \xi)$ of the operator \mathbf{N}_χ given by (33) is a rational homogeneous function of order zero in ξ . Therefore, applying the inclusion (32) and the mapping properties of the pseudodifferential operators with rational type symbols (see, eg, Hsiao and Wendland,³⁴ Theorem 8.4.13) and using Theorems 8 and 10 we deduce that the operators

$$\mathbf{M} : \mathbb{H} \rightarrow \mathbb{F}, \quad (71)$$

$$\mathbf{M} : \mathbb{X} \rightarrow \mathbb{Y} \quad (72)$$

are continuous for $\chi \in X_+^4$. Now, we prove the main theorem of this section.

Theorem 3. Let $\chi \in X_+^4$. Operator (72) is invertible.

Proof. Using Lemma 6, we can represent the matrix operator \mathbf{M} defined in (70) as a composition of two operators

$$\mathbf{M} = \mathbf{BC},$$

where

$$\mathbf{B} := \begin{bmatrix} r_{\Omega_2}(\beta I + \mathbf{N}_\chi) \mathring{E} & r_{\Omega_2}[-V_\chi + W_\chi \mathcal{M}_\omega^{-1} \mathcal{K}_\omega] & r_{\Omega_2} W_\chi \\ \mathbf{N}_\chi^+ \mathring{E} & -\mathcal{V}_\chi + [(\beta - \mu)I + \mathcal{W}_\chi] \mathcal{M}_\omega^{-1} \mathcal{K}_\omega & (\beta - \mu)I + \mathcal{W}_\chi \\ 0 & 0 & -\mathcal{M}_\omega \end{bmatrix}, \quad \mathbf{C} := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\mathcal{M}_\omega^{-1} \mathcal{K}_\omega & I \end{bmatrix}. \quad (73)$$

Evidently, the triangular matrix operator

$$\mathbf{C} : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S)$$

is invertible. Since the operator $\mathcal{M}_\omega : H^{\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S)$ is also invertible due to Lemma 6, from (73), it follows that the block-triangular matrix operator

$$\mathbf{B} : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S),$$

and consequently, operator (72) is invertible if and only if the following operator is invertible

$$\mathbf{D} : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{\frac{1}{2}}(S), \quad (74)$$

$$\mathbf{D} = [\mathbf{D}_{kj}]_{k,j=1}^2 := \begin{bmatrix} r_{\Omega_2}(\beta I + \mathbf{N}_\chi) \mathring{E} & r_{\Omega_2}[-V_\chi + W_\chi \mathcal{M}_\omega^{-1} \mathcal{K}_\omega] \\ \mathbf{N}_\chi^+ \mathring{E} & -\mathcal{V}_\chi + [(\beta - \mu)I + \mathcal{W}_\chi] \mathcal{M}_\omega^{-1} \mathcal{K}_\omega \end{bmatrix}. \quad (75)$$

Further, we apply the Vishik-Eskin approach, developed in Eskin,³⁵ and establish that operator (74) is invertible.

The proof is performed in four steps.

Step 1. Here, we show that the operator

$$\mathbf{D}_{11} = r_{\Omega_2}(\beta I + \mathbf{N}_\chi) \mathring{E} : H^1(\Omega_2) \rightarrow H^1(\Omega_2) \quad (76)$$

is Fredholm with zero index.

In view of (33), the principal homogeneous symbol of the operator $\beta I + \mathbf{N}_\chi$ can be written as

$$\mathfrak{S}_0(\mathbf{D}_{11}; y, \xi) = \mathfrak{S}_0(\beta I + \mathbf{N}_\chi; y, \xi) = \frac{A_2(y, \xi)}{\Delta(\xi)} = \frac{a_{kl}^{(2)}(y) \xi_k \xi_l}{|\xi|^2} > 0, \quad \Delta(\xi) := |\xi|^2, \quad y \in \overline{\Omega_2}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}. \quad (77)$$

Since the symbol $\mathfrak{S}_0(\mathbf{D}_{11}; y, \xi)$ given by (77) is an even rational homogeneous function of order 0 in ξ it follows that its factorisation index \varkappa equals to zero (see Eskin,³⁵ §6). Moreover, the operator $\beta I + \mathbf{N}_\chi$ possesses the transmission property. Therefore, we can apply the theory of pseudodifferential operators satisfying the transmission property to deduce that operator (76) is Fredholm (see Eskin,³⁵ Theorem 11.1 and Lemma 23.9; Boutet de Monvel³⁶).

To show that $\text{Ind} \mathbf{D}_{11} = 0$, we use the fact that the operators \mathbf{D}_{11} and $\mathbf{D}_{11,t}$, where

$$\mathbf{D}_{11,t} = r_{\Omega_2}[(1-t)I + t(\beta I + \mathbf{N}_\chi^+)] \mathring{E}, \quad t \in [0, 1],$$

are homotopic. Evidently $\mathbf{D}_{11,0} = I$ and $\mathbf{D}_{11,1} = \mathbf{D}_{11}$. In view of (33) and (77),

$$\mathfrak{S}_0(\mathbf{D}_{11,t}; y, \xi) = \frac{(1-t)\Delta(\xi) + t A_2(y, \xi)}{\Delta(\xi)} > 0$$

for all $t \in [0, 1]$, for all $y \in \overline{\Omega_2}$, and for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, and consequently the operator $\mathbf{D}_{11,t}$ is elliptic.

Since $\mathfrak{S}_0(\mathbf{D}_{11,t}; y, \xi)$ is rational, even, and homogeneous of order zero in ξ , we conclude that the operator $\mathbf{D}_{11,t} : H^1(\Omega_2) \rightarrow H^1(\Omega_2)$ is continuous Fredholm operator for all $t \in [0, 1]$. Therefore $\text{Ind} \mathbf{D}_{11,t}$ is the same for all $t \in [0, 1]$.

On the other hand, due to the equality $\mathbf{D}_{11,0} = I$, we get $\text{Ind} \mathbf{D}_{11} = \text{Ind} \mathbf{D}_{11,1} = \text{Ind} \mathbf{D}_{11,t} = \text{Ind} \mathbf{D}_{11,0} = 0$.

Step 2. Now we show that the operator \mathbf{D} defined by (74) and (75) is Fredholm. To this end, we apply the local principle (see, eg, Eskin,³⁵ §19 and §22).

Let U_j be an open neighbourhood of a fixed point $\tilde{y} \in \mathbb{R}^3$ and let $\psi_0^{(j)}, \varphi_0^{(j)} \in \mathcal{D}(U_j)$ be such that $\text{supp } \psi_0^{(j)} \cap \text{supp } \varphi_0^{(j)} \neq \emptyset$ contains some open neighbourhood $U'_j \subset U_j$ of the point y_0 . Consider the operator $\psi_0^{(j)} \mathbf{D} \varphi_0^{(j)}$. We separate two possible cases: (1) $\tilde{y} \in \Omega_2$ and (2) $\tilde{y} \in S$.

In the first case, when $\tilde{y} \in \Omega_2$, we can choose a neighbourhood \bar{U}_j of the point \tilde{y} such that $\bar{U}_j \subset \Omega_2$. Then the operator $\psi_0^{(j)} \mathbf{D} \varphi_0^{(j)}$ is equivalent to the operator $\psi_0^{(j)} \mathbf{D}_{11} \varphi_0^{(j)}$, where \mathbf{D}_{11} is defined by (76). As we have already shown in Step 1, this operator is Fredholm with zero index.

In the second case, when $\tilde{y} \in S$, we need to check that the Šapiro-Lopatinskiĭ type condition for the operator \mathbf{D} is fulfilled, ie, we have to show that the so-called *boundary symbol* that is constructed by means of the principal homogeneous symbols of the pseudodifferential operators involved in (75) is nonsingular (see Eskin,³⁵ §12). To write the boundary symbol function explicitly, we assume that the symbols are “frozen” at the point $\tilde{y} \in S$ considered as the origin O' of some local coordinate system. Denote by $\tilde{a}_{kl}^{(2)}(\tilde{y})$ the corresponding “frozen” coefficients of the principal part of the differential operator $A_2(y, \partial_y)$ subjected to a translation and an orthogonal transformation related to the local co-ordinate system. If the matrix of the transformation of the original co-ordinate system $Oy_1y_2y_3$ to the new one $O'\eta_1\eta_2\eta_3$ with $O' = \tilde{y}$ is an orthogonal matrix $\Lambda(\tilde{y}) := [\lambda_{kl}(\tilde{y})]_{3 \times 3}$, which transforms the outward unit normal vector $n^\top(\tilde{y})$ into the vector $\mathbf{e}_3 = (0, 0, -1)^\top$ (the outward unit normal vector to \mathbb{R}_+^3), ie, $n^\top(\tilde{y}) = \Lambda(\tilde{y})\mathbf{e}_3$, then $y = \tilde{y} + \Lambda(\tilde{y})\eta$, $\nabla_y = \Lambda(\tilde{y})\nabla_\eta$, and

$$\lambda_{k3}(\tilde{y}) = -n_k(\tilde{y}), \quad \tilde{a}_{kl}^{(2)}(\tilde{y}) = \lambda_{pk}(\tilde{y})a_{pq}^{(2)}(\tilde{y})\lambda_{ql}(\tilde{y}) = \{\Lambda^\top(\tilde{y})\mathbf{a}_2(\tilde{y})\Lambda(\tilde{y})\}_{kl}, \quad k, l = 1, 2, 3. \tag{78}$$

Evidently, the matrix $\tilde{\mathbf{a}}_2(\tilde{y}) = [\tilde{a}_{kl}^{(2)}(\tilde{y})]_{k,j=1}^3 := \Lambda^\top(\tilde{y})\mathbf{a}_2(\tilde{y})\Lambda(\tilde{y})$ is positive definite, since $\mathbf{a}_2(\tilde{y})$ is positive definite and for arbitrary $\tilde{y} \in S$, we have

$$\begin{aligned} \tilde{\beta}(\tilde{y}) &= \frac{1}{3} \left[\tilde{a}_{11}^{(2)}(\tilde{y}) + \tilde{a}_{22}^{(2)}(\tilde{y}) + \tilde{a}_{33}^{(2)}(\tilde{y}) \right] > 0, \quad \tilde{a}_{33}^{(2)}(\tilde{y}) = \lambda_{p3}a_{pq}^{(2)}(\tilde{y})\lambda_{q3} = a_{pq}^{(2)}(\tilde{y})n_p(\tilde{y})n_q(\tilde{y}) = 2\tilde{\mu}(\tilde{y}) > 0, \\ T_2(\tilde{y}, \partial_y) &= a_{pl}^{(2)}(\tilde{y})n_p(\tilde{y})\partial_{y_l} = n_p(\tilde{y})a_{pl}^{(2)}(\tilde{y})\lambda_{lq}(\tilde{y})\partial_{\eta_q} = -\lambda_{p3}(\tilde{y})a_{pl}^{(2)}(\tilde{y})\lambda_{lq}(\tilde{y})\partial_{\eta_q} = -\tilde{a}_{3q}^{(2)}(\tilde{y})\partial_{\eta_q}, \end{aligned}$$

due to (78) and (B2).

Further, let us note that the layer potentials can be represented by means of the volume potential (see, eg, Chkadua et al²⁶)

$$V_\chi \psi(y) = -\mathbf{P}_\chi(\gamma^* \psi)(y), \quad y \in \mathbb{R}^3 \setminus S, \tag{79}$$

$$W_\chi \varphi(y) = -\partial_{y_j} V_\chi(a_{kj}^{(2)} n_k \varphi) = \partial_{y_j} \mathbf{P}_\chi(\gamma^*(a_{kj}^{(2)} n_k \varphi))(y), \quad y \in \mathbb{R}^3 \setminus S, \tag{80}$$

where $\gamma^* : H^{\frac{1}{2}-t}(S) \rightarrow H_S^{-t}$, $t > 1/2$ is the adjoint operator to the trace operator γ , ie, $\langle \gamma^* \psi, h \rangle_{\mathbb{R}^3} := \langle \psi, \gamma h \rangle_S$ for all $h \in \mathcal{D}(\mathbb{R}^3)$. Here, $H_S^{-t} := \{f \in H^{-t}(\mathbb{R}^3) : \text{supp } f \subset S\}$, and H_S^{-t} does not contain nonzero elements, when $t \leq \frac{1}{2}$ (see Lemma 3.39 in McLean,³⁷ Theorem 2.10(i) in Mikhailov¹⁷).

In view of (79) and (80), the operator \mathbf{D}_{12} in (75) can be represented as

$$\mathbf{D}_{12} = -V_\chi(\psi_2) + W_\chi(\mathcal{M}_\omega^{-1} \mathcal{K}_\omega \psi_2) = \mathbf{P}_\chi(\gamma^* \psi_2) + \partial_{y_j} \mathbf{P}_\chi(\gamma^*(a_{kj}^{(2)} n_k \mathcal{M}_\omega^{-1} \mathcal{K}_\omega \psi_2)), \tag{81}$$

and its principal homogeneous symbol due to the above formulas and Remark 6 in Appendix C can be written as

$$\mathfrak{S}(\mathbf{D}_{12}; \tilde{y}, \xi) \equiv \mathbf{R}_{12}(\tilde{y}, \xi) := -\frac{1}{|\xi|^2} + \frac{i\xi_l \tilde{a}_{3l}^{(2)}(\tilde{y})}{|\xi|^2} 2\mathfrak{S}_0(\mathcal{V}_\omega; \tilde{y}, \xi'), \quad \xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}, \tag{82}$$

since the principal homogeneous symbol of the operator \mathbf{P}_χ reads as $\mathfrak{S}_0(\mathbf{P}; \xi) = -\mathcal{F}_{z \rightarrow \xi}[(4\pi|z|)^{-1}] = -|\xi|^{-2}$. Due to the Vishik-Eskin approach, now we have to construct the following matrix associated with the principal homogeneous symbols of the operators involved in \mathbf{D} at the local co-ordinate system introduced above

$$\mathbf{R}(\tilde{y}, \xi) := \begin{bmatrix} \mathbf{R}_{11}(\tilde{y}, \xi) & \mathbf{R}_{12}(\tilde{y}, \xi) \\ \mathbf{R}_{21}(\tilde{y}, \xi) & \mathbf{R}_{22}(\tilde{y}, \xi) \end{bmatrix}, \tag{83}$$

where $\mathbf{R}_{11}(\tilde{y}, \xi)$ is the principal homogeneous symbol of the operator $\mathbf{D}_{11} = \beta I + \mathbf{N}_\chi$,

$$\mathbf{R}_{11}(\tilde{y}, \xi) = \mathfrak{S}_0(\mathbf{D}_{11}; \tilde{y}, \xi) \equiv \mathfrak{S}_0(\beta I + \mathbf{N}_\chi; \tilde{y}, \xi) = \frac{A_2(\xi)}{\Delta(\xi)} = \frac{\tilde{a}_{kl}^{(2)}(\tilde{y})\xi_k\xi_l}{|\xi|^2} > 0, \quad \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{84}$$

$\mathbf{R}_{12}(\tilde{y}, \xi)$ is the principal homogeneous symbol of operator (81) and is given by (82), $\mathbf{R}_{21}(\tilde{y}, \xi)$ is the principal homogeneous symbol of the operator \mathbf{N}_χ ,

$$\mathbf{R}_{21}(\tilde{y}, \xi) := \mathfrak{S}_0(\mathbf{N}_\chi; \tilde{y}, \xi) = \frac{A_2(\tilde{y}, \xi)}{\Delta(\xi)} - \tilde{\beta}(\tilde{y}) = \frac{\tilde{a}_{kl}^{(2)}(\tilde{y})\xi_k\xi_l - \tilde{\beta}(\tilde{y})|\xi|^2}{|\xi|^2}, \tag{85}$$

$\mathbf{R}_{22}(\tilde{y}, \xi)$ is the principal homogeneous symbol of the boundary operator \mathbf{D}_{22} , which due to (75), (B4), (B5), and (C5) is written as

$$\begin{aligned} \mathbf{R}_{22}(\tilde{y}, \xi') &:= \mathfrak{S}_0(-\mathcal{V}_\chi + [(\beta - \mu)I + \mathcal{W}_\chi]\mathcal{M}_\omega^{-1}\mathcal{K}_\omega; \tilde{y}, \xi') \\ &= -\mathfrak{S}_0(\mathcal{V}_\chi; \tilde{y}, \xi') + \frac{1}{2}\mathfrak{S}_0((\beta - \mu)I + \mathcal{W}_\chi; \tilde{y}, \xi')\mathfrak{S}_0(\mathcal{M}_\omega^{-1}; \tilde{y}, \xi') \\ &= -\frac{1}{2|\xi'|} - [2\tilde{\beta}(\tilde{y}) - \tilde{a}_{33}^{(2)}(\tilde{y}) - i\sum_{l=1}^2\tilde{a}_{3l}^{(2)}(\tilde{y})\frac{\xi'_l}{|\xi'|}]\mathfrak{S}_0(\mathcal{V}_\omega; \tilde{y}, \xi'). \end{aligned} \tag{86}$$

Below, we drop the arguments \tilde{y} and ξ when it does not lead to misunderstanding.

Now, we show that the Šapiro-Lopatinskiĭ type condition for the operator \mathbf{D} is satisfied, ie, the boundary symbol (see Eskin,³⁵§12, formulas (12.25), (12.27))

$$\mathbf{S}_\mathbf{D}(\xi') = -\Pi'[\frac{\mathbf{R}_{21}}{\mathbf{R}_{11}^{(+)}} \Pi^+(\frac{\mathbf{R}_{12}}{\mathbf{R}_{11}^{(-)}})](\xi') + \mathbf{R}_{22}(\xi') \tag{87}$$

associated with the operator \mathbf{D} does not vanish for $\xi' \neq 0$. Here, $\mathbf{R}_{11}^{(+)}(\xi', \xi_3)$ and $\mathbf{R}_{11}^{(-)}(\xi', \xi_3)$ denote the “plus” and “minus” factors, respectively, in the factorisation of the symbol $\mathbf{R}_{11}(\xi', \xi_3)$ with respect to the variable ξ_3 in the complex ξ_3 plane, while Π^+ is a Cauchy type integral operator

$$\Pi^+(h)(\xi) = \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{h(\xi', \eta_3)d\eta_3}{\xi_3 + it - \eta_3},$$

and Π' is the operator defined on the set of rational functions

$$\Pi'(g)(\xi') = -\frac{1}{2\pi} \int_{\ell^-} g(\xi', \xi_3)d\xi_3,$$

where ℓ^- is a contour in the lower complex half-plane orientated counterclockwise and enclosing all poles of the rational function g with respect to ξ_3 .

Denote the roots of the equation $A_2(\xi) \equiv \tilde{a}_{kl}^{(2)}\xi_k\xi_l = 0$ with respect to ξ_3 by $\tau(\xi') = \alpha_1 - i\alpha_2$ and $\overline{\tau(\xi')} = \alpha_1 + i\alpha_2$, where we assume that $\alpha_2 > 0$. Then

$$A_2(\xi) = \tilde{a}_{kl}^{(2)}\xi_k\xi_l = \tilde{a}_{33}^{(2)}[\xi_3 - \tau(\xi')][\xi_3 - \overline{\tau(\xi')}] = A_2^{(+)}(\xi)A_2^{(-)}(\xi), \tag{88}$$

$$A_2^{(+)}(\xi) := \tilde{a}_{33}^{(2)}[\xi_3 - \tau(\xi')], \quad A_2^{(-)}(\xi) := \xi_3 - \overline{\tau(\xi')}, \tag{89}$$

$$\overline{\tau(\xi')} = \alpha_1(\xi') + i\alpha_2(\xi'), \quad \alpha_2(\xi') > 0, \quad \xi' \in \mathbb{R}^2 \setminus \{0\}. \tag{90}$$

Since $\Delta(\xi) = |\xi|^2 = \Delta^{(+)}(\xi)\Delta^{(-)}(\xi)$ with $\Delta^{(\pm)}(\xi) := \xi_3 \pm i|\xi'|$, we get the following factorisation of the symbol $\mathbf{R}_{11}(\xi)$,

$$\mathbf{R}_{11}(\xi) = \mathbf{R}_{11}^{(+)}(\xi)\mathbf{R}_{11}^{(-)}(\xi), \quad \mathbf{R}_{11}^{(+)}(\xi) := \frac{A_2^{(+)}(\xi)}{\Delta^{(+)}(\xi)}, \quad \mathbf{R}_{11}^{(-)}(\xi) := \frac{A_2^{(-)}(\xi)}{\Delta^{(-)}(\xi)}. \tag{91}$$

Using formulas (84) - (86) and (88) - (91), we rewrite (87) as

$$\begin{aligned} \mathbf{S}_D(\xi') &= -\Pi' \left\{ \left(\frac{A_2(\xi)}{\Delta(\xi)} - \tilde{\beta} \right) \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \Pi^+ \left[\left(-\frac{1}{\Delta(\xi)} - \frac{i \xi_l \tilde{a}_{3l}^{(2)}}{\Delta(\xi)} [-2\mathfrak{E}_0(\mathcal{V}_\omega; \xi')] \right) \frac{\Delta^{(-)}(\xi)}{A_2^{(-)}(\xi)} \right] \right\} \\ &\quad - \frac{1}{2|\xi'|} + \frac{1}{2} \left([2\tilde{\beta} - \tilde{a}_{33}^{(2)}] - i \sum_{l=1}^2 \tilde{a}_{3l}^{(2)}(\tilde{y}) \frac{\xi_l}{|\xi'|} \right) [-2\mathfrak{E}_0(\mathcal{V}_\omega; \xi')] = \mathbf{S}_D^{(1)}(\xi') + \mathbf{S}_D^{(2)}(\xi') [-2\mathfrak{E}_0(\mathcal{V}_\omega; \xi')], \end{aligned} \tag{92}$$

where

$$\begin{aligned} \mathbf{S}_D^{(1)}(\xi') &:= -\Pi' \left[\left(\frac{A_2(\xi)}{\Delta(\xi)} - \tilde{\beta} \right) \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \Pi^+ \left(-\frac{1}{\Delta(\xi)} \frac{\Delta^{(-)}(\xi)}{A_2^{(-)}(\xi)} \right) \right] - \frac{1}{2|\xi'|} \\ &= \Pi' \left[\left(\frac{A_2^{(-)}(\xi)}{\Delta^{(-)}(\xi)} - \tilde{\beta} \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \right) \Pi^+ \left(\frac{1}{\Delta^{(+)}(\xi) A_2^{(-)}(\xi)} \right) \right] - \frac{1}{2|\xi'|}, \end{aligned} \tag{93}$$

$$\begin{aligned} \mathbf{S}_D^{(2)}(\xi') &:= -\Pi' \left[\left(\frac{A_2(\xi)}{\Delta(\xi)} - \tilde{\beta} \right) \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \Pi^+ \left(-\frac{i \xi_l \tilde{a}_{3l}^{(2)}}{\Delta(\xi)} \frac{\Delta^{(-)}(\xi)}{A_2^{(-)}(\xi)} \right) \right] + \frac{1}{2} \left[(2\tilde{\beta} - \tilde{a}_{33}^{(2)}) - i \sum_{l=1}^2 \tilde{a}_{3l}^{(2)} \frac{\xi_l}{|\xi'|} \right] \\ &= \Pi' \left[\left(\frac{A_2^{(-)}(\xi)}{\Delta^{(-)}(\xi)} - \tilde{\beta} \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \right) \Pi^+ \left(\frac{i \xi_l \tilde{a}_{3l}^{(2)}}{\Delta^{(+)}(\xi) A_2^{(-)}(\xi)} \right) \right] + \frac{1}{2} \left[(2\tilde{\beta} - \tilde{a}_{33}^{(2)}) - i \sum_{l=1}^2 \tilde{a}_{3l}^{(2)} \frac{\xi_l}{|\xi'|} \right]. \end{aligned} \tag{94}$$

With the help of residue theorem, by direct calculations, we find

$$\begin{aligned} \Pi^+ \left(\frac{1}{\Delta^{(+)} A_2^{(-)}} \right) (\xi) &= \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{d\eta_3}{\Delta^{(+)}(\xi', \eta_3) A_2^{(-)}(\xi', \eta_3) (\xi_3 + it - \eta_3)} = \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{d\eta_3}{(\eta_3 + i|\xi'|)(\eta_3 - \tau(\xi'))(\xi_3 + it - \eta_3)} \\ &= -\frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{\ell^-} \frac{d\zeta}{(\zeta + i|\xi'|)(\zeta - \tau(\xi'))(\xi_3 + it - \zeta)} = -\frac{i}{2\pi} \lim_{t \rightarrow 0^+} \frac{2\pi i}{(-i|\xi'| - \tau(\xi'))(\xi_3 + it + i|\xi'|)} \\ &= -\frac{1}{(i|\xi'| + \tau(\xi'))(\xi_3 + i|\xi'|)}; \end{aligned} \tag{95}$$

$$\begin{aligned} \Pi' \left[\left(\frac{A_2^{(-)}}{\Delta^{(-)}} \right) \Pi^+ \left(\frac{1}{\Delta^{(+)} A_2^{(-)}} \right) \right] (\xi') &= \frac{1}{2\pi} \int_{\ell^-} \frac{\xi_3 - \overline{\tau(\xi')}}{\xi_3 - i|\xi'|} \frac{d\xi_3}{(i|\xi'| + \tau(\xi'))(\xi_3 + i|\xi'|)} = \frac{1}{2\pi [i|\xi'| + \tau(\xi')]} \int_{\ell^-} \frac{\xi_3 - \overline{\tau(\xi')}}{\xi_3^2 + |\xi'|^2} d\xi_3 \\ &= \frac{1}{2\pi [i|\xi'| + \tau(\xi')]} \int_{\ell^-} \left[\frac{1}{\xi_3 - i|\xi'|} - \frac{i|\xi'|}{\xi_3^2 + |\xi'|^2} - \frac{\overline{\tau(\xi')}}{\xi_3^2 + |\xi'|^2} \right] d\xi_3 = -\frac{1}{2\pi} \int_{\ell^-} \frac{d\xi_3}{\xi_3^2 + |\xi'|^2} = \frac{1}{2|\xi'|}, \end{aligned} \tag{96}$$

$$\begin{aligned} \tilde{\beta} \Pi' \left[\left(\frac{\Delta^{(+)}}{A_2^{(+)}} \right) \Pi^+ \left(\frac{1}{\Delta^{(+)} A_2^{(-)}} \right) \right] (\xi') &= \frac{\tilde{\beta}}{2\pi} \int_{\ell^-} \frac{\xi_3 + i|\xi'|}{\tilde{a}_{33}^{(2)} [\xi_3 - \tau(\xi')]} \frac{d\xi_3}{(i|\xi'| + \tau(\xi'))(\xi_3 + i|\xi'|)} = \frac{\tilde{\beta}}{2\pi \tilde{a}_{33}^{(2)} [i|\xi'| + \tau(\xi')]} \int_{\ell^-} \frac{d\xi_3}{\xi_3 - \tau(\xi')} \\ &= \frac{\tilde{\beta}}{2\pi \tilde{a}_{33}^{(2)} [i|\xi'| + \tau(\xi')]} \int_{\ell^-} \left[\frac{1}{\xi_3 + \tau(\xi')} + \frac{2\tau(\xi')}{\xi_3^2 - \tau^2(\xi')} \right] d\xi_3 = \frac{\tilde{\beta}}{2\pi \tilde{a}_{33}^{(2)} [i|\xi'| + \tau(\xi')]} \int_{\ell^-} \frac{2\tilde{\beta} \tau(\xi')}{\xi_3^2 - \tau^2(\xi')} d\xi_3 \\ &= \frac{2\tilde{\beta} \tau(\xi')}{2\pi \tilde{a}_{33}^{(2)} [i|\xi'| + \tau(\xi')]} \frac{2\pi i}{2\tau(\xi')} = \frac{i\tilde{\beta}}{\tilde{a}_{33}^{(2)} [i|\xi'| + \tau(\xi')]} \end{aligned} \tag{97}$$

Therefore, from (93) in view of (95) - (97) and (90), we get

$$\mathbf{S}_D^{(1)}(\xi') = -\frac{i\tilde{\beta}}{\tilde{a}_{33}^{(2)} [i|\xi'| + \tau(\xi')]} = -\frac{\tilde{\beta}(\alpha_2 + |\xi'|) + i\alpha_1 \tilde{\beta}}{\tilde{a}_{33}^{(2)} [\alpha_1^2 + (\alpha_2 + |\xi'|)^2]} \text{ for } \xi' \neq 0. \tag{98}$$

Now, we evaluate the function $S_D^{(2)}$. Let $\vartheta(\xi') := \sum_{l=1}^2 \tilde{a}_{3l}^{(2)} \xi_l$. Since τ and $\bar{\tau}$ are roots of the quadratic equation

$$A_2(\xi) \equiv \sum_{k,l=1}^3 \tilde{a}_{kl}^{(2)} \xi_k \xi_l = \tilde{a}_{33}^{(2)} \xi_3^2 + 2\vartheta(\xi') \xi_3 + \sum_{k,l=1}^2 \tilde{a}_{kl}^{(2)} \xi_k \xi_l = 0,$$

we have

$$2\vartheta(\xi') = -\tilde{a}_{33}^{(2)}(\tau + \bar{\tau}). \tag{99}$$

Again by direct calculations, we find

$$\begin{aligned} \Pi^+ \left(\frac{i \xi_l \tilde{a}_{3l}^{(2)}}{\Delta^{(+)}(\xi) A_2^{(-)}(\xi)} \right) &= \Pi^+ \left(\frac{i \tilde{a}_{3l}^{(2)} \xi_l}{\xi_3 + i|\xi'|} \frac{1}{\xi_3 - \bar{\tau}} \right) (\xi') = \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{[i\vartheta(\xi') + i\tilde{a}_{33}^{(2)}\eta_3] d\eta_3}{(\eta_3 + i|\xi'|)(\eta_3 - \bar{\tau})(\xi_3 + it - \eta_3)} \\ &= -\frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{(\bar{\tau} + i|\xi'|)(\xi_3 + i|\xi'|)}. \end{aligned}$$

Further, we have

$$\begin{aligned} \Pi' \left[\left(\frac{A_2^{(-)}(\xi)}{\Delta^{(-)}(\xi)} - \tilde{\beta} \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \right) \Pi^+ \left(\frac{i \xi_l \tilde{a}_{3l}^{(2)}}{\Delta^{(+)}(\xi) A_2^{(-)}(\xi)} \right) \right] &= \frac{1}{2\pi} \int_{\ell^-} \left(\frac{A_2^{(-)}(\xi)}{\Delta^{(-)}(\xi)} - \tilde{\beta} \frac{\Delta^{(+)}(\xi)}{A_2^{(+)}(\xi)} \right) \frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{(\bar{\tau} + i|\xi'|)(\xi_3 + i|\xi'|)} d\xi_3 \\ &= \frac{1}{2\pi} \frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{\bar{\tau} + i|\xi'|} \int_{\ell^-} \left[\frac{\xi_3 - \bar{\tau}}{\xi_3^2 + |\xi'|^2} - \frac{\tilde{\beta}}{\tilde{a}_{33}^{(2)}(\xi_3 - \tau)} \right] d\xi_3 = \frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{2|\xi'|} - \frac{i\tilde{\beta}}{\tilde{a}_{33}^{(2)}} \frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{\bar{\tau} + i|\xi'|}. \end{aligned} \tag{100}$$

Now, from (94), (99), and (100), we get

$$\begin{aligned} S_D^{(2)}(\xi') &= \frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{2|\xi'|} - \frac{i\tilde{\beta}}{\tilde{a}_{33}^{(2)}} \frac{i\vartheta(\xi') + \tilde{a}_{33}^{(2)}|\xi'|}{\bar{\tau} + i|\xi'|} + \frac{1}{2} \left[2\tilde{\beta} - \tilde{a}_{33}^{(2)} - i \frac{\vartheta(\xi')}{|\xi'|} \right] \\ &= \frac{\tilde{\beta}(\vartheta(\xi') + \tilde{a}_{33}^{(2)}\bar{\tau})}{\tilde{a}_{33}^{(2)}(\bar{\tau} + i|\xi'|)} = \frac{\tilde{\beta}(\bar{\tau} - \tau)}{2(\bar{\tau} + i|\xi'|)} = \frac{i\tilde{\beta}\alpha_2}{\bar{\tau} + i|\xi'|} \text{ for } \xi' \neq 0. \end{aligned} \tag{101}$$

Finally, from (92) in view of (98) and (101), we have

$$\begin{aligned} S_D(\xi') &= -\frac{\tilde{\beta}(\alpha_2 + |\xi'|) + i\alpha_1 \tilde{\beta}}{\tilde{a}_{33}^{(2)}[\alpha_1^2 + (\alpha_2 + |\xi'|)^2]} + \frac{i\tilde{\beta}\alpha_2}{\bar{\tau} + i|\xi'|} [-2\mathfrak{S}_0(\mathcal{V}_\omega; \xi')] \\ &= -\frac{\tilde{\beta}(\alpha_2 + |\xi'|)[1 + 2\alpha_2 \tilde{a}_{33}^{(2)} \mathfrak{S}_0(\mathcal{V}_\omega; \xi')] + i\alpha_1 \tilde{\beta}[1 - 2\alpha_2 \tilde{a}_{33}^{(2)} \mathfrak{S}_0(\mathcal{V}_\omega; \xi')]}{\tilde{a}_{33}^{(2)}[\alpha_1^2 + (\alpha_2 + |\xi'|)^2]}, \end{aligned}$$

whence the following inequality follows:

$$\operatorname{Re} S_D(\xi') = -\frac{\tilde{\beta}(\alpha_2 + |\xi'|)[1 + 2\alpha_2 \tilde{a}_{33}^{(2)} \mathfrak{S}_0(\mathcal{V}_\omega; \xi')]}{\tilde{a}_{33}^{(2)}[\alpha_1^2 + (\alpha_2 + |\xi'|)^2]} < 0 \text{ for } \xi' \neq 0 \tag{102}$$

due to the relations (see (C5))

$$\tilde{\beta} > 0, \quad \tilde{a}_{33}^{(2)} > 0, \quad |\xi'| > 0, \quad \alpha_2 > 0, \quad \mathfrak{S}_0(\mathcal{V}_\omega; \xi') > 0 \quad \forall \xi' \neq 0. \tag{103}$$

Thus, the Šapiro-Lopatinskii type condition for the “boundary symbol” S_D defined by (87) is satisfied and the operator D in (74) and (75) is Fredholm.

Step 3. Here, we prove that the index of the operator D equals to zero. To this end, let us consider the operator

$$D_t := \begin{bmatrix} r_{\Omega_2}(\beta I + N_\chi) \mathring{E} & r_{\Omega_2}[-V_\chi + W_\chi \mathcal{M}_\omega^{-1} \mathcal{K}_\omega] \\ t N_\chi^+ \mathring{E} & (t-1)\beta I + t\{-\mathcal{V}_\chi + [(\beta - \mu)I + \mathcal{W}_\chi] \mathcal{M}_\omega^{-1} \mathcal{K}_\omega\} \end{bmatrix}, \tag{104}$$

with $t \in [0, 1]$, and establish that it is homotopic to the operator D .

Evidently, $\mathbf{D}_1 = \mathbf{D}$ and $\mathbf{D}_t : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{\frac{1}{2}}(S)$ is continuous. First, we show that for the operator \mathbf{D}_t , the Šapiro-Lopatinskiĭ condition is satisfied for all $t \in [0, 1]$. The counterpart of the matrix (83) now reads as

$$\mathbf{R}_t(\tilde{y}, \xi) := \begin{bmatrix} \mathbf{R}_{11}(\tilde{y}, \xi) & \mathbf{R}_{12}(\tilde{y}, \xi) \\ t \mathbf{R}_{21}(\tilde{y}, \xi) & \mathbf{R}_{22,t}(\tilde{y}, \xi') \end{bmatrix},$$

where \mathbf{R}_{11} , \mathbf{R}_{12} , and \mathbf{R}_{21} are defined by formulas (84), (82), and (85), respectively, while in accordance with (104) and (86),

$$\mathbf{R}_{22,t}(\tilde{y}, \xi') := \mathfrak{C}_0((t-1)\beta I + t\{-\mathcal{V}_\chi + [(\beta - \mu)I + \mathcal{W}_\chi] \mathcal{M}_\omega^{-1} \mathcal{K}_\omega\}); \tilde{y}, \xi' = t \mathbf{R}_{22}(\tilde{y}, \xi') + (t-1)\beta.$$

The corresponding boundary symbol associated with the Šapiro-Lopatinskiĭ condition, the counterpart of (87), has the form

$$\begin{aligned} \mathbf{S}_{\mathbf{D},t}(\xi') &= -\Pi' \left[\frac{t \mathbf{R}_{21}}{\mathbf{R}_{11}^{(+)}} \Pi^+ \left(\frac{\mathbf{R}_{12}}{\mathbf{R}_{11}^{(-)}} \right) \right] (\xi') + \mathbf{R}_{22}^{(t)}(\xi') = -t \Pi' \left[\frac{\mathbf{R}_{21}}{\mathbf{R}_{11}^{(+)}} \Pi^+ \left(\frac{\mathbf{R}_{12}}{\mathbf{R}_{11}^{(-)}} \right) \right] (\xi') + t \mathbf{R}_{22}(\tilde{y}, \xi') - (1-t)\tilde{\beta} \\ &= t \mathbf{S}_{\mathbf{D}}(\xi') - (1-t)\tilde{\beta}, \end{aligned}$$

and due to the inequalities (102) and (103), we have

$$\operatorname{Re} \mathbf{S}_{\mathbf{D},t}(\xi') = t \operatorname{Re} \mathbf{S}_{\mathbf{D}}(\xi') - (1-t)\tilde{\beta} < 0 \quad \forall \xi' \neq 0 \quad \forall t \in [0, 1].$$

Thus, the Šapiro-Lopatinskiĭ condition for the operator \mathbf{D}_t is satisfied for all $t \in [0, 1]$. Therefore, as in the case of the operator \mathbf{D} , it follows that the operator $\mathbf{D}_t : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{\frac{1}{2}}(S)$ is Fredholm and has the same index for all $t \in [0, 1]$.

On the other hand, the upper triangular matrix operator \mathbf{D}_0 has zero index since one of the operators in the main diagonal, $-\beta I : H^{-\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S)$ is invertible, while the second operator, $\mathbf{D}_{11} = r_{\Omega_2}(\beta I + \mathbf{N}_\chi) \mathring{E} : H^1(\Omega_2) \rightarrow H^1(\Omega_2)$ is Fredholm with zero index as it has been shown in Step 1. Consequently, $\operatorname{Ind} \mathbf{D} = \operatorname{Ind} \mathbf{D}_1 = \operatorname{Ind} \mathbf{D}_t = \operatorname{Ind} \mathbf{D}_0 = 0$. Step 4. Now, we show that the operator \mathbf{D} is injective, which will imply its invertibility.

Let $\tilde{U} = (\tilde{u}_2, \tilde{\psi}_2)^\top \in H^1(\Omega_2) \times H^{-\frac{1}{2}}(S)$ be a solution to the homogeneous equation

$$\mathbf{D} \tilde{U} = 0. \tag{105}$$

Since the operator \mathbf{D} is Fredholm with zero index, there exists a left regulariser $\mathfrak{R}_{\mathbf{D}}$ such that

$$\mathfrak{R}_{\mathbf{D}} : H^1(\Omega_2) \times H^{\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{-\frac{1}{2}}(S)$$

and $\mathfrak{R}_{\mathbf{D}} \mathbf{D} = I + \mathfrak{Z}_{\mathbf{D}}$, where $\mathfrak{Z}_{\mathbf{D}}$ is the operator of order -1 (cf, eg, the proof of Theorems 22.1 and 23.1 in Eskin³⁵),

$$\mathfrak{Z}_{\mathbf{D}} : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \rightarrow H^2(\Omega_2) \times H^{\frac{1}{2}}(S). \tag{106}$$

Therefore, for $\tilde{U} = (\tilde{u}_2, \tilde{\psi}_2)^\top \in H^1(\Omega_2) \times H^{-\frac{1}{2}}(S)$ from (105), we have

$$\mathfrak{R}_{\mathbf{D}} \mathbf{D} \tilde{U} = \tilde{U} + \mathfrak{Z}_{\mathbf{D}} \tilde{U} = 0, \tag{107}$$

and in view of (106) and (107), we deduce

$$\tilde{U} = (\tilde{u}_2, \tilde{\psi}_2)^\top \in H^2(\Omega_2) \times H^{\frac{1}{2}}(S).$$

Clearly, by \tilde{u}_2 and $\tilde{\psi}_2$ we can construct the vector $U^{(0)} = (\tilde{u}_2, \tilde{\psi}_2, \tilde{\varphi}_2, \tilde{\psi}_1, \tilde{\varphi}_1, \tilde{u}_1) \in \mathbf{H}$, a solution to the homogeneous system (64)-(69). Here \mathbf{H} is defined in (56).

Therefore by equivalence Theorem 2 and uniqueness Theorem 1, we conclude that $U^{(0)}$ is a zero vector. Thus, the null space of the operator \mathbf{D} is trivial in the class $H^1(\Omega_2) \times H^{-\frac{1}{2}}(S)$. Consequently, the operator $\mathbf{D} : H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \rightarrow H^1(\Omega_2) \times H^{\frac{1}{2}}(S)$ is invertible, implying that the operator (72) is invertible as well, which completes the proof. \square

For a cut-off function χ of infinite smoothness, we have the following result.

Corollary 2. *Let a cut-off function $\chi \in X_+^\infty$. Then the operators*

$$\mathbf{D} : H^{r+1}(\Omega_2) \times H^{r-\frac{1}{2}}(S) \rightarrow H^{r+1}(\Omega_2) \times H^{r+\frac{1}{2}}(S), \tag{108}$$

$$\mathbf{M} : H^{r+1}(\Omega_2) \times H^{r-\frac{1}{2}}(S) \times H^{r+\frac{1}{2}}(S) \rightarrow H^{r+1}(\Omega_2) \times H^{r+\frac{1}{2}}(S) \times H^{r-\frac{1}{2}}(S), \tag{109}$$

where the \mathbf{D} and \mathbf{M} are defined by (75) and (70), respectively, are invertible for all $r > -\frac{1}{2}$.

Proof. It can be carried out by the word for word arguments applied in the proof of Theorem 3 and using the counterparts of Theorems 8 and 10 describing the mapping and smoothness properties of the localised potentials for a cut-off function of infinite smoothness, which actually coincide with the properties of usual potentials without localisation.

In the final part, Step 4, one needs to apply the fact that the operator (108) possesses a common regulariser for all $r > -\frac{1}{2}$ (see, eg, Agranovich³⁸) implying that the null space of the operator \mathbf{D} is trivial for all $r > -\frac{1}{2}$, which yields that the operators (108) and (109) are invertible for all $r > -\frac{1}{2}$. \square

From Theorem 3 and Lemma 2, we derive also the invertibility result for operator (71).

Corollary 3. *Let a cut-off function $\chi \in X_+^4$. Then the operator $\mathbf{M} : \mathbb{H} \rightarrow \mathbb{F}$ is invertible.*

Summarising the above-obtained results, we can make the following conclusions. Consider LBDIE system (50) - (55) with arbitrary right hand sides,

$$\beta u_2 + \mathcal{N}_\chi u_2 - V_\chi \psi_2 + W_\chi \varphi_2 = h_1 \quad \text{in } \Omega_2, \tag{110}$$

$$\mathcal{N}_\chi^+ u_2 - \mathcal{V}_\chi \psi_2 + [(\beta - \mu)I + \mathcal{W}_\chi] \varphi_2 = h_2 \quad \text{on } S, \tag{111}$$

$$\mathcal{K}_\omega \psi_2 - \mathcal{M}_\omega \varphi_2 = h_3 \quad \text{on } S, \tag{112}$$

$$\psi_2 - \psi_1 = h_4 \quad \text{on } S, \tag{113}$$

$$\varphi_2 - \varphi_1 = h_5 \quad \text{on } S, \tag{114}$$

$$u_1 + V_\omega \psi_1 - W_\omega \varphi_1 = h_6 \quad \text{in } \Omega_1. \tag{115}$$

Theorem 3 and Corollaries 2 and 3 imply the following assertion.

Corollary 4.

(i) *LBDIE system (110) - (115) with arbitrary right hand side data*

$$(h_1, \dots, h_6) \in \mathbf{Y} := H^1(\Omega_2) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times H_{comp}^1(\Omega_1) \tag{116}$$

is uniquely solvable in the space

$$\mathbf{X} := H^1(\Omega_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times (H_{loc}^1(\Omega_1) \cap Z(\Omega_1)). \tag{117}$$

(ii) *LBDIE system (110) - (115) with arbitrary right hand side data*

$$(h_1, \dots, h_6) \in \mathbf{F} := H^{1,0}(\Omega_2; \Delta) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times H_{comp}^{1,0}(\Omega_1; A_1)$$

is uniquely solvable in the space \mathbf{H} defined in (56),

$$\mathbf{H} = H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \times (H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1)).$$

In particular, under conditions (18), system (50) - (55) is uniquely solvable in the space \mathbf{H} .

In both cases, (i) and (ii), the solution continuously depends on the right hand side data provided $\text{supp } h_6 \subset \overline{\Omega_0}$, where $\overline{\Omega_0}$ is a fixed compact subset of Ω_1 .

Finally, Corollary 4(ii), equivalence Theorem 2, and uniqueness Theorem 1 lead to the following assertion.

Theorem 4. *Let conditions (18) hold. Transmission problem (14) - (17) is uniquely solvable, and the solution continuously depends on the right hand side data provided $\text{supp } f_1 \subset \overline{\Omega_0}$, where Ω_0 is a fixed compact subset of Ω_1 .*

5 | COUPLING OF VARIATIONAL AND NONLOCAL BIE APPROACH

Here, we present an alternative approach for investigation of transmission problem (14) - (18). We apply the nonlocal approach and reformulate the transmission problem in variational form. To this end, we recall the first Green identity (9) in Ω_2 ,

$$\int_{\Omega_2} \left[a_{kj}^{(2)} \partial_j u_2 \overline{\partial_k v} - \omega^2 \kappa_2 u_2 \overline{v} \right] dx - \langle T_2^+ u_2, \overline{\gamma^+ v} \rangle_S = - \int_{\Omega_2} (A_2 u_2) \overline{v} dx, \quad \forall u_2 \in H^{1,0}(\Omega_2; A_2), v \in H^1(\Omega_2). \quad (118)$$

Assuming that a pair $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times (H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1))$ solves transmission problem (14) - (18) and implementing the Steklov-Poincaré type relation (48), we reduce (118) to equation

$$\mathfrak{B}(u_2, v) = \mathfrak{F}(v) \quad \forall v \in H^1(\Omega_2), \quad (119)$$

where \mathfrak{B} is a sesquilinear form and \mathfrak{F} is an antilinear functional defined, respectively, as

$$\mathfrak{B}(u_2, v) := \int_{\Omega_2} \left[a_{kj}^{(2)}(x) \partial_j u_2(x) \overline{\partial_k v(x)} - \omega^2 \kappa_2(x) u_2(x) \overline{v(x)} \right] dx - \langle \mathcal{K}_\omega^{-1} \mathcal{M}_\omega(\gamma^+ u_2), \overline{\gamma^+ v} \rangle_S, \quad (120)$$

$$\mathfrak{F}(v) := - \int_{\Omega_2} f_2(x) \overline{v(x)} dx + \langle \Phi_\omega, \overline{\gamma^+ v} \rangle_S, \quad (121)$$

with $\Phi_\omega := \mathcal{K}_\omega^{-1} [\Psi_\omega f_1 - \mathcal{M}_\omega \varphi_0] + \psi_0 \in H^{-\frac{1}{2}}(S)$. Here, the operators \mathcal{K}_ω , \mathcal{M}_ω , and Ψ_ω are defined by relations (45) - (47).

We associate with Equation 119 the following variational problem (in a wider space):

- Find a function $u_2 \in H^1(\Omega_2)$ satisfying (119).

Let us first prove the following equivalence theorem.

Theorem 5. *Let conditions (18) be fulfilled.*

- If a pair $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times (H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1))$ solves transmission problem (14) - (18), then the function u_2 solves variational Equation (119).
- Vice versa, if a function $u_2 \in H^1(\Omega_2)$ solves variational Equation 119, then the pair (u_2, u_1) , where

$$u_1(y) = \mathcal{P}_\omega f_1(y) - V_\omega(T_2^+ u_2 - \psi_0)(y) + W_\omega(\gamma^+ u_2 - \varphi_0)(y), \quad y \in \Omega_1, \quad (122)$$

belongs to the class $H^{1,0}(\Omega_2; A_2) \times (H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1))$ and solves transmission problem (14) - (18).

Proof.

- The first part of the theorem follows from the derivation of variational Equation (119).
- To prove the second part, we proceed as follows. If u_2 solves (119), then the equation particularly holds for $v \in \mathcal{D}(\Omega_2)$, which implies that u_2 is a solution of Equation (15) in the sense of distributions and evidently $u_2 \in H^{1,0}(\Omega_2; A_2)$ since $f_2 \in H^0(\Omega_2)$ in view of (18). Therefore, the canonical co-normal derivative $T_2^+ u_2 \in H^{-\frac{1}{2}}(S)$ is well defined in the sense of (7).

Further, it is easy to see that function (122) is well defined, solves the differential Equation (14) due to (40) and (41), and belongs to the space $H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1)$ in view of (18). Therefore, the canonical co-normal derivative $T_1^- u_1 \in H^{-\frac{1}{2}}(S)$ is well defined in the sense of (8) as well.

In order to show that transmission conditions (16) and (17) are also satisfied, we write Green's identity (118) for u_2 and arbitrary $v \in H^1(\Omega_2)$ and subtract it from (119) to obtain

$$\langle T_2^+ u_2 - \mathcal{K}_\omega^{-1} \mathcal{M}_\omega(\gamma^+ u_2) - \Phi_\omega, \overline{\gamma^+ v} \rangle_S = 0.$$

Whence $T_2^+ u_2 - \mathcal{K}_\omega^{-1} \mathcal{M}_\omega(\gamma^+ u_2) - \Phi_\omega = 0$ on S , ie, $T_2^+ u_2 - \psi_0 - \mathcal{K}_\omega^{-1} \mathcal{M}_\omega(\gamma^+ u_2) - \mathcal{K}_\omega^{-1} (\Psi_\omega f_1 - \mathcal{M}_\omega \varphi_0) = 0$ on S , which is equivalent to the condition

$$\mathcal{K}_\omega(T_2^+ u_2 - \psi_0) - \mathcal{M}_\omega(\gamma^+ u_2 - \varphi_0) - \Psi_\omega f_1 = 0 \quad \text{on } S.$$

In turn, in view of (45) - (47), the latter implies

$$T_1^+ w - i\alpha\gamma^+ w = 0 \quad \text{on } S, \tag{123}$$

where

$$w := V_\omega(T_2^+ u_2 - \psi_0) - W_\omega(\gamma^+ u_2 - \varphi_0) - \mathcal{P}_\omega f_1 \quad \text{in } \Omega_2. \tag{124}$$

The function w satisfies the homogeneous equation $A_1 w = 0$ in Ω_2 in view of (124) and the homogeneous Robin condition (123). As in the proof of Theorem 2, we can deduce that $\gamma^+ w = 0$ and $T_1^+ w = 0$ on S for real $\alpha \neq 0$, implying $w = 0$ in Ω_2 . Therefore, for the function u_1 defined in (122) by Lemma 5, we have

$$\gamma^- u_1 = \gamma^- u_1 + \gamma^+ w = \gamma^+ u_2 - \varphi_0, \quad T_1^- u_1 = T_1^- u_1 + T_1^+ w = T^+ u_2 - \psi_0,$$

which completes the proof. □

Corollary 5. *The homogeneous variational problem (119) (with $\mathfrak{F} = 0$) possesses only the trivial solution.*

Proof. It follows from the uniqueness and equivalence Theorems 1 and 5, respectively. □

Further, we analyse the coercivity properties of the sesquilinear form \mathfrak{B} .

Lemma 3. *For the sesquilinear form \mathfrak{B} defined in (120), there are real constants $C_1^* > 0$, $C_2^* > 0$, and C_3^* such that*

$$\begin{aligned} |\mathfrak{B}(u, v)| &\leq C_1^* \|u\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)} & \forall u, v \in H^1(\Omega_2), \\ \text{Re } \mathfrak{B}(u, u) &\geq C_2^* \|u\|_{H^1(\Omega_2)}^2 - C_3^* \|u\|_{H^0(\Omega_2)}^2 & \forall u \in H^1(\Omega_2). \end{aligned}$$

Proof. The first equality follows from (120) by the Cauchy-Schwartz inequality and the trace theorem. To prove the second inequality, we use the positive definiteness of the matrix $\mathbf{a}_2 = [a_{kj}^{(2)}]_{k,j=1}^3$, Remark 7, and the trace theorem to obtain

$$\begin{aligned} \text{Re } \mathfrak{B}(u, u) &\geq c_1 \|u\|_{H^1(\Omega_2)}^2 - c_2 \|u\|_{H^0(\Omega_2)}^2 + C_1 \|\gamma^+ u\|_{H^{\frac{1}{2}}(S)}^2 - C_2 \|\gamma^+ u\|_{H^0(S)}^2 \\ &\geq c_1 \|u\|_{H^1(\Omega_2)}^2 - c_2 \|u\|_{H^0(\Omega_2)}^2 - C_2 \|\gamma^+ u\|_{H^\delta(S)}^2 \geq c_1 \|u\|_{H^1(\Omega_2)}^2 - c_2 \|u\|_{H^0(\Omega_2)}^2 - c_3 \|u\|_{H^{\frac{1}{2}+\delta}(\Omega_2)}^2, \end{aligned}$$

where $c_1 > 0$, $c_2 = \omega^2 \max_{\Omega_2} \kappa_2(x)$, $C_1 > 0$ and $C_2 \geq 0$ are the constants involved in (C8), $c_3 > 0$, and δ is an arbitrarily small positive number. Now, by Ehrling's lemma, cf, eg, Theorem 7.30 in Renardy et al,³⁹ for arbitrarily small positive number ϵ , there is a positive constant $C(\epsilon)$, such that

$$\|u\|_{H^{\frac{1}{2}+\delta}(\Omega_2)} \leq \epsilon \|u\|_{H^1(\Omega_2)} + C(\epsilon) \|u\|_{H^0(\Omega_2)},$$

which completes the proof. □

Now, we prove the following existence results.

Theorem 6. *Let \mathfrak{F} be a bounded linear functional on $H^1(\Omega_2)$. Then variational Equation (119) is uniquely solvable in the space $H^1(\Omega_2)$.*

Proof. By Lemma 3, the sesquilinear functional $\mathfrak{B}_\lambda(u, v) := \mathfrak{B}(u, v) + \lambda \langle u, v \rangle_{\Omega_2}$ with $\lambda > |C_3^*|$ is positive and bounded below on the space $H^1(\Omega_2) \times H^1(\Omega_2)$. Due to the Lax-Milgram lemma, \mathfrak{B}_λ defines an invertible linear operator $\mathbf{T}_\lambda : H^1(\Omega_2) \rightarrow \tilde{H}^{-1}(\Omega_2)$ for $\lambda > |C_3^*|$. Therefore, for arbitrary λ , the operator \mathbf{T}_λ is Fredholm with zero index (see, eg, Theorem 2.33 in McLean³⁷), since the sesquilinear form $\lambda \langle u, v \rangle_{\Omega_2}$ defines a compact imbedding operator $\lambda I : H^1(\Omega_2) \rightarrow \tilde{H}^{-1}(\Omega_2)$, where I is the identity operator. By Corollary 5, the operator \mathbf{T}_0 defined by the sesquilinear form $\mathfrak{B}(u, v) = \mathfrak{B}_0(u, v)$ possesses the trivial null-space and consequently is invertible, which completes the proof. □

Theorem 7. Let conditions (18) be fulfilled. Then transmission problem (14) - (18) is uniquely solvable in the space $H^{1,0}(\Omega_2; A_2) \times (H_{loc}^{1,0}(\Omega_1; A_1) \cap Z(\Omega_1))$.

Proof. If conditions (18) are satisfied, then the linear functional \mathfrak{F} given by (121) is bounded,

$$|\mathfrak{F}(v)| \leq C \|v\|_{H^1(\Omega_2)} \quad \forall v \in H^1(\Omega_2),$$

which follows from the Cauchy-Schwartz inequality, trace theorem, and properties of the operators \mathcal{K}_ω , \mathcal{M}_ω , and Ψ_ω defined by relations (45) - (47).

Therefore, by equivalence Theorem 5 and existence Theorem 6 along with uniqueness Theorem 1, we conclude that the transmission problem (14) - (18) is uniquely solvable. \square

Remark 4. From the equivalence Theorem 2 and existence Theorem 7, it follows that the LBDIE system (50) - (55) possesses a unique solution in the space \mathbf{H} defined by (56). However, this does not imply the results obtained in Section 4 concerning neither the invertibility of the localised boundary-domain matrix integral operator generated by the left hand side expressions in (50) - (55) nor the solvability in the space \mathbf{X} of system (110) - (115) with arbitrary right hand side functions from the space \mathbf{Y} (see (116) and (117)). The case is that Theorems 2 and 7 yield unique solvability of system (50) - (55) with only special form right hand side functions represented by volume and surface integrals (see the right hand side functions in (50) - (55)).

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APPENDIX A: CLASSES OF CUT-OFF FUNCTIONS

Here, we present some classes of localising cut-off functions (for details, see Chkadua et al³³).

Definition 2. We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \check{\chi}(|x|)$, $\check{\chi} \in W_1^k(0, \infty)$ and $\rho\check{\chi}(\rho) \in L_1(0, \infty)$.

We say $\chi \in X_+^k$ for integer $k \geq 1$ if $\chi \in X^k$, $\chi(0) = 1$, and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \hat{\chi}_s(\omega) > 0 \text{ for } \omega \in \mathbb{R} \setminus \{0\} & , \\ \int_0^\infty \rho \check{\chi}(\rho) d\rho \text{ for } \omega = 0 & , \end{cases} \quad \hat{\chi}_s(\omega) := \int_0^\infty \check{\chi}(\rho) \sin(\rho\omega) d\rho.$$

The following lemma provides an easily verifiable sufficient condition for nonnegative nonincreasing functions to belong to the class X_+^k .

Lemma 4. (Chkadua et al,³³ Lemma 3.2). *Let $k \geq 1$. If $\chi \in X^k$, $\check{\chi}(0) = 1$, $\check{\chi}(\rho) \geq 0$ for all $\rho \in (0, \infty)$, and $\check{\chi}$ is a nonincreasing function on $[0, +\infty)$, then $\chi \in X_+^k$.*

Here are some particular examples of cut-off functions:

$$\chi_{1k}(x) = \begin{cases} \left[1 - \frac{|x|}{\epsilon}\right]^k & \text{for } |x| < \epsilon, \\ 0 & \text{for } |x| \geq \epsilon, \end{cases} \quad \chi_{2k}(x) = \begin{cases} \left[1 - \frac{|x|^2}{\epsilon^2}\right]^k & \text{for } |x| < \epsilon, \\ 0 & \text{for } |x| \geq \epsilon, \end{cases} \quad \chi_3(x) = \begin{cases} \exp\left[\frac{|x|^2}{|x|^2 - \epsilon^2}\right] & \text{for } |x| < \epsilon, \\ 0 & \text{for } |x| \geq \epsilon. \end{cases}$$

Due to Lemma 4, we have $\chi_{1k} \in X_+^k$, $\chi_{2k} \in X_+^k \cap C^{k-1}(\mathbb{R}^3)$, and $\chi_3 \in X_+^\infty \cap C^\infty(\mathbb{R}^3)$.

APPENDIX B: PROPERTIES OF LOCALISED POTENTIALS

Here, we collect some theorems describing mapping properties of the localised potentials (25) and (26), and the localised boundary operators generated by them

$$\mathcal{V}_\chi g(y) := - \int_S P_\chi(x-y)g(x)dS_x, \quad \mathcal{W}_\chi g(y) := - \int_S [T_2(x, \partial_x)P_\chi(x-y)]g(x)dS_x, \quad y \in S. \tag{B1}$$

Note that \mathcal{V}_χ is a weakly singular integral operator (pseudodifferential operator of order -1), while \mathcal{W}_χ is a singular integral operator (pseudodifferential operator of order 0).

Remark that if $S \in C^\infty$ and a cut-off function χ is infinitely differentiable, then the localised potentials and the corresponding boundary operators have the same mapping properties as the corresponding harmonic potentials (see, eg, Miranda¹⁰ and Hsiao and Wendland³⁴). However, for cut-off functions of finite smoothness, the localised potential operators possess quite different properties, in particular, their smoothness is reduced and the smoothness exponents depend on the smoothness of a cut-off function χ . Properties of the localised potentials needed in our analysis in the main text are presented below (detailed proofs can be found in Chkadua et al^{26,33}).

Theorem 8. (Chkadua et al,³³ Theorems 5.6 and 5.10). *The following operators are continuous*

$$\begin{aligned} P_\chi &: H^s(\Omega_2) \rightarrow H^{s+2,S}(\Omega_2; \Delta), \quad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1. \\ V_\chi &: H^{s-\frac{3}{2}}(S) \rightarrow H^s(\Omega_2), \quad \frac{1}{2} < s < k + \frac{1}{2}, \quad \text{if } \chi \in X^k, \quad k = 1, 2, \dots \\ &: H^{s-\frac{3}{2}}(S) \rightarrow H^{s,S-1}(\Omega_2; \Delta), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \text{if } \chi \in X^2, \\ W_\chi &: H^{s-\frac{1}{2}}(S) \rightarrow H^s(\Omega_2), \quad \frac{1}{2} < s < k - \frac{1}{2}, \quad \text{if } \chi \in X^k, \quad k = 2, 3, \dots \\ &: H^{s-\frac{1}{2}}(S) \rightarrow H^{s,S-1}(\Omega_2; \Delta), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \text{if } \chi \in X^3, \end{aligned}$$

where $H^{l,r}(\Omega_2; \Delta) := \{u \in H^l(\Omega_2) : \Delta u \in H^r(\Omega_2)\}$.

Theorem 9. (Chkadua et al,³³ Corollary 5.12 and Theorem 5.13). *Let $\chi \in X^2$, $\psi \in H^{-\frac{1}{2}}(S)$, and $\varphi \in H^{\frac{1}{2}}(S)$. Then there hold the following relations on S*

$$\gamma^+ V_\chi \psi = \mathcal{V}_\chi \psi, \quad \gamma^+ W_\chi \varphi = -\mu \varphi + \mathcal{W}_\chi \varphi \quad \text{with } \mu(y) := \frac{1}{2} a_{kj}^{(2)}(y) n_k(y) n_j(y) > 0, \quad y \in S. \tag{B2}$$

Theorem 10. (Chkadua et al,³³ Theorem 5.14). *Let $-\frac{1}{2} < s < \frac{1}{2}$. The following operators are continuous,*

$$\begin{aligned} \mathcal{V}_\chi &: H^{s-\frac{1}{2}}(S) \rightarrow H^{s+\frac{1}{2}}(S), \quad \chi \in X^1, \\ \mathcal{W}_\chi &: H^{s+\frac{1}{2}}(S) \rightarrow H^{s+\frac{1}{2}}(S), \quad \chi \in X^2. \end{aligned} \tag{B3}$$

Moreover, operator (B3) is Fredholm with zero index.

Remark 5. The principal homogeneous symbols of the boundary pseudodifferential operators \mathcal{V}_χ , $-\mu I + \mathcal{W}_\chi$, and $(\beta - \mu)I + \mathcal{W}_\chi$, calculated in a local coordinate system with the origin at a point $\tilde{y} \in S$ and the third axis coinciding with the normal vector at the point $\tilde{y} \in S$, read as

$$\mathfrak{S}_0(\mathcal{V}_\chi; \tilde{y}, \xi') = \frac{1}{2|\xi'|}, \quad \mathfrak{S}_0(-\mu I + \mathcal{W}_\chi; \tilde{y}, \xi') = -\frac{1}{2} \tilde{a}_{33}^{(2)}(\tilde{y}) - \frac{i}{2} \sum_{l=1}^2 \tilde{a}_{3l}^{(2)}(\tilde{y}) \frac{\xi'_l}{|\xi'|}, \quad (\text{B4})$$

$$\mathfrak{S}_0((\beta - \mu)I + \mathcal{W}_\chi; \tilde{y}, \xi') = \frac{1}{2} \left[2\tilde{\beta} - \tilde{a}_{33}^{(2)}(\tilde{y}) - i \sum_{l=1}^2 \tilde{a}_{3l}^{(2)}(\tilde{y}) \frac{\xi'_l}{|\xi'|} \right], \quad \xi' \in \mathbb{R}^2 \setminus \{0\}, \quad (\text{B5})$$

where $[\tilde{a}_{kj}^{(2)}(\tilde{y})]_{k,j=1}^3 = [\lambda_{pk}(\tilde{y}) a_{pq}^{(2)}(\tilde{y}) \lambda_{qj}(\tilde{y})]_{k,j=1}^3 = \Lambda(\tilde{y})^\top \mathbf{a}_2(\tilde{y}) \Lambda(\tilde{y})$ is a positive definite matrix and

$$\tilde{\beta}(\tilde{y}) = \frac{1}{3} [\tilde{a}_{11}^{(2)}(\tilde{y}) + \tilde{a}_{22}^{(2)}(\tilde{y}) + \tilde{a}_{33}^{(2)}(\tilde{y})] > 0.$$

Here, $\mathbf{a}_2(\tilde{y}) = [a_{kj}^{(2)}(\tilde{y})]_{k,j=1}^3$ and $\Lambda(\tilde{y}) = [\lambda_{kj}(\tilde{y})]_{3 \times 3}$ is an orthogonal matrix with the property $\Lambda(\tilde{y})^\top n(\tilde{y}) = (0, 0, -1)^\top$, where $n(\tilde{y})$ is the outward unit normal vector at the point $\tilde{y} \in S$. Therefore $\lambda_{p3}(\tilde{y}) = -n_p(\tilde{y})$, $p = 1, 2, 3$. In view of (B2), it is evident that $\frac{1}{2} \tilde{a}_{33}^{(2)}(\tilde{y}) = \frac{1}{2} \lambda_{p3}(\tilde{y}) a_{pq}^{(2)}(\tilde{y}) \lambda_{q3}(\tilde{y}) = \mu(\tilde{y}) > 0$.

APPENDIX C: PROPERTIES OF RADIATING POTENTIALS

The layer potentials defined by (36) and the volume potential (cf (37))

$$\mathbf{P}_\omega f(y) := \int_{\mathbb{R}^3} \Gamma(x - y, \omega) f(x) dx, \quad y \in \mathbb{R}^3,$$

have the following properties (for details see Jentsch et al²⁰).

Lemma 5.

(i) *The following operators are continuous*

$$\begin{aligned} V_\omega &: H^{-\frac{1}{2}}(S) \rightarrow H^1(\Omega_2, A_1) & [H^{-\frac{1}{2}}(S) \rightarrow H_{loc}^1(\Omega_1, A_1) \cap Z(\Omega_1)], \\ W_\omega &: H^{\frac{1}{2}}(S) \rightarrow H^1(\Omega_2, A_1) & [H^{\frac{1}{2}}(S) \rightarrow H_{loc}^1(\Omega_1, A_1) \cap Z(\Omega_1)], \\ \mathbf{P}_\omega &: H_{comp}^0(\mathbb{R}^3) \rightarrow H_{loc}^2(\mathbb{R}^3) \cap Z(\mathbb{R}^3). \end{aligned}$$

Moreover,

$$A_1 \mathbf{P}_\omega f = f \text{ in } \mathbb{R}^3 \text{ for } f \in H_{comp}^0(\mathbb{R}^3).$$

(ii) *For $h \in H^{-\frac{1}{2}}(S)$ and $g \in H^{\frac{1}{2}}(S)$, the following jump relations hold true*

$$\begin{aligned} \gamma^+ V_\omega h &= \gamma^- V_\omega(h) = \mathcal{V}_\omega(h), & T_1^\pm V_\omega h &= (\pm \frac{1}{2} I + \mathcal{W}'_\omega) h \text{ on } S, \\ \gamma^\pm W_\omega g &= (\mp \frac{1}{2} I + \mathcal{W}_\omega) g, & T_1^+ W_\omega g &= T_1^- W_\omega g =: \mathcal{L}_\omega g \text{ on } S, \end{aligned} \quad (\text{C1})$$

where I stands for the identity operator, and

$$\mathcal{V}_\omega h(y) := - \int_S \Gamma(x - y, \omega) h(x) dS_x, \quad y \in S, \quad (\text{C2})$$

$$\mathcal{W}_\omega g(y) := - \int_S [T_1(x, \partial_x) \Gamma(x - y, \omega)] g(x) dS_x, \quad y \in S, \quad (\text{C3})$$

$$\mathcal{W}'_\omega h(y) := - \int_S [T_1(y, \partial_y) \Gamma(x - y, \omega)] h(x) dS_x, \quad y \in S, \quad (\text{C4})$$

$\Gamma(x, \omega)$ is the radiating fundamental solution defined by (38).

(iii) *The following operators are continuous,*

$$\mathcal{V}_\omega : H^{-\frac{1}{2}}(S) \rightarrow H^{\frac{1}{2}}(S), \quad \mathcal{W}_\omega : H^{\frac{1}{2}}(S) \rightarrow H^{\frac{1}{2}}(S), \quad \mathcal{W}'_\omega : H^{-\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S), \quad \mathcal{L}_\omega : H^{\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S).$$

(iv) The operators \mathcal{W}_ω , and \mathcal{W}'_ω are compact, since they have weakly singular kernel-functions of the type $O(|x - y|^{-1})$, \mathcal{V}_ω is a pseudodifferential operator of order -1 with positive principal homogeneous symbol, $\mathfrak{S}_0(\mathcal{V}_\omega; y, \xi') > 0$, and \mathcal{L}_ω is a singular integro-differential operator (pseudodifferential operator of order 1) with negative principal homogeneous symbol, $\mathfrak{S}_0(\mathcal{L}_\omega; y, \xi') < 0$; moreover,

$$\mathfrak{S}_0(\mathcal{L}_\omega; y, \xi') = -[4\mathfrak{S}_0(\mathcal{V}_\omega; y, \xi')]^{-1} < 0, \quad \xi' \in \mathbb{R}^2 \setminus \{0\}, \quad y \in S.$$

Lemma 6. Let \mathcal{K}_ω and \mathcal{M}_ω be defined by (45) and (46) with $\alpha > 0$. The following operators are invertible

$$\mathcal{K}_\omega : H^{-\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S), \quad \mathcal{M}_\omega : H^{\frac{1}{2}}(S) \rightarrow H^{-\frac{1}{2}}(S).$$

Remark 6. The principal homogeneous symbols of the pseudodifferential operators \mathcal{K}_ω , \mathcal{M}_ω , $\mathcal{M}_\omega^{-1}\mathcal{K}_\omega$, and $\mathcal{K}_\omega^{-1}\mathcal{M}_\omega$ calculated in a local coordinate system described in Remark 5 satisfy the relations

$$\mathfrak{S}_0(\mathcal{K}_\omega; y, \xi') = 1/2, \quad \mathfrak{S}_0(\mathcal{M}_\omega; y, \xi') = -[4\mathfrak{S}_0(\mathcal{V}_\omega; y, \xi')]^{-1} < 0, \quad (\text{C5})$$

$$\mathfrak{S}_0(\mathcal{M}_\omega^{-1}\mathcal{K}_\omega; y, \xi') = -2\mathfrak{S}_0(\mathcal{V}_\omega; y, \xi') < 0, \quad (\text{C6})$$

$$\mathfrak{S}_0(\mathcal{K}_\omega^{-1}\mathcal{M}_\omega; y, \xi') = -[2\mathfrak{S}_0(\mathcal{V}_\omega; y, \xi')]^{-1} < 0, \quad \xi' \in \mathbb{R}^2 \setminus \{0\}, \quad y \in S. \quad (\text{C7})$$

Remark 7. The principal homogenous symbols of the operators \mathcal{V}_ω , \mathcal{M}_ω , and $-\mathcal{K}_\omega^{-1}\mathcal{M}_\omega$ are positive in view of Lemma 5(iv) and Remark 6. Therefore, it can be shown that there are constants $C_1 > 0$ and $C_2 \geq 0$ such that the following inequalities hold (cf, eg, Theorem 6.2.7 in Hsiao and Wendland³⁴)

$$\begin{aligned} \langle \psi, \mathcal{V}_\omega \psi \rangle_S &\geq C_1 \|\psi\|_{H^{-\frac{1}{2}}(S)}^2 - C_2 \|\psi\|_{H^{-\frac{3}{2}}(S)}^2 \quad \forall \psi \in H^{-\frac{1}{2}}(S), \\ \langle \mathcal{M}_\omega \psi, \psi \rangle_S &\geq C_1 \|\psi\|_{H^{\frac{1}{2}}(S)}^2 - C_2 \|\psi\|_{H^{-\frac{1}{2}}(S)}^2 \quad \forall \psi \in H^{\frac{1}{2}}(S), \\ \langle -\mathcal{K}_\omega^{-1}\mathcal{M}_\omega \psi, \psi \rangle_S &\geq C_1 \|\psi\|_{H^{\frac{1}{2}}(S)}^2 - C_2 \|\psi\|_{H^{-\frac{1}{2}}(S)}^2 \quad \forall \psi \in H^{\frac{1}{2}}(S). \end{aligned} \quad (\text{C8})$$