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Analysis of Some Boundary-Domain Integral Equations for Variable-Coefficient Problems with Cracks

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Abstract

Some segregated direct boundary-domain integral equation (BDIE) systems associated with mixed, Dirichlet and Neumann boundary value problems (BVPs) for a scalar "Laplace" PDE with variable coefficient are formulated and analysed for domains with interior cuts (cracks). The main results established in the paper are the BDIE equivalence to the original BVPs and invertibility of the BDIE operators in the corresponding Sobolev space.

1 Introduction

Partial Differential Equations (PDEs) with variable coefficients arise naturally in mathematical modelling of non-homogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electro-magnetics, thermo-conductivity, fluid flows trough porous media, and other areas of physics and engineering.

The Boundary Integral Equation Method (Boundary Element Method) is a well established tool for solution Boundary Value Problems (BVPs) with constant coefficients. The main ingredient for reducing a BVP for a PDE to a BIE is a fundamental solution to the original PDE. However, it is generally not available in an analytical and/or cheaply calculated form for PDEs with variable coefficients. Following Levi and Hilbert, one can use in this case a parametrix (Levi function) as a substitute for the fundamental solution. Parametrix is usually much wider available than a fundamental solution and correctly describes the main part of the fundamental solution although does not have to satisfy the original PDE. This reduces the problem not to a boundary integral equation but to a Boundary-Domain Integral Equation (BDIE) system, see e.g. [12, 13].

In this paper, extending approach of [2], we develop analysis of some direct segregated BDIEs for the Dirichlet, Neumann and mixed variable-coefficient BVPs in domains with interior cuts (cracks), whose faces are subject to the Neumann conditions. Our main goal is to prove
(i) equivalence of the BDIE to the original crack type BVPs and
(ii) invertibility of the corresponding boundary-domain integral operators in appropriate Sobolev (Bessel potential) spaces.

2 Formulation of the boundary value problems

Let $\Omega = \Omega^+$ be a bounded open three–dimensional region of $\mathbb{R}^3$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$. For simplicity, we assume that the boundary $\partial\Omega$ is a simply connected, closed, infinitely smooth surface. Moreover, $\partial\Omega = \overline{S}_D \cup \overline{S}_N$ where S_D and S_N are nonintersecting ($S_D \cap S_N = \varnothing$), simply connected sub-manifolds of $\partial\Omega$ with infinitely smooth boundary curve $\ell := \partial S_D = \partial S_N \in C^\infty$. If either $S_D = \varnothing$ or $S_N = \varnothing$, then $\ell = \varnothing$. Further, we assume that the region Ω contains an interior crack. We define the crack as a two-dimensional, two-sided open manifold Σ with the crack edge $\partial\Sigma$. We assume that Σ is a sub-manifold of a simply connected closed infinitely smooth surface $\partial\Omega_0 \subset \Omega$ which is the boundary of a domain $\overline{\Omega}_0 \subset \Omega$. Denote $\Omega_{\Sigma} := \Omega \setminus \overline{\Sigma}$. Throughout the paper $n = (n_1, n_2, n_3)$ stands for the unit normal vector to $\partial\Omega$ exterior to Ω and for the unit normal vector to $\partial\Omega_0$ exterior to Ω_0. This agreement defines the positive direction of the normal vector on the crack surface Σ.

Further, let $a \in C^\infty(\overline{\Omega})$, $a(x) > 0$ for $x \in \overline{\Omega}$. Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. We consider boundary-domain integral equations associated with the following scalar elliptic differential equation

$$Lu(x) := L(x, \partial_x)\, u(x) := \sum_{i=1}^{3} \partial_{x_i} \Big(a(x)\, \partial_{x_i} u(x) \Big) = f(x), \quad x \in \Omega_{\Sigma}, \tag{2.1}$$

where u is an unknown function and f is a given function in Ω_{Σ}.

In what follows $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\Omega_{\Sigma}) = H_2^s(\Omega_{\Sigma})$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$, $H^s(\partial\Omega_0) = H_2^s(\partial\Omega_0)$, $s \in \mathbb{R}$, denote the Sobolev–Slobodetski (the Bessel potential) spaces. For $S \subset \partial\Omega$, we will use the space $\widetilde{H}^s(S) = \{g :\ g \in H^s(\partial\Omega),\ \mathrm{supp}\, g \subset \overline{S}\}$, and the space $H^s(S) = \{r_S g :\ g \in H^s(\partial\Omega)\}$ of restriction on S of functions from $H^s(\partial\Omega)$, where r_S denotes the restriction operator on S. Similar spaces are defined also on $\Sigma \subset \partial\Omega_0$.

From the trace theorem (see, e.g., [7]) it follows that $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, $\gamma^\pm u \in H^{\frac{1}{2}}(\Sigma)$ for $u \in H^1(\Omega_{\Sigma})$, where $\gamma^\pm$ is the trace operator.

For $u \in H^2(\Omega_{\Sigma})$, we denote by $T^\pm$ the corresponding co-normal derivative operator on $\partial\Omega$ and Σ in the trace sense,

$$T^\pm u(x) := a(x)\, \partial_n^\pm u(x) := \sum_{i=1}^{3} a(x)\, n_i(x)\, \gamma^\pm[\partial_i u(x)], \tag{2.2}$$

where ∂_n denotes the corresponding normal derivative operator. If $T^+ u = T^- u$, we will write Tu.

For the linear operator L, we introduce the following subspace of $H^s(\Omega_{\Sigma})$, c.f. [6, 4, 10],

$$H^{s,0}(\Omega_{\Sigma}; L) := \{g :\ g \in H^s(\Omega_{\Sigma}),\ Lg \in L_2(\Omega_{\Sigma})\}$$

endowed with the norm

$$\|g\|_{H^{s,0}(\Omega_{\Sigma}; L)} := \|g\|_{H^s(\Omega_{\Sigma})} + \|Lg\|_{L_2(\Omega_{\Sigma})}.$$

For a couple of functions (g^+, g^-) defined on a surface S, we denote their difference (jump) as $[g]_S = g^+ - g^-$, their average as $g_S^0 = (g^+ + g^-)/2$, and introduce the space $\mathbb{H}^s(S) := \{(g^+, g^-) :\ g_S^0 \in H^s(S),\ [g]_S \in \tilde{H}^s(S)\}$.

For $u \in H^1(\Omega_\Sigma)$ the co–normal derivative operators on $\partial\Omega$ and Σ do not generally exist in the trace sense. However if $u \in H^{1,0}(\Omega_\Sigma; L)$, one can correctly define the generalized (canonical) co–normal derivatives $T^\pm u \in \mathbb{H}^{-\frac{1}{2}}(\Sigma)$, $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$, similar to [6, Theorem 1.5.3.10], [4, Lemma 3.2], [11, Definition 3], as

$$T^\pm u := T^0_\Sigma u \pm \frac{1}{2}[T]_\Sigma u \quad \text{on } \Sigma, \tag{2.3}$$

$$\big\langle T^+u\,,\,w_{\partial\Omega}\big\rangle_{\partial\Omega} + \big\langle [T]_\Sigma u\,,\,w^0_\Sigma\big\rangle_\Sigma + \big\langle T^0_\Sigma u\,,\,[w_\Sigma]_\Sigma\big\rangle_\Sigma := \int\limits_{\Omega_\Sigma} \big[\,\gamma_{-1}w\,Lu + E(u,\gamma_{-1}w)\,\big]\,dx$$
$$\forall\ w = (w_{\partial\Omega}, w^\pm_\Sigma) \in H^{\frac{1}{2}}(\partial\Omega) \times \mathbb{H}^{\frac{1}{2}}(\Sigma). \tag{2.4}$$

If $\mu u \in H^{1,0}(\Omega^-; L)$ for any $\mu \in C^\infty_{comp}(\overline{\Omega^-})$, then

$$\big\langle T^-u\,,\,w_{\partial\Omega}\big\rangle_{\partial\Omega} := -\int\limits_{\Omega^-} \big[\,\gamma^-_{-1}w\,Lu + E(u,\gamma^-_{-1}w)\,\big]\,dx \qquad \forall\ w = w_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega). \tag{2.5}$$

Here $\gamma_{-1} : H^{\frac{1}{2}}(\partial\Omega) \times \mathbb{H}^{\frac{1}{2}}(\Sigma) \to H^1(\Omega_\Sigma)$ and $\gamma^-_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1_{com}(\Omega^-)$ are continuous right inverse operators to the trace operators,

$$E(u,v) := \sum_{i=1}^{3} a(x)\,\partial_i u(x)\,\partial_i v(x),$$

$\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-s}(\partial\Omega)$ and $H^s(\partial\Omega)$, $\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma$ the duality brackets between the spaces $H^{-s}(\Sigma)$ and $\tilde{H}^s(\Sigma)$, $s \in \mathbb{R}$, which extend the usual $L_2(\partial\Omega)$ and $L_2(\Sigma)$ inner products. Further on, we will also use the notation

$$\big\langle T^+u\,,\,w^+_\Sigma\big\rangle_\Sigma - \big\langle T^-u\,,\,w^-_\Sigma\big\rangle_\Sigma := \big\langle [T]_\Sigma u\,,\,w^0_\Sigma\big\rangle_\Sigma + \big\langle T^0_\Sigma u\,,\,[w_\Sigma]_\Sigma\big\rangle_\Sigma\,,$$

which is well defined for $T^\pm u \in \mathbb{H}^{-s}(\Sigma)$, $w^\pm_\Sigma \in \mathbb{H}^s(\Sigma)$, $s \in \mathbb{R}$.

Similar to [6, Theorem 1.5.3.11], [4, Lemma 3.4], [11, Definition 3], one can prove that the co-normal derivatives do not depend on the choice of the operator γ_{-1}, the first Green identity

$$\int\limits_{\Omega_\Sigma} \big[\,v\,Lu + E(u,v)\,\big]\,dx = \big\langle T^+u\,,\,\gamma^+v\big\rangle_S + \big\langle T^+u\,,\,\gamma^+v\big\rangle_\Sigma - \big\langle T^-u\,,\,\gamma^-v\big\rangle_\Sigma\,, \tag{2.6}$$

holds for any functions $u \in H^{1,0}(\Omega_\Sigma; L)$, $v \in H^1(\Omega_\Sigma)$, while the second Green identity

$$\int\limits_{\Omega_\Sigma} \big[\,v\,Lu - Lv\;u\,\big]\,dx = \big\langle T^+u\,,\,\gamma^+v\big\rangle_{\partial\Omega} - \big\langle T^+v\,,\,\gamma^+u\big\rangle_{\partial\Omega}$$
$$+ \big\langle T^+u\,,\,\gamma^+v\big\rangle_\Sigma - \big\langle T^-u\,,\,\gamma^-v\big\rangle_\Sigma + \big\langle T^+v\,,\,\gamma^+u\big\rangle_\Sigma - \big\langle T^-v\,,\,\gamma^-u\big\rangle_\Sigma \tag{2.7}$$

holds for any functions $u, v \in H^{1,0}(\Omega_\Sigma; L)$.

We will consider the BDIE approach for the following three crack type boundary value problems.

Mixed BVP with crack, or Problem (MC): Find a function $u \in H^1(\Omega_\Sigma)$ satisfying the conditions

$$L\,u = f \quad \text{in} \quad \Omega_\Sigma, \tag{2.8}$$

$$
\begin{aligned}
& r_{S_D}\,\gamma^+ u = \varphi_0 \quad \text{on} \quad S_D, && (2.9)\\
& r_{S_N}\,T^+ u = \psi_0 \quad \text{on} \quad S_N, && (2.10)\\
& T^+ u = \psi^+_\Sigma, \quad T^- u = \psi^-_\Sigma \quad \text{on} \quad \Sigma. && (2.11)
\end{aligned}
$$

where

$$
\varphi_0 \in H^{\frac{1}{2}}(S_D), \;\; \psi_0 \in H^{-\frac{1}{2}}(S_N), \;\; \psi^\pm_\Sigma \in \mathbb{H}^{-\frac{1}{2}}(\Sigma), \;\; f \in H^0(\Omega_\Sigma). \tag{2.12}
$$

Note that we can replace the crack conditions (2.11) by the equivalent ones,

$$
[T]_\Sigma u = [\psi_\Sigma]_\Sigma, \quad T^0_\Sigma u = \psi^0_\Sigma \quad \text{on} \quad \Sigma. \tag{2.13}
$$

Equation (2.8) is understood in the distributional sense, condition (2.9) in the trace sense, while equality (2.10) and (2.11) in the functional sense (2.3)-(2.4).

Clearly, if $S_N = \varnothing$ in (2.8)-(2.11), we arrive at the *Dirichlet problem with crack, or Problem* (DC): Find $u \in H^1(\Omega_\Sigma)$ such that

$$
\begin{aligned}
& L\,u = f \quad \text{in} \quad \Omega_\Sigma, && (2.14)\\
& \gamma^+ u = \varphi_0 \quad \text{on} \quad \partial\Omega, && (2.15)\\
& T^+ u = \psi^+_\Sigma, \quad T^- u = \psi^-_\Sigma \quad \text{on} \quad \Sigma. && (2.16)
\end{aligned}
$$

where

$$
\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega), \;\; \psi^\pm_\Sigma \in \mathbb{H}^{-\frac{1}{2}}(\Sigma), \;\; f \in H^0(\Omega_\Sigma). \tag{2.17}
$$

If $S_D = \varnothing$ in (2.8)-(2.11), we have the *Neumann problem with crack, or Problem* (NC): Find $u \in H^1(\Omega_\Sigma)$ such that

$$
\begin{aligned}
& L\,u = f \quad \text{in} \quad \Omega_\Sigma, && (2.18)\\
& T^+ u = \psi_0 \quad \text{on} \quad \partial\Omega, && (2.19)\\
& T^+ u = \psi^+_\Sigma, \quad T^- u = \psi^-_\Sigma \quad \text{on} \quad \Sigma. && (2.20)
\end{aligned}
$$

where

$$
\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega), \;\; \psi^\pm_\Sigma \in \mathbb{H}^{-\frac{1}{2}}(\Sigma), \;\; f \in H^0(\Omega_\Sigma). \tag{2.21}
$$

We have (similar e.g. to [7]) the following well-known uniqueness and existence result.

THEOREM 2.1 *(i) The homogeneous Dirichlet and mixed BVPs with crack have only the trivial solution, while the homogeneous Neumann crack problem admits a constant as a general solution.*

(ii) The nonhomogeneous problem (DC) under condition (2.17), and the nonhomogeneous problem (MC) under condition (2.12) are uniquely solvable.

(iii) Let the inclusions (2.21) be satisfied. Then the problem (NC) is solvable if and only if

$$
\int\limits_{\Omega_\Sigma} f(x)\,dx = \int\limits_{\partial\Omega} \psi_0(x)\,dS + \int\limits_{\Sigma} [\psi^+_\Sigma(x) - \psi^-_\Sigma(x)]\,dS, \tag{2.22}
$$

and the solution u is defined modulo constant summand.

Proof. The uniqueness results immediately follow from the first Green identity (2.6) with $v = u$ as a solution of the corresponding homogeneous boundary value problem. The existence results directly follow from the Lax-Milgram theorem applied to the weak variational formulation of the above problems. $\square$

In the subsequent sections our main goal is to reduce the above BVPs to the equivalent *boundary-domain* integral (pseudodifferential) equations and prove invertibility of the corresponding non-standard integral operators in appropriate function spaces.

3 Some segregated boundary-domain integral equations

The function

$$P(x,y) = -\frac{1}{4\pi\, a(y)\, |x-y|}\,, \qquad x, y \in \mathbb{R}^3, \;\; x \neq y, \tag{3.1}$$

is a parametrix (Levi function) of the operator $L(x, \partial_x)$ with the property

$$L(x,\partial_x)\, P(x,y) = \delta(x-y) + R(x,y), \tag{3.2}$$

where $\delta(\cdot)$ is the Dirac distribution and the remainder

$$R(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\, a(y)\, |x-y|^3}\, \frac{\partial a(x)}{\partial x_i}\,, \quad x, y \in \mathbb{R}^3, \;\; x \neq y, \tag{3.3}$$

possesses a weak singularity of type $\mathcal{O}(|x-y|^{-2})$ for small $|x-y|$, see [8, 2].

Further we introduce parametrix based surface potential operators

$$V_{\partial\Omega}\, g(y) := -\int\limits_{\partial\Omega} P(x,y)\, g(x)\, dS_x, \quad y \in \mathbb{R}^3 \backslash \partial\Omega \tag{3.4}$$

$$W_{\partial\Omega}\, g(y) := -\int\limits_{\partial\Omega} \big[\, T_x\, P(x,y) \big]\; g(x)\, dS_x, \quad y \in \mathbb{R}^3 \backslash \partial\Omega \tag{3.5}$$

$$V_{\Sigma}\, g(y) := -\int\limits_{\Sigma} P(x,y)\, g(x)\, dS_x, \quad y \in \mathbb{R}^3 \backslash \Sigma \tag{3.6}$$

$$W_{\Sigma}\, g(y) := -\int\limits_{\Sigma} \big[\, T_x\, P(x,y) \big]\; g(x)\, dS_x, \quad y \in \mathbb{R}^3 \backslash \Sigma \tag{3.7}$$

and volume potential operators

$$\mathcal{P}\, g(y) := \int\limits_{\Omega_\Sigma} P(x,y)\, g(x)\, dx, \qquad \mathcal{R}\, g(y) := \int\limits_{\Omega_\Sigma} R(x,y)\, g(x)\, dx, \quad y \in \mathbb{R}^3. \tag{3.8}$$

The corresponding direct values of the surface potentials are denoted as

$$\begin{aligned}
\mathcal{V}_{\partial\Omega}\, g(y) &:= -\int\limits_{\partial\Omega} P(x,y)\, g(x)\, dS_x, \quad \mathcal{W}_{\partial\Omega}\, g(y) := -\int\limits_{\partial\Omega} \big[\, T_x\, P(x,y) \big]\; g(x)\, dS_x, \quad y \in \partial\Omega,\\
\mathcal{V}_{\Sigma}\, g(y) &:= -\int\limits_{\Sigma} P(x,y)\, g(x)\, dS_x, \quad \mathcal{W}_{\Sigma}\, g(y) := -\int\limits_{\Sigma} \big[\, T_x\, P(x,y) \big]\; g(x)\, dS_x, \quad y \in \Sigma,
\end{aligned} \tag{3.9}$$

and the co-normal derivatives of the surface potentials as

$$\mathcal{W}'_{\partial\Omega}\, g(y) := -\int\limits_{\partial\Omega} \big[\, T_y\, P(x,y) \,\big]\, g(x)\, dS_x, \quad \mathcal{L}^{\pm}_{\partial\Omega}\, g(y) := T^{\pm}\, W_{\partial\Omega} g(y), \quad y \in \partial\Omega,$$

$$\mathcal{W}'_{\Sigma}\, g(y) := -\int\limits_{\Sigma} \big[\, T_y\, P(x,y) \,\big]\, g(x)\, dS_x, \quad \mathcal{L}^{\pm}_{\Sigma}\, g(y) := T^{\pm}\, W_{\Sigma} g(y), \quad y \in \Sigma, \tag{3.10}$$

$$[\mathcal{L}]_{\Sigma}\, g(y) := \mathcal{L}^{+}_{\Sigma}\, g(y) - \mathcal{L}^{-}_{\Sigma}\, g(y), \quad \mathcal{L}^{0}_{\Sigma}\, g(y) := \frac{1}{2}\{\mathcal{L}^{+}_{\Sigma}\, g(y) + \mathcal{L}^{-}_{\Sigma}\, g(y)\}, \quad y \in \Sigma.$$

Mapping and jump properties of the left hand side operators in (3.4)-(3.10) in Sobolev spaces are given in [2] (see also the Appendix below).

Taking, as in [8, 2], $v(x) := P(x,y)$ and $u \in H^{1,0}(\Omega_{\Sigma}; L)$ in (2.7), we obtain by the standard limiting procedures (see e.g. [12]),

$$u + \mathcal{R}u - V_{\partial\Omega}(T^{+}u) + W_{\partial\Omega}(\gamma^{+}u) - V_{\Sigma}([T]_{\Sigma}u) + W_{\Sigma}([u]_{\Sigma}) = \mathcal{P}Lu \quad \text{in} \quad \Omega_{\Sigma}, \tag{3.11}$$

where $[u]_{\Sigma} := \gamma^{+}u - \gamma^{-}u$ on Σ.

3.1 BDIEs for the problem (MC)

To get a *segregated* boundary domain integral formulation for the problem (MC), we replace the unknown traces, co-normal derivatives and jumps of u on S_N, S_D and Σ with new unknown functions that will be treated as independent of u. First of all, we denote $\varphi^* := [u]_{\Sigma} \in \widetilde{H}^{\frac{1}{2}}(\Sigma)$. Let now $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ be a fixed extension of the given right hand side of the Dirichlet condition (2.9), $\varphi_0 \in H^{\frac{1}{2}}(S_D)$, onto the whole of $\partial\Omega$. Then $\gamma^{+}u = \Phi_0 + \varphi$ on $\partial\Omega$, where the unknown function φ belongs to $\widetilde{H}^{\frac{1}{2}}(S_N)$ due to (2.9). Analogously, let $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be a fixed extension of the given right hand side of the Neumann condition (2.10), $\psi_0 \in H^{-\frac{1}{2}}(S_D)$, onto the whole of $\partial\Omega$. Then $T^{+}u = \Psi_0 + \psi$, where the unknown function ψ belongs to $\widetilde{H}^{-\frac{1}{2}}(S_D)$ due to (2.10). If $\varphi_0 = 0$ or $\psi_0 = 0$ then we can take the canonical extensions $\Phi_0 = 0$ or $\Psi_0 = 0$, respectively, on $\partial\Omega$.

As shown in Theorems A.3, B.1 of the Appendix, for $f \in H^{0}(\Omega_{\Sigma})$, $u \in H^{1}(\Omega_{\Sigma})$, $\varphi^* \in \widetilde{H}^{\frac{1}{2}}(\Sigma)$ we have,

$$[\mathcal{P}f]_{\Sigma} = 0, \quad [T]_{\Sigma}\mathcal{P}f = 0, \quad [\mathcal{R}u]_{\Sigma} = 0, \quad [T]_{\Sigma}\mathcal{R}u = -(\partial_n a)\,[u]_{\Sigma},$$
$$\mathcal{L}^{+}_{\Sigma}(\varphi^*) - \mathcal{L}^{-}_{\Sigma}(\varphi^*) = (\partial_n a)\,\varphi^* \quad \text{on} \quad \Sigma. \tag{3.12}$$

Let now $u \in H^{1,0}(\Omega_{\Sigma}; L)$ be a solution of the problem (MC). Taking (3.11) in the domain, its trace on S_D, its co-normal derivative on S_N, the average of its co-normal derivatives, $T^{0}_{\Sigma} = \frac{1}{2}(T^{+}+T^{-})$, on Σ, and employing the relations

$$\psi = T^{+}u - \Psi_0 \in \widetilde{H}^{-\frac{1}{2}}(S_D), \quad \varphi = \gamma^{+}u - \Phi_0 \in \widetilde{H}^{\frac{1}{2}}(S_N), \quad \varphi^* = [u]_{\Sigma} \in \widetilde{H}^{\frac{1}{2}}(\Sigma), \tag{3.13}$$

we derive the boundary-domain integral equation system (MC11),

$$u + \mathcal{R}u - V_{\partial\Omega}\psi + W_{\partial\Omega}\varphi + W_{\Sigma}\varphi^* = \mathcal{P}f + V_{\Sigma}([\psi_{\Sigma}]_{\Sigma}) + V_{\partial\Omega}\Psi_0 - W_{\partial\Omega}\Phi_0 \quad \text{in} \quad \Omega_{\Sigma}, \tag{3.14}$$

$$r_{S_D}\left\{\gamma^{+}\mathcal{R}u - \mathcal{V}_{\partial\Omega}\psi + \mathcal{W}_{\partial\Omega}\varphi + \gamma^{+}W_{\Sigma}\varphi^*\right\} =$$

$$r_{S_D}\left\{\gamma^+\mathcal{P}f+\gamma^+V_\Sigma([\psi_\Sigma]_\Sigma)-\frac{1}{2}\varphi_0+\mathcal{V}_{\partial\Omega}\Psi_0-\mathcal{W}_{\partial\Omega}\Phi_0\right\} \quad \text{on} \quad S_D, \quad (3.15)$$

$$r_{S_N}\left\{T^+\mathcal{R}u-\mathcal{W}'_{\partial\Omega}\psi+\mathcal{L}^+_{\partial\Omega}\varphi+T^+W_\Sigma\varphi^*\right\}=$$
$$r_{S_N}\left\{T^+\mathcal{P}f+T^+V_\Sigma([\psi_\Sigma]_\Sigma)-\frac{1}{2}\psi_0+\mathcal{W}'_{\partial\Omega}\Psi_0-\mathcal{L}^+_{\partial\Omega}\Phi_0\right\} \quad \text{on} \quad S_N, \quad (3.16)$$

$$T^0_\Sigma\mathcal{R}u-T^0_\Sigma V_{\partial\Omega}\psi+T^0_\Sigma W_{\partial\Omega}\varphi+\mathcal{L}^0_\Sigma\varphi^*=$$
$$T^0_\Sigma\mathcal{P}f+\mathcal{W}'_\Sigma([\psi_\Sigma]_\Sigma)+T^0_\Sigma V_{\partial\Omega}\Psi_0-T^0_\Sigma W_{\partial\Omega}\Phi_0-\psi^0_\Sigma \quad \text{on} \quad \Sigma. \quad (3.17)$$

The notation (MC11) indicates that the BDIE system includes integral operators (3.15) and (3.16) of the first kind on the Dirichlet and Neumann parts of the boundary, respectively.

Now we formulate the basic equivalence theorem for the problem (MC) and BDIE system (MC11).

THEOREM 3.1 *Let conditions (2.12) hold and let $\Phi_0\in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0\in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of φ_0 and ψ_0, respectively.*

(i) If a function $u\in H^1(\Omega_\Sigma)$ solves the problem (MC), then the four-vector $(u,\psi,\varphi,\varphi^*)$, where ψ, φ, and φ^* are defined by (3.13), solves the BDIE system (3.14)-(3.17).

(ii) If a four-vector $(u,\psi,\varphi,\varphi^*)\in H^1(\Omega_\Sigma)\times\widetilde{H}^{-\frac{1}{2}}(S_D)\times\widetilde{H}^{\frac{1}{2}}(S_N)\times\widetilde{H}^{\frac{1}{2}}(\Sigma)$ solves the BDIE system (3.14)-(3.17), then this solution is unique, u solves the problem (MC) and relations (3.13) hold.

Proof. For a function $u\in H^1(\Omega_\Sigma)$ solving (2.8) under conditions (2.12), we have $u\in H^{1,0}(\Omega_\Sigma;L)$ since $f\in H^0(\Omega_\Sigma)$. Under hypothesis of item (i) this implies (3.11) and thus the claims of item (i).

Now, let a four-vector $(u,\psi,\varphi,\varphi^*)\in H^1(\Omega_\Sigma)\times\widetilde{H}^{-\frac{1}{2}}(S_D)\times\widetilde{H}^{\frac{1}{2}}(S_N)\times\widetilde{H}^{\frac{1}{2}}(\Sigma)$ solve the BDIE system (3.14)-(3.17). We have to show that u solves the problem (MC).

From (3.15) and the trace of (3.14) on $\partial\Omega$ we conclude that $r_{S_D}u^+=\varphi_0$ on S_D, while from (3.16) and the co-normal derivative of (3.14) on $\partial\Omega$ we have $r_{S_N}T^+u=\psi_0$ on S_N. Taking the jump of traces of (3.14) on Σ we get

$$[u]_\Sigma=\varphi^* \quad \text{on } \Sigma. \quad (3.18)$$

Further, take the co-normal derivatives T^+, T^- of the equation (3.14) on Σ, construct their difference, and compare their sum with (3.17) to obtain

$$r_\Sigma\left\{T^+u-T^-u-[u]_\Sigma\partial_n a-([\psi_\Sigma]_\Sigma)+\varphi^*\partial_n a\right\}=0, \quad r_\Sigma\left\{T^+u+T^-u\right\}=[\psi_\Sigma]_\Sigma,$$

i.e.,

$$[T]_\Sigma u=[\psi_\Sigma]_\Sigma, \quad T^0_\Sigma u=\psi^0_\Sigma \quad \text{on} \quad \Sigma. \quad (3.19)$$

These relations coincide with (2.13) thus implying (2.11).

Since $u\in H^1(\Omega_\Sigma)$, it follows from equation (3.14) and Theorems A.1, B.1 that $u\in H^{1,0}(\Omega_\Sigma;L)$, and we can write Green's third identity (3.11) for u. Comparing it with equation (3.14) and taking into account (3.18), (3.19) gives

$$-V_{\partial\Omega}(T^+u-\psi-\Psi_0)+W_{\partial\Omega}(u^+-\varphi-\Phi_0)=\mathcal{P}(Lu-f) \quad \text{in} \quad \Omega_\Sigma. \quad (3.20)$$

Since all the potentials in (3.20) are continuous on Ω (including Σ), equation (3.20) can be extended on the whole Ω. Then taking into account that $u^+ - \varphi - \Phi_0 = 0$ on S_D and $T^+ u - \psi - \Psi_0 = 0$ on S_N, we obtain by [2, Lemmas 4.1, 4.2] that $Lu - f = 0$ in Ω, while $u^+ - \varphi - \Phi_0 = 0$ and $T^+ u - \psi - \Psi_0 = 0$ on $\partial\Omega$.

We now have to prove uniqueness of the BDIE system solution. Let $(u, \psi, \varphi, \varphi^*) \in H^1(\Omega_\Sigma) \times \widetilde{H}^{\frac{1}{2}}(S_N) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$ solve homogeneous BDIE system (3.14)-(3.17), which zero right hand side can be considered as generated by the zero right hand side of problem (MC), $(\varphi_0, \psi_0, \psi_\Sigma^\pm, f) = 0$. Then already proved statements of item (ii) imply that u is a solution of the homogeneous problem (MC), which is zero by Theorem 2.1, and thus $(\psi, \varphi, \varphi^*) = 0$ by item (i). □

Further we study invertibility in appropriate function spaces of the 4×4 matrix operator $\mathcal{A}^{11}$, generated by the left hand side of the BDIE system (3.14)-(3.17),

$$\mathcal{A}^{11} := \begin{bmatrix} I + \mathcal{R} & -V_{\partial\Omega} & W_{\partial\Omega} & W_\Sigma \\ r_{S_D}\gamma^+\mathcal{R} & -r_{S_D}\mathcal{V}_{\partial\Omega} & r_{S_D}\mathcal{W}_{\partial\Omega} & r_{S_D}\gamma^+ W_\Sigma \\ r_{S_N} T^+\mathcal{R} & -r_{S_N}\mathcal{W}'_{\partial\Omega} & r_{S_N}\mathcal{L}^+_{\partial\Omega} & r_{S_N} T^+ W_\Sigma \\ T^0_\Sigma \mathcal{R} & -T^0_\Sigma V_{\partial\Omega} & T^0_\Sigma W_{\partial\Omega} & \mathcal{L}^0_\Sigma \end{bmatrix}. \tag{3.21}$$

Let

$$\begin{aligned} \mathbb{X} &:= H^1(\Omega_\Sigma) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \times \widetilde{H}^{\frac{1}{2}}(\Sigma), \\ \mathbb{F}^{11} &:= H^1(\Omega_\Sigma) \times H^{\frac{1}{2}}(S_D) \times H^{-\frac{1}{2}}(S_N) \times H^{-\frac{1}{2}}(\Sigma). \end{aligned}$$

In view of the mapping properties of the potential type operators (3.4)-(3.8), see Appendix, the operator

$$\mathcal{A}^{11} \ : \ \mathbb{X} \to \mathbb{F}^{11} \tag{3.22}$$

is continuous.

Note that we have the identity (see [2])

$$\mathcal{L}^\pm_S\, g = \hat{\mathcal{L}}_S\,(g) + (\partial_n a)\,\Big(\pm \frac{1}{2}\, g - \mathcal{W}_S\, g \Big),$$

with either $S = \partial\Omega$ or $S = \Sigma$. Here

$$\hat{\mathcal{L}}_S\,(g) := \mathcal{L}_{S,\Delta}\,(ag) = [TW_{S,\Delta}\,(ag)]^+ = [TW_{S,\Delta}\,(ag)]^- \quad \text{on} \quad S, \tag{3.23}$$

where $W_{S,\Delta}\,(ag)$ is the usual harmonic double layer potential over S with density ag,

$$W_{S,\Delta}\,(ag)\,(y) = \frac{1}{4\pi} \int\limits_S \frac{\partial}{\partial n(x)} \frac{1}{|x-y|}\, a(x)\, g(x)\, dS_x.$$

Equality (3.23) then represents the well-known Liapunov-Tauber theorem for a harmonic double layer potential.

By Theorem 3.1(ii) operator (3.22) is injective. Now we prove that it is surjective. To this end let us consider the operator

$$\mathcal{A}^{11}_0 := \begin{bmatrix} I & -V_{\partial\Omega} & W_{\partial\Omega} & W_\Sigma \\ 0 & -r_{S_D}\mathcal{V}_{\partial\Omega} & 0 & 0 \\ 0 & 0 & r_{S_N}\hat{\mathcal{L}}_{\partial\Omega} & 0 \\ 0 & 0 & 0 & \hat{\mathcal{L}}_\Sigma \end{bmatrix}. \tag{3.24}$$

As follows from Appendix, the operator $\mathcal{A}_0^{11} : \mathbb{X} \to \mathbb{F}^{11}$ is continuous and the operator $\mathcal{A}^{11} - \mathcal{A}_0^{11} : \mathbb{X} \to \mathbb{F}^{11}$ is compact. Since the diagonal operators

$$\begin{aligned} r_{S_D}\mathcal{V}_{\partial\Omega} &: \widetilde{H}^{-\frac{1}{2}}(S_D) \to H^{\frac{1}{2}}(S_D), \\ r_{S_N}\hat{\mathcal{L}}_{\partial\Omega} &: \widetilde{H}^{\frac{1}{2}}(S_N) \to H^{-\frac{1}{2}}(S_N), \\ \hat{\mathcal{L}}_{\Sigma} &: \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma), \end{aligned}$$

are invertible (see Theorems A.4, A.5), we conclude that the triangular operator $\mathcal{A}_0^{11} : \mathbb{X} \to \mathbb{F}^{11}$ is invertible, implying that (3.22) is a Fredholm operator with index zero. Therefore from injectivity of $\mathcal{A}^{11}$ it follows that (3.22) is invertible. Thus we have the following result.

THEOREM 3.2 *The operator $\mathcal{A}^{11} : \mathbb{X} \to \mathbb{F}^{11}$ is continuous and continuously invertible.*

3.2 BDIEs for the problem (DC)

A segregated BDIE system for problem (DC) is formulated by the same way as for the problem (MC) but with apparent simplifications. Let $u \in H^{1,0}(\Omega_\Sigma; L)$ be a solution of the problem (DC). Taking (3.11) in the domain, its trace on $\partial\Omega$, the average of its co-normal derivatives, T_Σ^0, on Σ, and introducing the notations

$$\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega), \qquad \varphi^* = [u]_\Sigma \in \widetilde{H}^{\frac{1}{2}}(\Sigma), \tag{3.25}$$

we derive the following boundary-domain integral equation system (DC1),

$$\begin{aligned} u + \mathcal{R}u - V_{\partial\Omega}\psi + W_\Sigma\varphi^* &= \mathcal{P}f + V_\Sigma([\psi_\Sigma]_\Sigma) - W_{\partial\Omega}\varphi_0 \quad \text{in } \Omega_\Sigma, & (3.26)\\ \gamma^+\mathcal{R}u - \mathcal{V}_{\partial\Omega}\psi + \gamma^+W_\Sigma\varphi^* &= \gamma^+\mathcal{P}f + V_\Sigma([\psi_\Sigma]_\Sigma) - \frac{1}{2}\varphi_0 - \mathcal{W}_{\partial\Omega}\varphi_0 \quad \text{on } \partial\Omega, & (3.27)\\ T_\Sigma^0\mathcal{R}u - T_\Sigma^0V_{\partial\Omega}\psi + \mathcal{L}_\Sigma^0\varphi^* &= T_\Sigma^0\mathcal{P}f + \mathcal{W}'_\Sigma([\psi_\Sigma]_\Sigma) - T_\Sigma^0W_{\partial\Omega}\varphi_0 - \psi_\Sigma^0 \quad \text{on } \Sigma. & (3.28) \end{aligned}$$

Let us denote the left hand side operator of the BDIE system (DC1) as

$$\mathcal{A}^{D1} := \begin{bmatrix} I + \mathcal{R} & -V_{\partial\Omega} & W_\Sigma \\ \gamma_{\partial\Omega}^+\mathcal{R} & -\mathcal{V}_{\partial\Omega} & \gamma_{\partial\Omega}^+W_\Sigma \\ T_\Sigma^0\mathcal{R} & -T_\Sigma^0V_{\partial\Omega} & \mathcal{L}_\Sigma^0 \end{bmatrix}, \tag{3.29}$$

where $\gamma_{\partial\Omega}^+ := r_{\partial\Omega}\gamma^+$.

Simplifying corresponding proofs of Theorems 3.1 and 3.2, we arrive at the following equivalence theorem for the problem (DC) and BDIE system (DC1), and the theorem of invertibility of the operator $\mathcal{A}^{D1}$.

THEOREM 3.3 *Let conditions (2.17) hold.*

(i) If a function $u \in H^1(\Omega_\Sigma)$ solves the problem (DC), then the triple (u, ψ, φ^*), where ψ and φ^* are defined by (3.25), solves BDIE system (3.26)-(3.28).

(ii) If a triple $(u, \psi, \varphi^*) \in H^1(\Omega_\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$ solves the BDIE system (3.26)-(3.28), then this solution is unique, u solves the problem (DC) and relations (3.25) hold.

THEOREM 3.4 *The operator,*

$$\mathcal{A}^{D1}: H^1(\Omega_\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^1(\Omega_\Sigma) \times H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\Sigma)$$

is continuous and continuously invertible.

3.3 BDIEs for the problem (NC)

Again, BDIEs for problem (NC) are formulated by the same way as for the problem (MC) but with apparent simplifications. Let $u \in H^{1,0}(\Omega_\Sigma; L)$ be a solution of the problem (NC). Taking (3.11) in the domain, its co-normal derivative on $\partial\Omega$, the average of its co-normal derivatives, T^0_Σ, on Σ, and introducing the notations

$$\varphi = \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega), \qquad \varphi^* = [u]_\Sigma \in \widetilde{H}^{\frac{1}{2}}(\Sigma), \tag{3.30}$$

we derive the following boundary-domain integral equation system (NC1),

$$\begin{aligned}
u + \mathcal{R}u + W_{\partial\Omega}\varphi + W_\Sigma\varphi^* &= \mathcal{P}f + V_\Sigma([\psi_\Sigma]_\Sigma) + V_{\partial\Omega}\psi_0 \quad \text{in } \Omega_\Sigma, && (3.31)\\
T^+\mathcal{R}u + \mathcal{L}^+_{\partial\Omega}\varphi + T^+W_\Sigma\varphi^* &= T^+\mathcal{P}f + T^+V_\Sigma([\psi_\Sigma]_\Sigma) - \frac{1}{2}\psi_0 + \mathcal{W}'_{\partial\Omega}\psi_0 \quad \text{on } \partial\Omega, && (3.32)\\
T^0_\Sigma\mathcal{R}u + T^0_\Sigma W_{\partial\Omega}\varphi + \mathcal{L}^0_\Sigma\varphi^* &= T^0_\Sigma\mathcal{P}f + \mathcal{W}'_\Sigma([\psi_\Sigma]_\Sigma) + T^0_\Sigma V_{\partial\Omega}\psi_0 - \psi^0_\Sigma \quad \text{on } \Sigma. && (3.33)
\end{aligned}$$

THEOREM 3.5 *Let conditions (2.21) hold.*

(i) If a function $u \in H^1(\Omega_\Sigma)$ solves the problem (NC), then the triple (u, φ, φ^*), where φ and φ^* are defined by (3.30), solves BDIE system (3.31)-(3.33).

(ii) If a triple $(u, \varphi, \varphi^*) \in H^1(\Omega_\Sigma) \times H^{\frac{1}{2}}(\partial\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$ solves BDIE system (3.31)-(3.33), then u solves the problem (NC) and relations (3.30) hold.

(iii) Homogeneous BDIE system (3.31)-(3.33) admits only one linearly independent solution $(u, \varphi, \varphi^*) = (1, 1, 0)$ in $H^1(\Omega_\Sigma) \times H^{\frac{1}{2}}(\partial\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$.

(iv) Condition (2.22) is necessary and sufficient for solvability of nonhomogeneous BDIE system (3.31)-(3.33) in $H^1(\Omega_\Sigma) \times H^{\frac{1}{2}}(\partial\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$.

Proof. Items (i) and (ii) are obtained by simplifying corresponding proofs of Theorems 3.1. Then items (iii) and (iv) follow from items (i) and (ii) and from Theorem 2.1(iii), similar to the last paragraph of the proof of Theorem 3.1. □

Let us denote the left hand side operator of the BDIE system (NC1) as

$$\mathcal{A}^{N1} := \begin{bmatrix} I + \mathcal{R} & W_{\partial\Omega} & W_\Sigma \\ T^+_{\partial\Omega}\mathcal{R} & \mathcal{L}^+_{\partial\Omega} & T^+_{\partial\Omega}W_\Sigma \\ T^0_\Sigma\mathcal{R} & T^0_\Sigma W_{\partial\Omega} & \mathcal{L}^0_\Sigma \end{bmatrix}, \tag{3.34}$$

where $T^+_{\partial\Omega} := r_{\partial\Omega}T^+$.

Let $\mathbb{X}^N := H^1(\Omega_\Sigma) \times H^{\frac{1}{2}}(\partial\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$, $\quad \mathbb{F}^{N1} := H^1(\Omega_\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\Sigma)$.

THEOREM 3.6 *The operator* $\mathcal{A}^{N1} : \mathbb{X}^N \to \mathbb{F}^{N1}$ *is a continuous Fredholm operator with zero index. It has one–dimensional null–space,* $\ker \mathcal{A}^{N1}$*, spanned over the element* $(u, \varphi, \varphi^*) = (1, 1, 0)$.

Proof. Let us consider the operator

$$\mathcal{A}_0^{N1} := \begin{bmatrix} I & W_{\partial\Omega} & W_{\Sigma} \\ 0 & \hat{\mathcal{L}}_{\partial\Omega} & 0 \\ 0 & 0 & \hat{\mathcal{L}}_{\Sigma} \end{bmatrix} . \tag{3.35}$$

It is evident from the Appendix that the operator $\mathcal{A}_0^{N1} : \mathbb{X}^N \to \mathbb{F}^{N1}$ is continuous and the operator $\mathcal{A}^{N1} - \mathcal{A}_0^{N1} : \mathbb{X}^N \to \mathbb{F}^{N1}$ is compact. By Theorem A.5 the operator $\hat{\mathcal{L}}_{\Sigma} : \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma)$ is continuously invertible and the operator $\hat{\mathcal{L}}_{\partial\Omega} : \widetilde{H}^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator with index zero. Then we conclude that the triangular operator $\mathcal{A}_0^{N1} : \mathbb{X}^N \to \mathbb{F}^{N1}$ is a Fredholm operator with index zero, implying that $\mathcal{A}^{N1} : \mathbb{X}^N \to \mathbb{F}^{N1}$ is also a Fredholm operator with index zero. By Theorem 3.5(iii), $\ker \mathcal{A}^{N1}$ is one-dimensional and is spanned over the element $(u, \varphi, \varphi^*) = (1, 1, 0)$. □

4 Concluding remarks

Three *segregated* direct boundary-domain integral equation systems associated with the Dirichlet, Neumann and mixed problems for a scalar "Laplace" PDE with *variable* coefficient on a three-dimensional bounded domain with a crack were formulated and analysed in the paper. The Neumann conditions were assumed on the crack surfaces. In all the considered BDIE systems, the operators on the boundary were of the first kind. Equivalence of the BDIE systems to the original BVPs was proved in the case when right-hand side of the PDE is from $L_2(\Omega)$, and the Dirichlet and the Neumann data from the spaces $H^{\frac{1}{2}}(S_D)$ and $H^{-\frac{1}{2}}(S_N)$, $\mathbb{H}^{-\frac{1}{2}}(\Sigma)$, respectively. The invertibility of the BDIE operators was proved in the corresponding Sobolev spaces.

Similar to [2], other segregated direct BDIE systems with boundary operators of the second or the third (mixed) kind can be formulated and analysed for the same BVPs, while using approach of [10] *united* direct boundary-domain integro-differential systems can be also formulated and analysed for the BVPs with crack. The BDIEs for unbounded domains with cracks can be analysed as well. The approach can be extended also to more general PDEs and to systems of PDEs, while smoothness of the variable coefficients and the boundary can be essentially relaxed, and the PDE right hand side can be considered in more general spaces, c.f. [9, 10].

Employing methods of [1] and [3], one can consider also the localised counterparts of the BDIEs for BVPs with cracks.

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APPENDIX

A Properties of surface potentials

The auxiliary facts collected in Theorems A.1-A.5 follow e.g. from [4, Theorem 2], [5, Ch. XI, Part B, §3,], [14, Theorem 2.7(ii)], [2, Section 3], [10, Appendix].

THEOREM A.1 *The following operators are continuous,*

$$V_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; L), \qquad W_{\partial\Omega}: H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; L),$$
$$V_{\Sigma}: \widetilde{H}^{-\frac{1}{2}}(\Sigma) \to H^{1,0}(\Omega_\Sigma; L), \qquad W_{\Sigma}: \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{1,0}(\Omega_\Sigma; L).$$

THEOREM A.2 *The following operators are continuous.*

$$\mathcal{V}_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \qquad \mathcal{V}_{\Sigma}: \widetilde{H}^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma), \tag{A.1}$$
$$\mathcal{W}_{\partial\Omega}: H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \qquad \mathcal{W}_{\Sigma}: \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma), \tag{A.2}$$
$$\mathcal{W}'_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \qquad \mathcal{W}'_{\Sigma}: \widetilde{H}^{-\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma), \tag{A.3}$$
$$\mathcal{L}^{\pm}_{\partial\Omega}: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \qquad \mathcal{L}^{\pm}_{\Sigma}: \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma) \tag{A.4}$$

Moreover the operators $(A.2)$-$(A.3)$ are compact.

THEOREM A.3 *Let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\psi^* \in \widetilde{H}^{-\frac{1}{2}}(\Sigma)$ and $\varphi^* \in \widetilde{H}^{\frac{1}{2}}(\Sigma)$. Then there hold the following jump relations on $\partial\Omega$,*

$$\gamma^{\pm} V_{\partial\Omega}\psi = \mathcal{V}_{\partial\Omega}\psi, \tag{A.5}$$
$$\gamma^{\pm} W_{\partial\Omega}\varphi = \mp\frac{1}{2}\,\varphi + \mathcal{W}_{\partial\Omega}\varphi, \tag{A.6}$$
$$T^{\pm} V_{\partial\Omega}\psi = \pm\frac{1}{2}\,\psi + \mathcal{W}'_{\partial\Omega}\psi, \tag{A.7}$$
$$T^{+} W_{\partial\Omega}\varphi - T^{-} W_{\partial\Omega}\varphi = \mathcal{L}^{+}_{\partial\Omega}\varphi - \mathcal{L}^{-}_{\partial\Omega}\varphi = \frac{\partial a}{\partial n}\,\varphi, \tag{A.8}$$

and similar jump relations on Σ,

$$\gamma^{\pm} V_{\Sigma}\psi^* = \mathcal{V}_{\Sigma}\psi^*, \tag{A.9}$$
$$\gamma^{\pm} W_{\Sigma}\varphi^* = \mp\frac{1}{2}\,\varphi^* + \mathcal{W}_{\Sigma}\varphi^*, \tag{A.10}$$
$$T^{\pm} V_{\Sigma}\psi^* = \pm\frac{1}{2}\,\psi^* + \mathcal{W}'_{\Sigma}\psi^*, \tag{A.11}$$
$$T^{+} W_{\Sigma}\varphi^* - T^{-} W_{\Sigma}\varphi^* = \mathcal{L}^{+}_{\Sigma}\varphi^* - \mathcal{L}^{-}_{\Sigma}\varphi^* = \frac{\partial a}{\partial n}\,\varphi^*. \tag{A.12}$$

THEOREM A.4 *Let $S = \partial\Omega$ or S be a nonempty, simply connected sub–manifold of $\partial\Omega$ with infinitely smooth boundary curve. Then the operators*

$$\mathcal{V}_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \qquad r_S\mathcal{V}_{\partial\Omega}: \widetilde{H}^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S)$$

are continuously invertible.

THEOREM A.5 *Let the operator $\hat{\mathcal{L}}_S$ be given by (3.23).*

(i) The operator $\hat{\mathcal{L}}_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is Fredholm operator with zero index and the operator $\mathcal{L}^{\pm}_{\partial\Omega} - \hat{\mathcal{L}}_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is compact.

(ii) Let $S = \Sigma$ or S along with $\partial\Omega\backslash\overline{S}$ be nonempty, open simply connected sub–manifolds of $\partial\Omega$ with an infinitely smooth boundary curve. Then the operator $\hat{\mathcal{L}}_S : \widetilde{H}^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$ is continuously invertible and the operator $\mathcal{L}^{\pm}_S - \hat{\mathcal{L}}_S : \widetilde{H}^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$ is compact.

THEOREM A.6 *The following operators are compact,*

$$\begin{aligned}
\gamma^{\pm}_{\Sigma} V_{\partial\Omega} &: H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\Sigma), & \gamma^{\pm}_{\Sigma} W_{\partial\Omega} &: H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\Sigma),\\
T^{\pm}_{\Sigma} V_{\partial\Omega} &: H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\Sigma), & T^{\pm}_{\Sigma} W_{\partial\Omega} &: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\Sigma),\\
\gamma^{+}_{\partial\Omega} V_{\Sigma} &: \widetilde{H}^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\partial\Omega), & \gamma^{+}_{\partial\Omega} W_{\Sigma} &: \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\partial\Omega),\\
T^{+}_{\partial\Omega} V_{\Sigma} &: \widetilde{H}^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\partial\Omega), & T^{+}_{\partial\Omega} W_{\Sigma} &: \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\partial\Omega).
\end{aligned}$$

Proof. Let S be either $\partial\Omega$ or Σ. As shown in [2],

$$V_S g = \frac{1}{a} V_{\Delta,S}\, g, \quad W_S g = \frac{1}{a} W_{\Delta,S}(ag), \tag{A.13}$$

where $V_{\Delta,S}$, $W_{\Delta,S}$ are the single and double layer potentials for the Laplace operator, with the densities having support on S. Since $\Delta V_{\Delta,S}\, g = 0$, $\Delta W_{\Delta,S}\, g = 0$ on $\mathbb{R}^3\backslash S$, we have $V_{\Delta,S}\, g, W_{\Delta,S}\, g \in C^\infty(\mathbb{R}^3\backslash S)$ by the solution regularity theorem for strongly elliptic equations, see e.g. [7], which by (A.13) implies also $V_S g, W_S g \in C^\infty(\mathbb{R}^3\backslash S)$. Since $\partial\Omega$ and Σ do not intersect, employing the Rellich compact imbedding theorem completes the proof. □

B Properties of volume potentials

THEOREM B.1 *The following operators are continuous*

$$\begin{aligned}
\mathcal{P} &: H^0(\Omega) \to H^2(\Omega) \subset H^{1,0}(\Omega; L), && \text{(B.1)}\\
\mathcal{R} &: H^0(\Omega) \to H^1(\Omega), && \text{(B.2)}\\
&: H^1(\Omega_\Sigma) \to H^{1,0}(\Omega_\Sigma; L). && \text{(B.3)}
\end{aligned}$$

Moreover, for $f \in H^0(\Omega_\Sigma)$ and $u \in H^1(\Omega_\Sigma)$ we have,

$$[\mathcal{P}f]_\Sigma = 0, \quad [T]_\Sigma \mathcal{P}f = 0, \quad [\mathcal{R}u]_\Sigma = 0, \quad [T]_\Sigma \mathcal{R}u = -(\partial_n a)\,[u]_\Sigma. \tag{B.4}$$

Proof. The continuity of operators (B.1) and (B.2) is proved in [2, Theorem 3.8]. Similar to the proof of [2, Theorem 3.8], integrating by parts we have the following relation for $g \in H^1(\Omega_\Sigma)$,

$$\partial_j\, \mathcal{P}_\Delta\, g = \mathcal{P}_\Delta\left(\partial_j g\right) + V_{\Delta,\partial\Omega}\left(n_j \gamma^+ g\right) + V_{\Delta,\Sigma}\left(n_j [g]_\Sigma\right) \qquad \text{in } \Omega_\Sigma. \tag{B.5}$$

Taking into account that expressions (3.1) and (3.3) give

$$\mathcal{R}\, u = -\frac{1}{a} \sum_{j=1}^{3} \partial_j \left[\mathcal{P}_\Delta\left(u\, \partial_j a\right)\right],$$

we have from (B.5),

$$\mathcal{R}\, u = -\frac{1}{a} \sum_{j=1}^{3} \mathcal{P}_\Delta\, \partial_j\big(u\, \partial_j a\big) - V_{\Delta,\partial\Omega}(\gamma^+ u\ \partial_n a) - V_{\Delta,\Sigma}\left([u]_\Sigma \partial_n a\right) \qquad \text{in } \Omega_\Sigma \tag{B.6}$$

which along with (B.1) and Theorem A.1 implies (B.3). The first two relations in (B.4) follow from (B.1) and imply the last two by (B.6) and Theorem A.3. □

The following statement follows from [2, Corollary 3.9].

THEOREM B.2 *Let $S = \partial\Omega$, or S be a nonempty, open sub–manifold $\partial\Omega$ with an infinitely smooth boundary curve, or $S = \Sigma$. The operators*

$$\mathcal{R} \ : \ H^1(\Omega_\Sigma) \to H^1(\Omega_\Sigma), \tag{B.7}$$

$$r_S\, \gamma^{\pm}\mathcal{R} \ : \ H^1(\Omega_\Sigma) \to H^{\frac{1}{2}}(S), \tag{B.8}$$

$$r_S\, T^{\pm}\mathcal{R} \ : \ H^1(\Omega_\Sigma) \to H^{-\frac{1}{2}}(S) \tag{B.9}$$

are compact.

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