# Analysis of Some Localized Boundary–Domain Integral Equations for Transmission Problems with Variable Coefficients

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## 1 Introduction

We consider the basic and mixed transmission problems for scalar second order elliptic partial differential equations with variable coefficients and use the localized parametrices to reduce the problems to direct segregated boundary–domain integral equations.

The treatment, by variational methods, of the transmission problems considered in this paper have been investigated in the research literature, and the corresponding uniqueness and existence results are well known (see, e.g., [HW08, LiMa72]).

For the special cases, when the fundamental solution is available, the Dirichlet- and Neumann-type boundary value problems have also been investigated by the classical potential method (see [Mir70, HW08] and the references therein).

Our goal here is to show that the problems can be equivalently reduced to some *localized boundary–domain integral equations* (LBDIEs) and the corresponding *localized boundary–domain integral operators* (LBDIOs) are invertible which, beside a pure mathematical interest, may also have some applications in numerical analysis for construction of efficient numerical algorithms (see, e.g., [Mik02, MN05, SSA00, ZZA98, ZZA99] and the references therein).

In our case, the localized parametrix is represented as the product of a Levi function of the differential operator under consideration and an appropriately chosen

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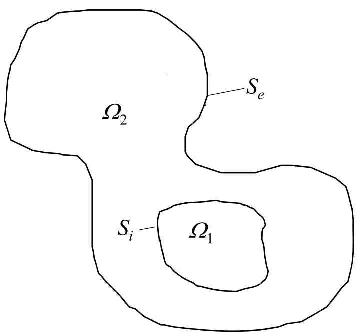
localizing function, e.g., a function supported on some neighborhood of the singularity point of the Levi function. Although the kernels of the corresponding localized potentials do not solve the original PDEs, the localized potentials preserve almost all mapping properties of the usual non-localized ones (cf. [CMN09-1, Mik06, CMN11]). However, some unusual properties of the localized potentials appear due to the localization of the kernel functions which have no counterparts in classical potential theory and which need special consideration and analysis.

By the direct approach based on Green's representation formula, we reduce the Dirichlet and mixed transmission problems to the LBDIE system. First we establish the equivalence between the original transmission problems and the corresponding LBDIE systems, which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we investigate the Fredholm properties of the LBDIOs and prove their invertibility in appropriate function spaces. In this paper we present analysis for a wider classes of the localizing functions than in [CMN09-L].

## 2 Reduction to Localized Boundary–Domain Integral Equations

### *2.1 Formulation of the Interface Problems*

Let $\Omega$ and $\Omega_1$ be bounded open domains in $\mathbb{R}^3$, $\overline{\Omega}_1 \subset \Omega$ and $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. We assume that the *interface surface* $S_i = \partial\Omega_1$ and the *exterior boundary* $S_e = \partial\Omega$ of the composite body $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ are sufficiently smooth, say $C^\infty$-regular if not otherwise stated. Clearly, $\partial\Omega_2 = S_i \cup S_e$. Throughout the paper $n^{(q)} = n^{(q)}(x)$ denotes the unit normal vector to $\partial\Omega_q$ directed outward from the corresponding domain $\Omega_q$. Clearly, $n^{(1)}(x) = -n^{(2)}(x)$ for $x \in S_i$.




By $H^r(\Omega') = H_2^r(\Omega')$ and $H^r(S) = H_2^r(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain $\Omega'$ and on a closed manifold $S$ without boundary. The subspace of $H^r(\mathbb{R}^3)$ of functions with compact support is denoted by $H^r_{comp}(\mathbb{R}^3)$. Recall that $H^0(\Omega') = L_2(\Omega')$ is a space of square integrable functions in $\Omega$.