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LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATION METHOD FOR THE DIRICHLET BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS

We consider the Dirichlet boundary value problem (BVP) for second order elliptic partial differential equations with variable coefficients and develop the approach based on the *localized parametrix method.*

Consider a uniformly elliptic second order scalar partial differential operator

$$A(x,\partial_x)\,u=\frac{\partial}{\partial x_k}\Big(a_{kj}(x)\,\frac{\partial u}{\partial x_j}\Big),$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$, $a_{kj}\in C^\infty$ and $a_{kj}=a_{jk}$, $j,k=1,2,3$. We assume that $a_{kj}(x)=\delta_{kj}$ outside of some compact set, where δ_{kj} is the Kronecker's delta. Clearly, $\lim\limits_{|x|\to\infty}a_{kj}(x)=\delta_{kj}$. Moreover, due to the uniform ellipticity, there are positive constants c_1 and c_2 such that $c_1\,|\xi|^2\leq a_{kj}(x)\,\xi_k\,\xi_j\leq c_2\,|\xi|^2\ \ \forall\, x\in\mathbb{R}^3,\ \ \forall\,\xi\in\mathbb{R}^3$. Here and in what follows, under repeated indices we assume summation from 1 to 3, unless otherwise stated.

Further, let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega^+=S\in C^\infty$, $\overline{\Omega^+}=\Omega^+\cup S$. Throughout the paper, $n=(n_1,n_2,n_3)$ denotes the unit normal vector to S directed outward with respect to the domain Ω^+. Set $\Omega^-:=\mathbb{R}^3\setminus\overline{\Omega^+}$.

By $H^r(\Omega)=H^r_2(\Omega)$ and $H^r(S)=H^r_2(S)$, $r\in\mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for C^∞ functions in $\mathbb{R}^3$ with compact support, and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega)=L_2(\Omega)$ is a space of square integrable functions in Ω.

We also need the following subspace of $H^1(\Omega)$, $H^{1,\,0}(\Omega;A):=\{u\in H^1(\Omega)\,:\,A(x,\partial)u\in H^0(\Omega)\}$.

The Dirichlet boundary value problem is formulated as follows: Find a function $u\in H^{1,\,0}(\Omega^+,A)$ satisfying the differential equation

$$A(x,\partial_x)u=f\quad\text{in}\quad\Omega^+\tag{1}$$

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and the boundary condition

$$u^+ = \varphi_0 \quad \text{on} \quad S, \tag{2}$$

where $\varphi_0 \in H^{1/2}(S)$, $f \in H^0(\Omega^+)$.

The BVP treated in the paper is well investigated in the scientific literature both by the variational and by the conventional classical potential methods when the corresponding fundamental solution is available in an explicit form (see, e.g., [5], [6], [7], [9]).

Our goal here is to show that solutions of the problem can be represented by *localized potentials* and that the *localized boundary-domain integral operator (LBDIO)* corresponding to the Dirichlet problem is invertible, which is very important from the point of view of numerical analysis, since the LBDIE leads to a very convenient numerical schemes in applications (for details see [8], [11], [12], [13]).

In our case, the localized parametrix $P_\chi(x,y)$ is represented as the product of the corresponding Lewy function $P_1(y, x-y)$ of the differential operator under consideration by an appropriately chosen cut-off function $\chi(x,y)$ supported in some neighborhood of the origin. Clearly, the kernels of the corresponding localized potentials are supported in some neighborhood of the reference point y (assuming that x is an integration variable) and they do not solve the original differential equation, while the localized potentials preserve almost all mapping properties of the usual non-localized ones (cf.[2], [3]).

Using the direct approach, we reduce the BVP to the *localized boundary-domain integral equations (LBDIE) system.* First, we establish the equivalence between the original boundary value problem and the corresponding LBDIEs system which proved to be a quite nontrivial problem and plays a crucial role in our analysis.

Afterwards, we establish that the localized boundary domain integral operator obtained belongs to the Boutet de Monvel algebra of pseudodifferential operators. With the help of the Vishik-Eskin theory, based on the factorization method (Wiener-Hopf method) (see [1], [4], [10]), we investigate Fredholm properties and prove invertibility of the corresponding localized boundary-domain operator in appropriate function spaces.

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