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LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR DIRICHLET PROBLEM FOR SECOND ORDER STRONGLY ELLIPTIC SYSTEMS WITH VARIABLE COEFFICIENTS

O. CHKADUA, S. MIKHAILOV AND D. NATROSHVILI

Abstract. Employing a localized parametrix the Dirichlet boundary value problem for second order strongly elliptic systems with variable coefficients is reduced to a *localized boundary-domain integral equations* (*LBDIE*) *system*. The equivalence between the Dirichlet problem and the LBDIE system is studied. It is established that the localized boundary-domain integral operator obtained in the paper belongs to the Boutet de Monvel algebra. The Fredholm property of this operator and its invertibility are investigated by the Wiener-Hopf factorization method.

რეზიუმე. ლოკალიზებული პარამეტრიქსის მეთოდის გამოყენებით დირიხლეს სასაზღვრო ამოცანა მეორე რიგის ცვლადკოეფიციენტებიანი ძლიერად ელიფსური სისტემებისათვის დაყვანილია ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემაზე. შესწავლილია დირიხლეს სასაზღვრო ამოცანისა და მიღებულ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემის ეკვივალენტობა. ნაჩვენებია, რომ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემით წარმოშობილი ოპერატორი ეკუთვნის ბუტე დე მონველის ალგებრას. გამოკვლეულია ამ ოპერატორის ფრედჰოლმურობა და დადგენილია მისი შებრუნებადობა ვინერ-ჰოფის ფაქტორიზაციის მეთოდით.

1. Introduction

We consider the Dirichlet boundary-value problem (BVP) for second order strongly elliptic systems of partial differential equations in the divergence form with variable coefficients and develop the generalized potential method based on the *localized parametrix method*.

The BVP treated in the paper is well investigated in the scientific literature by the variational and also by the usual classical potential methods

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when the corresponding fundamental solution is available in explicit form (see, e.g., [13], [16], [17], [21]).

Our goal here is to show that solutions of the problem can be represented by *localized potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, which seems very important from the point of view of numerical analysis, since they lead to very convenient numerical schemes in applications (for details see [18], [24], [25], [26], [27], [20]).

By means of the localized layer and volume potentials we reduce the Dirichlet BVP to the *localized boundary-domain integral equations* (*LBDIE*) *system*. First we establish the equivalence between the original boundary value problem and the corresponding LBDIEs system which proved to be a quite nontrivial problem and plays a crucial role in our analysis.

Afterwards we establish that the localized boundary domain integral operator obtained belongs to the Boutet de Monvel algebra of pseudo-differential operators and with the help of the Vishik-Eskin theory, based on the factorization method (Wiener-Hopf method), we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate function spaces. This paper develops methods and results of [4–11], [19].

2. Formulation of the Boundary Value Problems and Localized Green's Third Formula

Consider a uniformly strongly elliptic second order matrix partial differential operator

$$A(x,\partial_x)=\big[A_{pq}(x,\partial_x)\big]_{3\times 3}=\left[\frac{\partial}{\partial x_k}\left(a_{kj}^{pq}(x)\,\frac{\partial}{\partial x_j}\right)\right]_{3\times 3}, \tag{2.1}$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$, $a_{kj}^{pq}=a_{jk}^{qp}=a_{pj}^{kq}\in C^\infty$, $j,k,p,q=1,2,3$. Here and in what follows by repeated indices summation from 1 to 3 is meant if not otherwise stated.

We assume that the coefficients a_{kj}^{pq} are real and the quadratic from $a_{kj}^{pq}(x)\,\eta_{kj}\,\eta_{pq}$ is uniformly positive definite in $\mathbb{R}^3$ with respect to symmetric variables $\eta_{kj}=\eta_{jk}\in\mathbb{R}$, which implies that the principal homogeneous symbol of the operator $A(x,\partial_x)$ with opposite sign, $A(x,\xi)=[a_{kj}^{pq}(x)\xi_k\,\xi_j]_{3\times3}$ is uniformly positive definite, i.e. there are positive constants c_1 and c_2 such that

$$c_1\,|\xi|^2|\zeta|^2\le\big(A(x,\xi)\zeta\,,\,\zeta\big)\le c_2\,|\xi|^2|\zeta|^2,\ \ \forall\ x\in\mathbb{R}^3,\ \ \forall\ \xi\in\mathbb{R}^3,\ \ \forall\ \zeta\in\mathbb{C}^3, \tag{2.2}$$

where $(\cdot\,,\,\cdot)$ denotes the usual scalar product in $\mathbb{C}^3$.

Further, let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega^+=S\in C^\infty$, $\overline{\Omega^+}=\Omega^+\cup S$. Throughout the paper $n=$

(n_1, n_2, n_3) denotes the unit normal vector to S directed outward with respect to the domain Ω^+. Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

By $H^r(\Omega) = H^r_2(\Omega)$ and $H^r(S) = H^r_2(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for C^∞ functions in $\mathbb{R}^3$ with compact support and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω.

For a vector $u = (u_1, u_2, u_3)^\top$ the inclusion $u = (u_1, u_2, u_3)^\top \in H^r$ means that each component u_j belongs to the space H^r.

Let us denote $u^\pm \equiv \{u\}^\pm = \gamma^\pm u$, where γ^+ and γ^- are the trace operators on S from the interior and exterior of Ω^+ respectively.

We also need the following subspace of $H^1(\Omega)$,

$$H^{1,0}(\Omega; A) := \Big\{ u = (u_1, u_2, u_3)^\top \in H^1(\Omega) \, : \, A(x, \partial)u \in H^0(\Omega) \Big\}. \quad (2.3)$$

The Dirichlet boundary-value problem reads as follows.

Dirichlet problem: Find a vector-function $u = (u_1, u_2, u_3)^\top \in H^{1,0}(\Omega^+, A)$ satisfying the differential equation

$$A(x, \partial_x)u = f \ \text{ in } \ \Omega^+ \quad (2.4)$$

and the Dirichlet boundary condition

$$u^+ = \varphi_0 \ \text{ on } \ S, \quad (2.5)$$

where $\varphi_0 = (\varphi_{01}, \varphi_{02}, \varphi_{03})^\top \in H^{1/2}(S)$ and $f = (f_1, f_2, f_3)^\top \in H^0(\Omega^+)$.

Equation (2.4) is understood in the distributional sense, while the Dirichlet-type boundary condition (2.5) is understood in the usual trace sense.

Now, we introduce the *co-normal derivative operator* associated with the differential operator $A(x, \partial_x)$,

$$T(x, \partial_x) = \big[T_{pq}(x, \partial_x) \big]_{3\times 3} := \Big[a^{pq}_{kj}(x)\, n_k(x)\, \frac{\partial}{\partial x_j} \Big]_{3\times 3}. \quad (2.6)$$

Evidently, the co-normal derivative for a smooth vector-function u, say $u \in H^2(\Omega^+)$, reads as follows

$$\begin{aligned} \big[T^\pm(x, \partial_x)\, u(x) \big]_p &:= \big[\{ T(x, \partial_x)\, u(x) \}^\pm \big]_p := \\ &= a^{pq}_{kj}(x)\, n_k(x)\, \{ \partial_{x_j} u_q(x) \}^\pm, \ \ x \in S, \ \ p = 1, 2, 3, \end{aligned} \quad (2.7)$$

which is understood in the usual traces sense.

Note that the co-normal derivative operator defined in (2) can be extended by continuity to the space $H^{1,0}(\Omega^+; A)$ with the help of Green's first identity,

$$\langle T^+ u \, , \, g \rangle_S := \int\limits_{\Omega^+} A(x, \partial_x)u(x)\, v(x)\, dx + \int\limits_{\Omega^+} E(u(x), v(x))\, dx, \quad (2.8)$$

where $E(u(x), v(x)) = a_{kj}^{pq}(x)\, \partial_{x_j} u_q(x)\, \partial_{x_k} v_p(x)$, $g \in H^{1/2}(S)$ is an arbitrary vector-function and $v \in H^1(\Omega)$ is an extension of g from S onto the whole of Ω^+, i.e., $v^+ = g$ on S, while $\langle \cdot\,, \cdot \rangle_S$ denotes the duality between the adjoint spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$ which extends the usual bilinear $L_2(S)$ inner product. Clearly the definition (2.8) does not depend on the extension operator.

Let us define the following class of cut-off functions (see[7]).

Definition 2.1. We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W_1^k(0,\infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0,\infty)$. We say $\chi \in X_+^k$ for integer $k \geq 1$ if $\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\hat{\chi}_s(\omega)}{\omega} > 0 \ \text{ for } \ \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho \breve{\chi}\,(\varrho)\, d\varrho \ \text{ for } \ \omega = 0, \end{cases} \tag{2.9}$$

and $\hat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$

$$\hat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}\,(\varrho)\, \sin(\varrho\, \omega)\, d\varrho. \tag{2.10}$$

We say $\chi \in X_{1+}^k$ for integer $k \geq 1$ if $\chi \in X_+^k$ and

$$\omega \hat{\chi}_s(\omega) \leq 1, \quad \forall\, \omega \in \mathbb{R}. \tag{2.11}$$

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$ and $X_+^{k_1} \subset X_+^{k_2}$, $X_{1+}^{k_1} \subset X_{1+}^{k_2}$ for $k_1 > k_2$. The class X_+^k is defined in terms of the sine-transform. The following lemma provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class (for details see [7]).

Lemma 2.2. *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0,\infty)$, and $\breve{\chi}$ is a non-increasing function on $[0,+\infty)$, then $\chi \in X_+^k$.*

The following examples for χ are presented in [7],

$$\chi_1(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{2.12}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{2.13}$$

$$\chi_3(x) = \begin{cases} \left(1 - \dfrac{|x|}{\varepsilon}\right)^2 \left(1 - 2\dfrac{|x|}{\varepsilon}\right) & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon. \end{cases} \tag{2.14}$$

One can observe that $\chi_1 \in X^k_+$, while $\chi_2 \in X^\infty_+$ due to Lemma 2.2, and $\chi_3 \in X^2_+$. Moreover, $\chi_1 \in X^k_{1+}$ for $k = 2$ and $k = 3$, and $\chi_3 \in X^2_{1+}$, while $\chi_1 \notin X^1_{1+}$ and $\chi_2 \notin X^\infty_{1+}$ (for details see [7]).

Define a *localized matrix parametrix* corresponding to the fundamental solution function $F_1(x) := -[\,4\,\pi\,|x|\,]^{-1}$ of the Laplace operator, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$,

$$P(x) \equiv P_\chi(x) := F_\chi(x)\, I = \chi(x)\, F_1(x)\, I = -\frac{\chi(x)}{4\,\pi\,|x|}\, I \quad \text{with} \quad \chi(0)=1, \tag{2.15}$$

where $F_\chi(x) := \chi(x)\, F_1(x)$, I is the identity 3×3 matrix and χ is a localizing function

$$\chi \in X^k_+\,, \qquad k \geq 3. \tag{2.16}$$

Throughout the paper we assume that the condition (2.16) is satisfied and χ has a compact support if not otherwise stated.

Denote by $B(y,\varepsilon)$ a ball centered at the point y and radius $\varepsilon > 0$ and let $\Sigma(y,\varepsilon) := \partial B(y,\varepsilon)$.

There holds Green's second identity

$$\int\limits_{\Omega^+} \big[v\ A(x,\partial)u - A(x,\partial)v\ u\big]dx = \int\limits_S \big[\{v\}^+\{Tu\}^+ - \{Tv\}^+\ \{u\}^+\big]dS \tag{2.17}$$

for smooth vector-functions u and v, say $u, v \in C^2(\overline{\Omega^+})$.

Let us take in the role of $v(x)$ successively the columns of the matrix $P(x-y)$, where y is an arbitrarily fixed interior point in Ω^+, and write the identity (2.17) for the region $\Omega^+_\varepsilon := \Omega^+ \setminus B(y,\varepsilon)$ with $\varepsilon > 0$ such that $\overline{B(y,\varepsilon)} \subset \Omega^+$. Keeping in mind that $P^\top(x-y) = P(x-y)$ and $[A(x,\partial_x)P(x-y)]^\top = [A(x,\partial_x)P(x-y)]$, we arrive at the equality,

$$\begin{aligned}
&\int\limits_{\Omega^+_\varepsilon} \big[P(x-y)\ A(x,\partial_x)u(x) - A(x,\partial_x)P(x-y)\ u(x)\big]\, dx = \\
&= \int\limits_S \big[P(x-y)\ \{T(x,\partial_x)u(x)\}^+ - \{T(x,\partial_x)P(x-y)\}^\top \{u(x)\}^+\big]dS - \\
&- \int\limits_{\Sigma(y,\varepsilon)} \big[P(x-y)T(x,\partial_x)u(x) - \{T(x,\partial_x)P(x-y)\}^\top u(x)\big]d\Sigma(y,\varepsilon).
\end{aligned} \tag{2.18}$$

The direction of the normal vector on $\Sigma(y,\varepsilon)$ is chosen as outward.

It is clear that the operator

$$\mathcal{A}\, u(y) := \lim_{\varepsilon\to 0} \int\limits_{\Omega^+_\varepsilon} [\,A(x,\partial)P(x-y)\,]\, u(x)\, dx =$$

$$= \text{v.p.} \int_{\Omega^+} [\, A(x, \partial_x) P(x-y) \,]\, u(x)\, dx \tag{2.19}$$

is a singular integral operator, "v.p" means the Cauchy principal value integral. If the domain of integration in (2.19) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{A}\, u \equiv \mathbf{A}\, u$, i.e.,

$$\mathbf{A}\, u(y) := \text{v.p.} \int_{\mathbb{R}^3} [\, A(x, \partial_x) P(x-y) \,]\, u(x)\, dx\,, \tag{2.20}$$

$$\big[A(x, \partial_x) P(x-y)\big]_{pq} =$$

$$= \mathbf{b}_{pq}(x)\, \delta(x-y) + \text{v.p.}\Big[- \frac{a_{kj}^{pq}(x)}{4\,\pi} \frac{\partial^2}{\partial x_k\, \partial x_j} \frac{1}{|x-y|} \Big] + R_{pq}(x,y) \tag{2.21}$$

$$= \mathbf{b}_{pq}(y)\, \delta(x-y) + \text{v.p.}\Big[- \frac{a_{kj}^{pq}(y)}{4\,\pi} \frac{\partial^2}{\partial x_k\, \partial x_j} \frac{1}{|x-y|} \Big] + R_{pq}^{(1)}(x,y), \tag{2.22}$$

where

$$\mathbf{b}(x) = [\, \mathbf{b}_{pq}(x) \,]_{3\times 3} = \frac{1}{3}\, [\, a_{kj}^{pq}(x) \delta_{kj} \,]_{3\times 3} = \frac{1}{3}\, [\, a_{kk}^{pq}(x) \,]_{3\times 3} =$$

$$= \frac{1}{3}\, [\, a_{11}^{pq}(x) + a_{22}^{pq}(x) + a_{33}^{pq}(x) \,]_{3\times 3}, \tag{2.23}$$

$$R(x,y) = [R_{pq}(x,y)]_{3\times 3}\,, \quad R_1(x,y) = [R_{pq}^{(1)}(x,y)]_{3\times 3}, \tag{2.24}$$

$$R_{pq}(x,y) := -\frac{a_{kj}^{pq}(x)}{4\pi} \Big\{ \big[\chi(x-y) - 1\big] \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|} + \\ + \frac{\partial^2 \chi(x-y)}{\partial x_k \partial x_j} \frac{1}{|x-y|} + 2\, \frac{\partial \chi(x-y)}{\partial x_j} \frac{\partial}{\partial x_k} \frac{1}{|x-y|} \Big\} - \\ - \frac{1}{4\pi} \frac{\partial a_{kj}^{pq}(x)}{\partial x_k} \Big[\frac{\partial \chi(x-y)}{\partial x_j} \frac{1}{|x-y|} + \chi(x-y) \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \Big], \tag{2.25}$$

$$R_{pq}^{(1)}(x,y) := R_{pq}(x,y) - \frac{a_{kj}^{pq}(x) - a_{kj}^{pq}(y)}{4\,\pi} \frac{\partial^2}{\partial x_k\, \partial x_j} \frac{1}{|x-y|}\,, \tag{2.26}$$

$$p, q = 1, 2, 3.$$

Clearly the entries of the matrix-functions $R(x,y)$ and $R^{(1)}(x,y)$ possess weak singularities of type $\mathcal{O}(|x-y|^{-2})$ as $x \to y$.

Further, by direct calculations one can easily verify that

$$\lim_{\varepsilon \to 0} \int_{\Sigma(y,\varepsilon)} P(x-y)\, T(x, \partial_x) u(x)\, d\Sigma(y,\varepsilon) = 0, \tag{2.27}$$

$$\lim_{\varepsilon \to 0} \int_{\Sigma(y,\varepsilon)} \{T(x, \partial_x) P(x-y)\}\, u(x)\, d\Sigma(y,\varepsilon) =$$

$$= \Big[\frac{a_{kj}^{pq}(y)}{4\,\pi} \int\limits_{\Sigma_1} \eta_k\, \eta_j \, d\Sigma_1 \Big]_{3\times 3} u(y) =$$

$$= \Big[\frac{a_{kj}^{pq}(y)}{4\,\pi} \; \frac{4\,\pi\,\delta_{kj}}{3} \Big]_{3\times 3} u(y) = \mathbf{b}(y)\, u(y), \tag{2.28}$$

where Σ_1 is a unit sphere, $\eta = (\eta_1, \eta_2, \eta_3) \in \Sigma_1$ and $\mathbf{b}$ is defined by (2.23).

Passing to the limit in (2.18) as $\varepsilon \to 0$ and using the relations (2.19), (2.27), and (2.28) we obtain

$$\mathbf{b}(y)\, u(y) + \mathcal{A}\, u(y) - V(T^+ u)(y) + W(u^+)(y) = \mathcal{P}\big(A(x,\partial_x) u\big)(y), \quad y \in \Omega^+, \tag{2.29}$$

where $\mathcal{A}$ is a *localized singular integral operator* given by (2.19), while V, W, and $\mathcal{P}$ are the *localized single layer, double layer and Newtonian volume potentials*,

$$V(g)(y) := -\int\limits_S P(x-y)\, g(x)\, dS_x, \tag{2.30}$$

$$W(g)(y) := -\int\limits_S \big[\, T(x,\partial_x)\, P(x-y) \,\big]\; g(x)\, dS_x, \tag{2.31}$$

$$\mathcal{P}(h)(y) := \int\limits_{\Omega^+} P(x-y)\, h(x)\, dx. \tag{2.32}$$

If the domain of integration in the Newtonian volume potential (2.32) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{P}\, h \equiv \mathbf{P}\, h$, i.e.,

$$\mathbf{P}(h)(y) := \int\limits_{\mathbb{R}^3} P(x-y)\, h(x)\, dx. \tag{2.33}$$

Mapping properties of the above potentials are investigated in [7].

Denote by ℓ_0 the extension operator by zero from Ω^+ onto Ω^-. It is evident that for a function $u \in H^1(\Omega^+)$ we have

$$\big(\mathcal{A}\, u\big)(y) = \big(\mathbf{A}\ell_0 u\big)(y) \quad \text{for} \;\; y \in \Omega^+.$$

Now we rewrite Green's third formula (2.29) in a more convenient form for our further purposes

$$[\mathbf{b}{+}\mathbf{A}]\ell_0 u\,(y) - V(T^+u)(y) + W(u^+)(y) = \mathcal{P}\big(A(x,\partial_x)u\big)(y), \;\; y \in \Omega^+. \tag{2.34}$$

The principal homogeneous symbols of the singular integral operators $\mathbf{A}$ and $\mathbf{b} + \mathbf{A}$ read as

$$\mathfrak{S}_0(\mathbf{A})(y,\xi) = |\xi|^{-2} A(y,\xi) - \mathbf{b} \qquad \forall\, y \in \overline{\Omega^+}, \qquad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{2.35}$$

$$\mathfrak{S}_0(\mathbf{b}+\mathbf{A})(y,\xi) = |\xi|^{-2} A(y,\xi) \quad \forall\, y \in \overline{\Omega^+}, \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{2.36}$$

It is evident that the symbol matrix (2.36) is positive definite due to (2.2),

$$\big(\mathfrak{S}_0(\mathbf{b}+\mathbf{A})(y,\xi)\,\zeta,\zeta\big) = |\xi|^{-2}\big(A(y,\xi)\,\zeta,\zeta\big) \geq c_1\,|\zeta|^2, \tag{2.37}$$

$$\forall\, y \in \overline{\Omega^+}, \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \ \ \forall\, \zeta \in \mathbb{C}^3, \tag{2.38}$$

where c_1 is the same positive constant as in (2.2).

Using the properties of localized potentials and taking the trace of equation (2.34) on S we arrive at the relation:

$$\mathbf{A}^+\ell_0 u - \mathcal{V}(T^+u) + (\mathbf{b}-\mu)\,u^+ + \mathcal{W}(u^+) = \mathcal{P}^+\big(A(x,\partial_x)u\big) \ \text{ on } \ S, \tag{2.39}$$

where the localized boundary integral operators $\mathcal{V}$ and $\mathcal{W}$ are direct values of the localized single and double layer potentials and μ is the following matrix

$$\mu(y) = [\mu^{pq}(y)]_{3\times 3} := \frac{1}{2}\,\big[\,a_{kj}^{pq}(y)\,n_k(y)\,n_j(y)\,\big]_{3\times 3}, \quad y \in S, \tag{2.40}$$

which is positive definite due to (2.2), while

$$\mathbf{A}^+\ell_0 u \equiv \gamma^+\mathbf{A}\ell_0 u := \{\,\mathbf{A}\,\ell_0 u\,\}^+ \ \text{ on } \ S, \tag{2.41}$$

$$\mathcal{P}^+(f) \equiv \gamma^+\mathcal{P}(f) := \{\,\mathcal{P}(f)\}^+ \quad \text{on } \ S. \tag{2.42}$$

Now, we are in the position to reduce the above formulate Dirichlet boundary value problem to the LBDIEs system equivalently.

3. LBDIE Formulation of the Dirichlet Problem and the Equivalence Theorem

Let $u \in H^{1,0}(\Omega^+, A)$ be a solution to the Dirichlet BVP (2.4)–(2.5) with $\varphi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega^+)$. As we have derived above there hold the relations (2.34) and (2.39), which now can be rewritten in the form

$$[\,\mathbf{b}+\mathbf{A}\,]\,\ell_0 u - V(\psi) = \mathcal{P}(f) - W(\varphi_0) \ \text{ in } \ \Omega^+, \tag{3.1}$$

$$\mathbf{A}^+\ell_0 u - \mathcal{V}(\psi) = \mathcal{P}^+(f) - (\mathbf{b}-\mu)\,\varphi_0 - \mathcal{W}(\varphi_0) \ \text{ on } \ S, \tag{3.2}$$

where $\psi := T^+u \in H^{-\frac{1}{2}}(S)$ and μ is defined by (2.40).

One can consider these relations as the LBDIEs system with respect to the unknown vector-functions u and ψ. The following equivalence theorem holds.

Theorem 3.1. *The Dirichlet boundary value problem* (2.4)–(2.5) *is equivalent to LBDIEs system* (3.1)–(3.2) *in the following sense:*

(i) *If a vector-function* $u \in H^{1,0}(\Omega^+, A)$ *solves the Dirichlet BVP* (2.4)–(2.5)*, then it is unique and the pair* $(u,\psi) \in H^{1,0}(\Omega^+, A) \times H^{-\frac{1}{2}}(S)$ *with*

$$\psi = T^+u\,, \tag{3.3}$$

solves the LBDIEs system (3.1)–(3.2) *and, vice versa,*

(ii) *If a pair* $(u,\psi)\in H^{1,0}(\Omega^+,A)\times H^{-\frac{1}{2}}(S)$ *solves the LBDIEs system* (3.1)–(3.2)*, then it is unique and the vector-function* u *solves the Dirichlet BVP* (2.4)–(2.5)*, and relation* (3.3) *holds.*

4. Invertibility of the Dirichlet LBDIO

From Theorem 3.1 it follows that the LBDIEs system (3.1)–(3.2), which has a special right hand side, is uniquely solvable in the class $H^{1,0}(\Omega^+,A)\times H^{-1/2}(S)$. Let us investigate the localized boundary-domain integral operator generated by the left hand side expressions in (3.1)–(3.2) in appropriate functional spaces.

The LBDIEs system (3.1)–(3.2) with an arbitrary right hand side vector-functions from the space $H^1(\Omega^+)\times H^{1/2}(S)$ can be written as

$$(\mathbf{b}+\mathbf{A})\ell_0 u\,-V\psi=F_1\ \text{ in }\ \Omega^+, \tag{4.1}$$

$$\mathbf{A}^+\ell_0 u-\mathcal{V}\psi=F_2\ \text{ on }\ S, \tag{4.2}$$

where $F_1\in H^1(\Omega^+)$ and $F_2\in H^{1/2}(S)$.

Denote

$$\mathbf{B}:=(\mathbf{b}+\mathbf{A}). \tag{4.3}$$

Evidently, the principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as (see (2.36))

$$\mathfrak{S}_0(\mathbf{B})(y,\xi)=|\xi|^{-2}A(y,\xi)\quad\text{for}\quad y\in\overline{\Omega^+},\quad \xi\in\mathbb{R}^3\setminus\{0\}, \tag{4.4}$$

is even rational homogeneous matrix-function of order 0 in ξ and due to (2.2) it is positive definite,

$$\big(\mathfrak{S}_0(\mathbf{B})(y,\xi)\zeta,\zeta\big)\geq\ c_1\,|\zeta|^2\quad\text{for all}\quad y\in\overline{\Omega^+},\quad \xi\in\mathbb{R}^3\setminus\{0\}\text{and}\zeta\in\mathbb{C}^3.$$

Consequently, $\mathbf{B}$ is a strongly elliptic pseudodifferential operator of zero order (i.e., singular integral operator) and the partial indices of factorization of the symbol (4.4) equal to zero (cf. [23], [2], [3]).

Since (4.4) is a rational matrix-function in ξ, we can apply the theory of pseudodifferential operators with symbol satisfying the transmission conditions (see [12], [1], [22], [2], [23]).

We need some auxiliary assertions in our further analysis. To formulate them, let $y_0\in\partial\Omega^+$ be some fixed point and consider the frozen symbol $\mathfrak{S}_0(\mathbf{B})(y_0,\xi)\equiv\mathfrak{S}_0(\mathbf{B})(\xi)$. Further, let $\widehat{\mathbf{B}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}_0(\mathbf{B})(\xi',\xi_3):=\mathfrak{S}_0(\mathbf{B})\big((1+|\xi'|)\omega,\xi_3\big)$$
$$\text{with}\ \ \omega=\frac{\xi'}{|\xi'|},\ \ \xi=(\xi',\xi_3),\ \ \xi'=(\xi_1,\xi_2).$$

The principal homogeneous symbol matrix $\mathfrak{S}_0(\mathbf{B})(\xi)$ of the operator $\widehat{\mathbf{B}}$ can be factorized with respect to the variable ξ_3 as:

$$\mathfrak{S}_0(\mathbf{B})(\xi) = \mathfrak{S}^-(\mathbf{B})(\xi)\ \mathfrak{S}^+(\mathbf{B})(\xi), \tag{4.5}$$

where

$$\mathfrak{S}^\pm(\mathbf{B})(\xi) = \frac{1}{\xi_3 \pm i\ |\xi\,'|}\ A^\pm(\xi\,', \xi_3),$$

$A^\pm(\xi\,', \xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in ξ_3 of the positive definite polynomial symbol matrix $A(\xi\,', \xi_3) \equiv A(y_0, \xi\,', \xi_3)$ (see [12], [14], [15]), i.e.

$$A(\xi\,', \xi_3) = A^-(\xi\,', \xi_3)\ A^+(\xi\,', \xi_3) \tag{4.6}$$

with $\det A^+(\xi\,', \tau) \neq 0$ for $\mathrm{Re}\tau > 0$ and $\det A^-(\xi\,', \tau) \neq 0$ for $\mathrm{Re}\tau < 0$. Moreover, the entries of the matrices $A^\pm(\xi\,', \xi_3)$ are homogeneous functions in $\xi = (\xi\,', \xi_3)$ of order 1.

Denote, by $a^\pm(\xi\,')$ the coefficients at ξ_3^3 in the determinants $\det A^\pm(\xi\,', \xi_3)$. Evidently,

$$a^-(\xi\,')\, a^+(\xi\,') = \det\, A(0, 0, 1) > 0 \quad \text{for} \quad \xi\,' \neq 0. \tag{4.7}$$

It is easy to see that the factor-matrices $A^\pm(\xi\,', \xi_3)$ have the following structure

$$\left[A^\pm(\xi\,', \xi_3)\right]^{-1} = \frac{1}{\det A^\pm(\xi\,', \xi_3)}\, [\, p^\pm_{ij}(\xi\,', \xi_3)\,]_{3\times 3}, \tag{4.8}$$

where $p^\pm_{ij}(\xi\,', \xi_3)$ are the co-factors of the matrix $A^\pm(\xi\,', \xi_3)$, which can be written in the form

$$p^\pm_{ij}(\xi\,', \xi_3) = c^\pm_{ij}(\xi\,')\, \xi_3^2 + b^\pm_{ij}(\xi\,')\, \xi_3 + d^\pm_{ij}(\xi\,'). \tag{4.9}$$

Here $c^\pm_{ij}$, $b^\pm_{ij}$ and $d^\pm_{ij}$, $i, j = 1, 2, 3$, are homogeneous functions in $\xi\,'$ of order 0, 1, and 2, respectively.

The following assertions hold.

Lemma 4.1. *Let ℓ_0 be the extension operator by zero from $\mathbb{R}^3_+$ onto the half-space $\mathbb{R}^3_-$. The operator*

$$r_{\mathbb{R}^3_+} \widehat{\mathbf{B}} \ell_0 \ :\ H^s(\mathbb{R}^3_+) \to H^s(\mathbb{R}^3_+)$$

is invertible for all $s \geq 0$.

Lemma 4.2. *Let the factor matrix $A^+(\xi\,', \tau)$ be as in* (4.6)*, and a^+ and c^+_{ij} be as in* (4.7) *and* (4.9) *respectively. Then the following equality holds*

$$\frac{1}{2\pi i} \int\limits_{\gamma^-} \left[A^+(\xi\,', \tau)\right]^{-1} d\tau = \frac{1}{a^+(\xi\,')} \left[\, c^+_{ij}(\xi\,')\,\right]_{3\times 3}, \tag{4.10}$$

and

$$\det [\, c^{+}_{ij}(\xi\,')\,]_{3\times 3} \neq 0 \quad for \quad \xi\,' \neq 0. \tag{4.11}$$

Here γ^{-} *is a contour in the lower complex half-plane enclosing all the roots of the polynomial* $\det A^{+}(\xi\,', \tau)$ *with respect to* τ.

Denote by $\mathcal{A}$ the localized boundary-domain integral operator generated by the left hand side expressions in LBDIEs system (4.1)–(4.2) as

$$\mathfrak{D} := \begin{bmatrix} r_{\Omega^+}\mathbf{B}\ell_0 & -r_{\Omega^+}V \\ \mathbf{A}^{+}\ell_0 & -\mathcal{V} \end{bmatrix}.$$

The following theorem holds.

Theorem 4.3. *Let a cut-off function* $\chi \in X^{\infty}_{+}$ *and* $r \geq 0$. *Then the following operator*

$$\mathfrak{D} \,:\, H^{r+1}(\Omega^+) \times H^{r-1/2}(S) \to H^{r+1}(\Omega^+) \times H^{r+1/2}(S) \tag{4.12}$$

is invertible.

Corollary 4.4. *Let a cut-off function* $\chi \in X^{3}_{+}$. *Then the operator*

$$\mathfrak{D} \,:\, H^{1}(\Omega^+) \times H^{-1/2}(S) \to H^{1}(\Omega^+) \times H^{1/2}(S)$$

is invertible.

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Authors' addresses:

O. Chkadua

Iv. Javakhishvili Tbilisi State University
A. Razmadze Mathematical Institute
2, University str., Tbilisi 0186, Georgia

Sokhumi State University
9, Anna Politkovskaia str., Tbilisi 0186, Georgia
E-mail: chkadua7@yahoo.com

S. Mikhailov

Department of Mathematics, Brunel University
Uxbridge, UB8 3PH, UK
E-mail: sergey.mikhailov@brunel.ac.uk

D. Natroshvili

Department of Mathematics, Georgian Technical University
77, M. Kostava str., Tbilisi 0175, Georgia
E-mail: natrosh@hotmail.com