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LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR DIRICHLET PROBLEM OF THE THEORY OF PIEZO-ELASTICITY FOR INHOMOGENEOUS SOLIDS

Dedicated to the 110-th birthday anniversary
of academician V. Kupradze

Abstract. The paper deals with the three-dimensional Dirichlet boundary-value problem (BVP) of piezo-elasticity theory for anisotropic inhomogeneous solids and develops the generalized potential method based on the localized parametrix method. Using Green's integral representation formula and properties of the localized layer and volume potentials we reduce the Dirichlet BVP to the localized boundary-domain integral equations (LBDIE) system. The equivalence between the Dirichlet BVP and the corresponding LBDIE system is studied. We establish that the obtained localized boundary-domain integral operator belongs to the Boutet de Monvel algebra and with the help of the Wiener–Hopf factorization method we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate function spaces.

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Key words and phrases. Piezo-elasticity, strongly elliptic systems, variable coefficients, boundary value problem, localized parametrix, localized boundary-domain integral equations, pseudodifferential operators.

რეზიუმე. ნაშრომი ეძღვნება ლოკალიზებული პარამეტრიქსის მეთოდის განვითარებას პიეზო-დრეკადობის თეორიის დირიხლეს სამგანზომილებიანი ამოცანისთვის არაერთგვაროვანი ანიზოტროპული სხეულების შემთხვევაში. გრინის ინტეგრალური წარმოდგენის ფორმულისა და ლოკალიზებული პოტენციალების გამოყენებით დირიხლეს ამოცანა დაიყვანება ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემაზე. შესწავლილია დირიხლეს სასაზღვრო ამოცანისა და მიღებულ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემის ეკვივალენტობა. ვინერ–ჰოფის ფაქტორიზაციის მეთოდის გამოყენებით ნაჩვენებია, რომ ლოკალიზებული სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა ოპერატორი, რომელიც ეკუთვნის ბუტე დე მონველის ალგებრას, არის ფრედჰოლმური და დადგენილია მისი შებრუნებადობა შესაბამის სივრცეებში.

1. Introduction

We consider the three-dimensional Dirichlet boundary-value problem (BVP) of piezo-elasticity for anisotropic inhomogeneous solids and develop the generalized potential method based on the *localized parametrix method.*

Due to great theoretical and practical importance, problems of piezo-elasticity became very popular among mathematicians and engineers (for details see, e.g., [26]–[34], [42], [50]).

The BVPs and various type interface problems of piezo-elasticity for *homogeneous anisotropic solids*, i.e., when the material parameters are constants and the corresponding fundamental solution is available in explicit form, by the usual classical potential methods are investigated in [4]–[9], [41]. Unfortunately this classical potential method is not applicable in the case of inhomogeneous solids since for the corresponding system of differential equations with variable coefficients a fundamental solution is not available in explicit form in general.

Therefore, in our analysis we apply the so-called *localized parametrix method* which leads to the localized boundary-domain integral equations system.

Our main goal here is to show that solutions of the boundary value problem can be represented by *localized potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, which seems very important from the point of view of numerical analysis, since they lead to very convenient numerical schemes in applications (for details see [37], [43], [46]–[49]).

To this end, using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Dirichlet BVP of piezo-elasticity to the *localized boundary-domain integral equations (LBDIE) system.* First we establish the equivalence between the original boundary value problem and the corresponding LBDIE system which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we establish that the localized boundary domain matrix integral operator generated by the LBDIE belongs to the Boutet de Monvel algebra and with the help of the Vishik–Eskin theory, based on the factorization method (Wiener–Hopf factorization method), we investigate Fredholm properties and prove invertibility of the localized operator in appropriate function spaces.

Note that the operator, generated by the system of piezo-elasticity for inhomogeneous anisotropic solids, is second order nonself-adjoint strongly elliptic partial differential operator with variable coefficients. In [21], the LBDIE method has been developed for the Dirichlet problem in the case of self-adjoint second order strongly elliptic systems with variable coefficients, while the same method for the case of scalar elliptic second order partial differential equations with variable coefficients is justified in [11]–[20], [38].

2. Reduction to LBDIE System and the Equivalence Theorem

2.1. **Formulation of the boundary value problem and localized Green's third formula.** Consider the system of static equations of piezoelectricity for an inhomogeneous anisotropic medium [42]:

$$A(x,\partial_x)U + X = 0,$$

where $U := (u_1,u_2,u_3,u_4)^\top$, $u = (u_1,u_2,u_3)^\top$ is the displacement vector, $u_4 = \varphi$ is the electric potential, $X = (X_1,X_2,X_3,X_4)^\top$, $(X_1,X_2,X_3)^\top$ is a given mass force density, X_4 is a given charge density, $A(x,\partial_x)$ is a formally nonself-adjoint matrix differential operator

$$A(x,\partial_x) = \big[A_{jk}(x,\partial_x)\big]_{4\times4} := \\ := \begin{bmatrix} \big[\partial_i(c_{ijlk}(x)\partial_l)\big]_{3\times3} & \big[\partial_i(e_{lij}(x)\partial_l)\big]_{3\times1} \\ \big[-\partial_i(e_{ikl}(x)\partial_l)\big]_{1\times3} & \partial_i(\varepsilon_{il}(x)\partial_l) \end{bmatrix}_{4\times4},$$

where $\partial_x = (\partial_1,\partial_2,\partial_3)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$. Here and in what follows by repeated indices summation from 1 to 3 is meant if not otherwise stated.

The variable coefficients involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl} = c_{jikl} = c_{klij} \in C^\infty,\quad e_{ijk} = e_{ikj} \in C^\infty,\quad \varepsilon_{ij} = \varepsilon_{ji} \in C^\infty, \\ i,j,k,l = 1,2,3.$$

In view of these symmetry relations, the formally adjoint differential operator $A^*(x,\partial_x)$ reads as

$$A^*(x,\partial_x) = \big[A^*_{jk}(x,\partial_x)\big]_{4\times4} := \\ := \begin{bmatrix} \big[\partial_i(c_{ijlk}(x)\partial_l)\big]_{3\times3} & \big[-\partial_i(e_{lij}(x)\partial_l)\big]_{3\times1} \\ \big[\partial_i(e_{ikl}(x)\partial_l)\big]_{1\times3} & \partial_i(\varepsilon_{il}(x)\partial_l) \end{bmatrix}_{4\times4}.$$

Moreover, from physical considerations it follows that (see, e.g., [42]):

$$c_{ijkl}(x)\xi_{ij}\xi_{kl} \geqslant c_0\xi_{ij}\xi_{ij} \ \text{ for all } \ \xi_{ij} = \xi_{ji} \in \mathbb{R}, \tag{2.1}$$

$$\varepsilon_{ij}(x)\eta_i\eta_j \geqslant c_1\eta_i\eta_i \ \text{ for all } \ \eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3, \tag{2.2}$$

where c_0 and c_1 are positive constants.

With the help of the inequalities (2.1) and (2.2) it can easily be shown that the operator $A(x,\partial_x)$ is uniformly strongly elliptic, that is,

$$\operatorname{Re} A(x,\xi)\zeta\cdot\zeta \geqslant c|\xi|^2|\zeta|^2 \ \text{ for all } \ \xi \in \mathbb{R}^3 \ \text{ and for all } \ \zeta \in \mathbb{C}^4, \tag{2.3}$$

where $A(x,\xi)$ is the principal homogeneous symbol matrix of the operator $A(x,\partial_x)$ with opposite sign:

$$A(x,\xi) = \big[A_{jk}(x,\xi)\big]_{4\times4} := \\ := \begin{bmatrix} \big[c_{ijlk}(x)\xi_i\xi_l\big]_{3\times3} & \big[e_{lij}(x)\xi_i\xi_l\big]_{3\times1} \\ \big[-e_{ikl}(x)\xi_i\xi_l\big]_{1\times3} & \varepsilon_{il}(x)\xi_i\xi_l \end{bmatrix}_{4\times4}. \tag{2.4}$$

Here and in what follows $a \cdot b$ denotes the scalar product of two vectors $a, b \in \mathbb{C}^4$, $a \cdot b = \sum\limits_{j=1}^{4} a_j \overline{b_j}$.

In the theory of piezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{ij} n_i = c_{ijlk} n_i \partial_l u_k + e_{lij} n_i \partial_l \varphi \;\text{ for }\; j = 1, 2, 3,$$

while the normal component of the electric displacement vector (with opposite sign) reads as

$$-D_i n_i = -e_{ikl} n_i \partial_l u_k + \varepsilon_{il} n_i \partial_l \varphi.$$

Let us introduce the following matrix differential operator

$$\mathcal{T} = \mathcal{T}(x, \partial_x) = \big[\mathcal{T}_{jk}(x, \partial_x)\big]_{4\times4} :=$$
$$:= \begin{bmatrix} \big[c_{ijlk}(x) n_i \partial_l\big]_{3\times3} & \big[e_{lij}(x) n_i \partial_l\big]_{3\times1} \\ \big[-e_{ikl}(x) n_i \partial_l\big]_{1\times3} & \varepsilon_{il}(x) n_i \partial_l \end{bmatrix}_{4\times4}.$$

For a four–vector $U = (u, \varphi)^\top$ we have

$$\mathcal{T} U = \big(\sigma_{i1} n_i,\ \sigma_{i2} n_i,\ \sigma_{i3} n_i,\ -D_i n_i\big)^\top. \tag{2.5}$$

Clearly, the components of the vector $\mathcal{T}U$ given by (2.5) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-elasticity, and the forth one is the normal component of the electric displacement vector (with opposite sign).

In Green's formulae there also appear the following boundary operator associated with the adjoint differential operator $A^*(x, \partial_x)$:

$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(x, \partial_x) = \big[\widetilde{\mathcal{T}}_{jk}(x, \partial_x)\big]_{4\times4} :=$$
$$:= \begin{bmatrix} \big[c_{ijlk}(x) n_i \partial_l\big]_{3\times3} & \big[-e_{lij}(x) n_i \partial_l\big]_{3\times1} \\ \big[e_{ikl}(x) n_i \partial_l\big]_{1\times3} & \varepsilon_{il}(x) n_i \partial_l \end{bmatrix}_{4\times4}.$$

Further, let $\Omega = \Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega = S \in C^\infty$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper $n = (n_1, n_2, n_3)$ denotes the unit normal vector to S directed outward with respect to the domain Ω. Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$.

By $H^r(\Omega) = H^r_2(\Omega)$ and $H^r(S) = H^r_2(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for C^∞ functions in $\mathbb{R}^3$ with compact support and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω.

For a vector $U = (u_1, u_2, u_3, u_4)^\top$ the inclusion $U = (u_1, u_2, u_3, u_4)^\top \in H^r$ means that all components u_j, $j = \overline{1, 4}$, belong to H^r.

Let us denote by $U^+ \equiv \{U\}^+$ and $U^- \equiv \{U\}^-$ the traces of U on S from the interior and exterior of Ω, respectively.

We also need the following subspace of $H^1(\Omega)$:

$$H^{1,0}(\Omega; A) := \Big\{U = (u_1, u_2, u_3, u_4)^\top \in H^1(\Omega) : \ A(x, \partial)U \in H^0(\Omega)\Big\}.$$

Assume that the domain Ω is filled with an anisotropic inhomogeneous piezoelectric material.

The Dirichlet boundary-value problem reads as follows:

Find a vector-function $U = (u, \varphi)^\top = (u_1, u_2, u_3, u_4)^\top \in H^{1,0}(\Omega, A)$ satisfying the differential equation

$$A(x, \partial_x)U = f \ \text{ in } \ \Omega \tag{2.6}$$

and the Dirichlet boundary condition

$$U^+ = \Phi_{\scriptscriptstyle 0} \ \text{ on } \ S, \tag{2.7}$$

where $\Phi_0 = (\Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{04})^\top \in H^{1/2}(S)$ and $f = (f_1, f_2, f_3, f_4)^\top \in L_2(\Omega)$ are given vector-functions.

The equation (2.6) is understood in the distributional sense, while the Dirichlet-type boundary condition (2.7) is understood in the usual trace sense.

For arbitrary complex-valued vector-functions $U = (u_1, u_2, u_3, u_4)^\top \in H^2(\Omega)$ and $V = (v_1, v_2, v_3, v_4)^\top \in H^2(\Omega)$, we have the following Green's formulae [8]:

$$\int\limits_\Omega \Big[A(x, \partial_x)U \cdot V + E(U, V)\Big]\, dx = \int\limits_S \{\mathcal{T}U\}^+ \cdot \{V\}^+ \, dS, \tag{2.8}$$

$$\int\limits_\Omega \Big[A(x, \partial_x)U \cdot V - U \cdot A^*(x, \partial_x)V\Big]\, dx =$$

$$= \int\limits_S \Big[\{\mathcal{T}U\}^+ \cdot \{V\}^+ - \{U\}^+ \cdot \{\widetilde{\mathcal{T}}V\}^+\Big]\, dS, \tag{2.9}$$

where

$$E(U, V) = c_{ijlk}\partial_i u_j \overline{\partial_l v_k} + e_{lij}\big(\partial_i u_j \overline{\partial_l v_4} - \partial_l u_4 \overline{\partial_i v_j}\,\big) + \varepsilon_{jl}\partial_j u_4 \overline{\partial_l v_4} \tag{2.10}$$

with $u = (u_1, u_2, u_3)^\top$ and $v = (v_1, v_2, v_3)^\top$, and the overbar denotes complex conjugation.

Note that the above Green's formulae can be generalized, by a standard limiting procedure, to Lipschitz domains and to vector–functions $U \in H^1(\Omega)$ and $V \in H^1(\Omega)$ with $A(x, \partial_x)U \in L_2(\Omega)$ and $A^*(x, \partial_x)V \in L_2(\Omega)$.

With the help of Green's formula (2.8) we can determine a *generalized trace vector* $\mathcal{T}^+U \equiv \{\mathcal{T}U\}^+ \in H^{-1/2}(\partial\Omega)$ for a vector-function $U \in H^{1,0}(\Omega; A)$ (cf. [39])

$$\big\langle \mathcal{T}^+U, V^+\big\rangle_{\partial\Omega} := \int\limits_\Omega A(\partial, \tau)U \cdot V\, dx + \int\limits_\Omega E(U, V)\, dx, \tag{2.11}$$

where $V \in H^1(\Omega)$ is an arbitrary vector-function.

Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the function spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ which extends the usual L_2-scalar product

$$\langle f,g\rangle_S = \int\limits_S \sum_{j=1}^N f_j\overline{g_j}\,dS \;\text{ for }\; f,g\in [L_2(S)]^N.$$

Remark 2.1. From the conditions (2.1) and (2.2) it follows that for complex-valued vector-functions the sesquilinear form $E(U,V)$ defined by (2.10) satisfies the inequality

$$\operatorname{Re} E(U,U) \geq c(s_{ij}\overline{s_{ij}} + \eta_j\overline{\eta_j}) \;\; \forall\, U = (u_1,u_2,u_3,u_4)^\top \in H^1(\Omega)$$

with $s_{ij} = 2^{-1}\big(\partial_i u_j(x) + \partial_j u_i(x)\big)$, $\eta_j = \partial_j u_4(x)$, where c is a positive constant. Therefore Green's first formula (2.8) and the Lax–Milgram lemma imply that the above formulated Dirichlet BVP is uniquely solvable in the space $H^{1,0}(\Omega;A)$ (see, e.g., [25], [35], [36]).

As it has already been mentioned, our goal here is to develop a generalized potential method and justify the LBDIE approach for the Dirichlet boundary value problem.

Define a *localized matrix parametrix* corresponding to the fundamental solution function $F_1(x) := -[4\pi|x|\,]^{-1}$ of the Laplace operator, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$,

$$P(x) \equiv P_\chi(x) := F_\chi(x)I = \\ = \chi(x)F_1(x)I = -\frac{\chi(x)}{4\pi|x|}\,I \;\text{ with }\; \chi(0)=1, \quad (2.12)$$

where $F_\chi(x) := \chi(x)F_1(x)$, I is the unit 4×4 matrix, while χ is a localizing function (see Appendix A)

$$\chi \in X_+^k, \quad k \geq 3. \quad (2.13)$$

Throughout the paper we assume that the condition (2.13) is satisfied and χ has a compact support if not otherwise stated.

Denote by $B(y,\varepsilon)$ a ball centered at the point y and radius $\varepsilon > 0$ and let $\Sigma(y,\varepsilon) := \partial B(y,\varepsilon)$.

In Green's second formula (2.9), let us take in the role of $V(x)$ successively the columns of the matrix $P(x-y)$, where y is an arbitrarily fixed interior point in Ω, and write the identity (2.9) for the region $\Omega_\varepsilon := \Omega\setminus\overline{B(y,\varepsilon)}$ with $\varepsilon > 0$ such that $\overline{B(y,\varepsilon)} \subset \Omega$. Keeping in mind that $P^\top(x-y) = P(x-y)$, we arrive at the equality

$$\int\limits_{\Omega_\varepsilon} \Big[P(x-y)A(x,\partial_x)U(x) - \big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\Big]\,dx = \\ = \int\limits_S \Big[P(x-y)\{\mathcal{T}(x,\partial_x)U(x)\}^+ - \big\{\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big\}^\top\{U(x)\}^+\Big]\,dS -$$

$$-\int\limits_{\Sigma(y,\varepsilon)} \Big[P(x-y)\mathcal{T}(x,\partial_x)U(x)-\big\{\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big\}^\top U(x)\Big]\,d\Sigma(y,\varepsilon). \quad (2.14)$$

The direction of the normal vector on $\Sigma(y,\varepsilon)$ is chosen as outward.

It is clear that the operator

$$\mathcal{A}U(y) := \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\,dx =$$

$$= \text{v.p.}\int\limits_{\Omega}\big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\,dx \quad (2.15)$$

is a singular integral operator, "v.p." means the Cauchy principal value integral. If the domain of integration in (2.15) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{A}U \equiv \mathbf{A}U$, i.e.,

$$\mathbf{A}U(y) := \text{v.p.}\int\limits_{\mathbb{R}^3}\big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\,dx.$$

Note that

$$\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|} = -\frac{4\pi\delta_{il}}{3}\,\delta(x-y) + \text{v.p.}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\,, \quad (2.16)$$

where δ_{il} is the Kronecker delta, while $\delta(\,\cdot\,)$ is the Dirac distribution. The left-hand side in (2.16) is understood in the distributional sense. In view of (2.12) and (2.16), and taking into account that $\chi(0)=1$ we can write the following equality in the distributional sense

$$\big[A^*(x,\partial_x)P(x-y)\big]^\top =$$

$$= \begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\Big(c_{ijlk}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\Big]_{3\times3} & \Big[\dfrac{\partial}{\partial x_i}\Big(e_{ikl}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\Big]_{3\times1} \\ \Big[-\dfrac{\partial}{\partial x_i}\Big(e_{lij}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\Big]_{1\times3} & \dfrac{\partial}{\partial x_i}\Big(\varepsilon_{il}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big) \end{bmatrix}_{4\times4} =$$

$$= \begin{bmatrix} \Big[c_{ijlk}(x)\,\dfrac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l}\Big]_{3\times3} & \Big[e_{ikl}(x)\,\dfrac{\partial^2 F_\chi(x-y}{\partial x_i\partial x_l}\Big]_{1\times3} \\ \Big[-e_{lij}(x)\,\dfrac{\partial^2 F_\chi(x-y)}{\partial x_l\partial x_i}\Big]_{3\times1} & \varepsilon_{il}(x)\,\dfrac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l} \end{bmatrix}_{4\times4} +$$

$$+ \begin{bmatrix} \Big[\dfrac{\partial c_{ijlk}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times3} & \Big[\dfrac{\partial e_{ikl}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times3} \\ \Big[-\dfrac{\partial e_{lij}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times1} & \dfrac{\partial \varepsilon_{il}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l} \end{bmatrix}_{4\times4} =$$

$$= \begin{bmatrix} \big[c_{ijlk}(x)k_{il}(x,y)\big]_{3\times3} & \big[e_{ikl}(x)k_{il}(x,y)\big]_{1\times3} \\ \big[-e_{lij}(x)k_{il}(x,y)\big]_{3\times1} & \varepsilon_{il}(x)k_{il}(x,y) \end{bmatrix}_{4\times4} +$$

$$+\begin{bmatrix} \Big[\dfrac{\partial c_{ijlk}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\dfrac{\partial e_{ikl}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3} \\ \Big[-\dfrac{\partial e_{lij}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1} & \dfrac{\partial \varepsilon_{il}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l} \end{bmatrix}_{4\times 4},$$

where

$$k_{il}(x,y):=\frac{\delta_{il}}{3}\,\delta(x-y)+\text{v.p.}\,\frac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l}=$$

$$=\frac{\delta_{il}}{3}\,\delta(x-y)-\frac{1}{4\pi}\,\text{v.p.}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}+m_{il}(x,y),$$

$$m_{il}(x,y):=-\frac{1}{4\pi}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{\chi(x-y)-1}{|x-y|}\,.$$

Therefore,

$$\big[A^*(x,\partial_x)P(x-y)\big]^\top=$$

$$=\mathbf{b}(x)\delta(x-y)+\text{v.p.}\big[A^*(x,\partial)P(x-y)\big]^\top=$$

$$=\mathbf{b}(x)\delta(x-y)+R(x,y)-\frac{1}{4\pi}\times$$

$$\times\text{v.p.}\begin{bmatrix} \Big[c_{ijlk}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[e_{ikl}(x)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1} \\ \Big[-e_{lij}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & \varepsilon_{il}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|} \end{bmatrix}_{4\times 4}=$$

$$=\mathbf{b}(x)\delta(x-y)+R^{(1)}(x,y)-\frac{1}{4\pi}\times$$

$$\times\text{v.p.}\begin{bmatrix} \Big[c_{ijlk}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[e_{ikl}(y)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1} \\ \Big[-e_{lij}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & \varepsilon_{il}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|} \end{bmatrix}_{4\times 4}, \tag{2.17}$$

where

$$\mathbf{b}(x):=\frac{1}{3}\begin{bmatrix} [c_{ljlk}(x)]_{3\times 3} & [e_{lkl}(x)]_{3\times 1} \\ [-e_{llj}(x)]_{1\times 3} & \varepsilon_{ll}(x) \end{bmatrix}_{4\times 4}, \tag{2.18}$$

$$R(x,y)=\begin{bmatrix} \big[c_{ijlk}(x)m_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)m_{il}(x,y)\big]_{1\times 3} \\ \big[-e_{lij}(x)m_{il}(x,y)\big]_{3\times 1} & \varepsilon_{il}(x)m_{il}(x,y) \end{bmatrix}_{4\times 4}+$$

$$+\begin{bmatrix} \Big[\dfrac{\partial c_{ijlk}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\dfrac{\partial e_{ikl}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3} \\ \Big[-\dfrac{\partial e_{lij}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1} & \dfrac{\partial \varepsilon_{il}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l} \end{bmatrix}_{4\times 4},$$

$$R^{(1)}(x,y)=R(x,y)-$$
$$-\frac{1}{4\pi}\begin{bmatrix}\Big[c_{ijlk}(x,y)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\Big]_{3\times3} & \Big[-e_{lij}(x,y)\,\frac{\partial^2}{\partial x_l\partial x_i}\,\frac{1}{|x-y|}\Big]_{3\times1}\\ \Big[e_{ikl}(x,y)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\Big]_{1\times3} & \varepsilon_{il}(x,y)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\end{bmatrix}_{4\times4},$$
$$c_{ijlk}(x,y):=c_{ijlk}(x)-c_{ijlk}(y),$$
$$e_{lij}(x,y):=e_{lij}(x)-e_{lij}(y),$$
$$\varepsilon_{il}(x,y):=\varepsilon_{il}(x)-\varepsilon_{il}(y).$$

Clearly, the entries of the matrix-functions $R(x,y)$ and $R^{(1)}(x,y)$ possess weak singularities of type $\mathcal{O}(|x-y|^{-2})$ as $x\to y$. Therefore we get

$$\text{v.p.}A^\top(x,\partial_x)P(x-y)=R(x,y)+$$
$$+\text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix}\Big[-c_{ijlk}(x)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\Big]_{3\times3} & \Big[e_{lij}(x)\,\frac{\partial^2}{\partial x_l\partial x_i}\,\frac{1}{|x-y|}\Big]_{3\times1}\\ \Big[-e_{ikl}(x)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\Big]_{1\times3} & -\varepsilon_{il}(x)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\end{bmatrix}_{4\times4},$$
$$\text{v.p.}A^\top(x,\partial_x)P(x-y)=R^{(1)}(x,y)+\tag{2.19}$$
$$+\text{v.p.}\frac{1}{4\pi}\begin{bmatrix}\Big[-c_{ijlk}(y)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\Big]_{3\times3} & \Big[e_{lij}(y)\,\frac{\partial^2}{\partial x_l\partial x_i}\,\frac{1}{|x-y|}\Big]_{3\times1}\\ \Big[-e_{ikl}(y)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\Big]_{1\times3} & -\varepsilon_{il}(y)\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\end{bmatrix}_{4\times4}.$$

Further, by direct calculations one can easily verify that

$$\lim_{\varepsilon\to0}\int\limits_{\Sigma(y,\varepsilon)}P(x-y)\mathcal{T}(x,\partial_x)U(x)\,d\Sigma(y,\varepsilon)=0,\tag{2.20}$$
$$\lim_{\varepsilon\to0}\int\limits_{\Sigma(y,\varepsilon)}\big\{\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big\}^\top U(x)\,d\Sigma(y,\varepsilon)=$$
$$=\frac{1}{4\pi}\int\limits_{\Sigma_1}\begin{bmatrix}\big[c_{ijlk}(y)\eta_i\eta_l\big]_{3\times3} & \big[e_{ikl}(y)\eta_l\eta_i\big]_{3\times1}\\ \big[-e_{lij}(y)\eta_i\eta_l\big]_{1\times3} & \varepsilon_{il}(y)\eta_i\eta_l\end{bmatrix}_{4\times4}d\Sigma_1\,U(y)=$$
$$=\frac{1}{4\pi}\begin{bmatrix}\Big[c_{ijlk}(y)\,\frac{4\pi\delta_{il}}{3}\Big]_{3\times3} & \Big[e_{ikl}(y)\,\frac{4\pi\delta_{li}}{3}\Big]_{3\times1}\\ \Big[-e_{lij}(y)\,\frac{4\pi\delta_{il}}{3}\Big]_{1\times3} & \varepsilon_{il}(y)\,\frac{4\pi\delta_{il}}{3}\end{bmatrix}_{4\times4}U(y)=$$
$$=\mathbf{b}(y)U(y),\tag{2.21}$$

where Σ_1 is a unit sphere, $\eta=(\eta_1,\eta_2,\eta_3)\in\Sigma_1$, and $\mathbf{b}$ is defined by (2.18).

Passing to the limit in (2.14) as $\varepsilon \to 0$ and using the relations (2.15), (2.20), and (2.21) we obtain

$$\mathbf{b}(y)U(y) + \mathcal{A}U(y) - V(\mathcal{T}^{+}U)(y) + W(U^{+})(y) = \\ = \mathcal{P}\big(A(x,\partial_x)U\big)(y), \quad y \in \Omega, \tag{2.22}$$

where $\mathcal{A}$ is the *localized singular integral operator* given by (2.15), while V, W, and $\mathcal{P}$ are the *localized single layer, double layer, and Newtonian volume vector-potentials*:

$$V(g)(y) := -\int\limits_{S} P(x-y)g(x)\,dS_x, \tag{2.23}$$

$$W(g)(y) := -\int\limits_{S} \big[\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big]^{\top} g(x)\,dS_x,$$

$$\mathcal{P}(h)(y) := \int\limits_{\Omega} P(x-y)h(x)\,dx. \tag{2.24}$$

Here the densities g and h are four dimensional vector-functions.

Let us also introduce the scalar volume potential

$$\mathbb{P}(\mu)(y) := \int\limits_{\Omega} F_{\chi}(x-y)\mu(x)\,dx \tag{2.25}$$

with μ beeing a scalar density function.

If the domain of integration in the Newtonian volume potential (2.24) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{P}h \equiv \mathbf{P}h$, i.e.,

$$\mathbf{P}(h)(y) := \int\limits_{\mathbb{R}^3} P(x-y)h(x)\,dx.$$

Mapping properties of the above potentials are investigated in [14].

We refer to the relation (2.22) as *Green's third formula.* It is evident that by a standard limiting procedure we can extend Green's third formula to functions from the space $H^{1,0}(\Omega, A)$. In particular, it holds true for solutions of the above formulated Dirichlet BVP. In this case, the generalized trace vector $\mathcal{T}^{+}U$ is understood in the sense of the definition (2.11).

For $U = (u_1, \dots, u_4) \in H^1(\Omega)$ one can easily derive the following relation

$$\mathcal{A}U(y) = -\mathbf{b}(y)U(y) - W(U^{+})(y) + \mathcal{Q}U(y), \quad \forall\, y \in \Omega, \tag{2.26}$$

where

$$\mathcal{Q}U(y) := \frac{\partial}{\partial y_l} \begin{bmatrix} \big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y) + \mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times 1} \\ -\mathbb{P}(e_{lij}\partial_i u_j)(y) + \mathbb{P}(\varepsilon_{il}\partial_i u_4)(y) \end{bmatrix}_{4\times 1} \tag{2.27}$$

and $\mathbb{P}$ is defined in (2.25).

In what follows, in our analysis we need explicit expression of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A})(y,\xi)$ of the singular integral operator $\mathcal{A}$. This matrix coincides with the Fourier transform of the singular matrix kernel defined by (2.19). Let $\mathcal{F}$ denote the Fourier transform operator,

$$\mathcal{F}_{z\to\xi}[g]=\int\limits_{\mathbb{R}^3} g(z)e^{iz\cdot\xi}\,dz,$$

and set

$$h_{il}(z):=\text{v.p.}\,\frac{\partial^2}{\partial z_i\partial z_l}\,\frac{1}{|z|}\,,$$
$$\widehat{h}_{il}(\xi):=\mathcal{F}_{z\to\xi}(h_{il}(z)),\quad i,l=1,2,3.$$

In view of (2.16) and taking into account the relations $\mathcal{F}_{z\to\xi}\delta(z)=1$ and $\mathcal{F}_{z\to\xi}(|z|^{-1})=4\pi|\xi|^{-2}$ (see, e.g., [23]), we easily derive

$$\widehat{h}_{il}(\xi)=\mathcal{F}_{z\to\xi}(h_{il}(z))=\mathcal{F}_{z\to\xi}\Big(\frac{4\pi\delta_{li}}{3}\,\delta(z)+\frac{\partial^2}{\partial z_i\partial z_l}\,\frac{1}{|z|}\Big)=$$
$$=\frac{4\pi\delta_{li}}{3}+(-i\xi_i)(-i\xi_l)\mathcal{F}_{z\to\xi}\Big(\frac{1}{|z|}\Big)=\frac{4\pi\delta_{il}}{3}-\frac{4\pi\xi_i\xi_l}{|\xi|^2}\,.$$

Now, for arbitrary $y\in\overline{\Omega}$ and $\xi\in\mathbb{R}^3\setminus\{0\}$, due to (2.19) we get

$$\begin{aligned}\mathfrak{S}(\mathcal{A})(y,\xi)&=-\frac{1}{4\pi}\,\mathcal{F}_{z\to\xi}\begin{bmatrix}\big[c_{ijlk}(y)h_{il}(z)\big]_{3\times3} & \big[e_{ikl}(y)h_{il}(z)\big]_{3\times1}\\ \big[-e_{lij}(y)h_{il}(z)\big]_{1\times3} & \varepsilon_{il}(y)h_{il}(z)\end{bmatrix}_{4\times4}=\\
&=-\frac{1}{4\pi}\begin{bmatrix}\big[c_{ijlk}(y)\widehat{h}_{il}(\xi)\big]_{3\times3} & \big[e_{ikl}(y)\widehat{h}_{il}(\xi)\big]_{3\times1}\\ \big[-e_{lij}(y)\widehat{h}_{il}(\xi)\big]_{1\times3} & -\varepsilon_{il}(y)\widehat{h}_{il}(\xi)\end{bmatrix}_{4\times4}=\\
&=-\mathbf{b}(y)+\frac{1}{|\xi|^2}\begin{bmatrix}\big[c_{ijlk}(y)\xi_i\xi_l\big]_{3\times3} & \big[e_{ikl}(y)\xi_l\xi_i\big]_{3\times1}\\ \big[-e_{lij}(y)\xi_i\xi_l\big]_{1\times3} & \varepsilon_{il}(y)\xi_i\xi_l\end{bmatrix}_{4\times4}=\\
&=\frac{1}{|\xi|^2}\,A(y,\xi)-\mathbf{b}(y),\end{aligned}\tag{2.28}$$

where $A(y,\xi)$ is the matrix defined in (2.4), while $\mathbf{b}(y)$ is given by (2.18).

As we see the entries of the symbol matrix $\mathfrak{S}(\mathcal{A})(y,\xi)$ of the operator $\mathcal{A}$ are even rational homogeneous functions in ξ of order 0. It can be easily verified that both the characteristic function of the singular kernel in (2.17) and the Fourier transform (2.28) satisfy the Tricomi condition, i.e., their integral averages over the unit sphere vanish (cf. [40]).

Denote by ℓ_0 the extension operator by zero from Ω onto Ω^-. It is evident that for a function $U\in H^1(\Omega)$ we have

$$\big(\mathcal{A}U\big)(y)=\big(\mathbf{A}\ell_0U\big)(y)\ \text{ for }\ y\in\Omega.$$

Now we rewrite Green's third formula (2.22) in a more convenient form for our further purposes

$$[\mathbf{b}+\mathbf{A}]\ell_0U(y)-V(\mathcal{T}^+u)(y)+W(U^+)(y)=\mathcal{P}\big(A(x,\partial_x)U\big)(y),\quad y\in\Omega.\tag{2.29}$$

The relation (2.28) implies that the principal homogeneous symbols of the singular integral operators $\mathbf{A}$ and $\mathbf{b}+\mathbf{A}$ read as

$$\mathfrak{S}(\mathbf{A})(y,\xi)=|\xi|^{-2}A(y,\xi)-\mathbf{b}(y)\ \ \forall\,y\in\overline{\Omega},\ \ \forall\,\xi\in\mathbb{R}^3\setminus\{0\},\tag{2.30}$$

$$\mathfrak{S}(\mathbf{b}+\mathbf{A})(y,\xi)=|\xi|^{-2}A(y,\xi)\ \ \forall\,y\in\overline{\Omega},\ \ \forall\,\xi\in\mathbb{R}^3\setminus\{0\}.\tag{2.31}$$

It is evident that the symbol matrix (2.31) is strongly elliptic due to (2.3),

$$\operatorname{Re}\mathfrak{S}(\mathbf{b}+\mathbf{A})(y,\xi)\zeta\cdot\zeta=|\xi|^{-2}\operatorname{Re}A(y,\xi)\zeta\cdot\zeta\geq c|\zeta|^2$$
$$\forall\,y\in\overline{\Omega},\ \ \forall\,\xi\in\mathbb{R}^3\setminus\{0\},\ \ \forall\,\zeta\in\mathbb{C}^4,$$

where c is the same positive constant as in (2.3).

From the decomposition (2.17) and the equality (2.28) it follows that (see, e.g., [2], [25, Theorem 8.6.1])

$$r_{\Omega}\mathbf{A}\ell_0:H^1(\Omega)\to H^1(\Omega),$$

since the symbol (2.30) is rational and the operators with the kernel functions either $R(x,y)$ or $R_1(x,y)$ maps $H^1(\Omega)$ into $H^2(\Omega)$ for $\chi\in X^2$ (cf. [14, Theorem 5.6]). Here and throughout the paper r_{Ω} denotes the restriction operator to Ω.

Using the properties of localized potentials described in the Appendix B (see Theorems B.1 and B.4) and taking the trace of the equation (2.29) on S we arrive at the relation:

$$\mathbf{A}^+\ell_0U-\mathcal{V}(\mathcal{T}^+U)+(\mathbf{b}-\mathbf{d})U^++\mathcal{W}(U^+)=\mathcal{P}^+(A(x,\partial_x)U)\ \text{ on }\ S,\tag{2.32}$$

where the localized boundary integral operators $\mathcal{V}$ and $\mathcal{W}$ are generated by the localized single and double layer potentials and are defined in (B.1) and (B.2), the matrix $\mathbf{d}$ is defined by (B.3), while

$$\mathbf{A}^+\ell_0U\equiv\gamma^+\mathbf{A}\ell_0U:=\{\mathbf{A}\ell_0U\}^+\ \text{ on }\ S,$$
$$\mathcal{P}^+(f)\equiv\gamma^+\mathcal{P}(f):=\{\mathcal{P}(f)\}^+\ \text{ on }\ S.$$

Now we prove the following technical lemma.

Lemma 2.2. *Let $\chi\in X^3$ and*

$$f=(f_1,f_2,f_3,f_4)^\top\in H^0(\Omega),\quad F=(F_1,F_2,F_3,F_4)^\top\in H^{1,0}(\Omega,\Delta),$$
$$\Psi=(\psi_1,\psi_2,\psi_3,\psi_4)^\top\in H^{-\frac12}(S),\quad \Phi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)^\top\in H^{\frac12}(S).$$

Moreover, let $U=(u_1,u_2,u_3,u_4)^\top\in H^1(\Omega)$ and the following equation hold

$$\mathbf{b}(y)U(y)+\mathcal{A}U(y)-V(\Psi)(y)+W(\Phi)(y)=F(y)+\mathcal{P}(f)(y),\ \ y\in\Omega.\tag{2.33}$$

Then $U\in H^{1,0}(\Omega,A)$.

Proof. Note that by Theorem B.1 $\mathcal{P}\big(f\big)\in H^2(\Omega)$ for arbitrary $f\in H^0(\Omega)$, while by Theorem B.2 the inclusions $V(\Psi),W(\Phi)\in H^{1,0}(\Omega,\Delta)$ hold for arbitrary $\Psi \in H^{-\frac{1}{2}}(S)$ and $\Phi \in H^{\frac{1}{2}}(S)$. Using the relations (2.26)–(2.27), the equation (2.33) can be rewritten as

$$\frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y)+\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times1}\\ -\mathbb{P}(e_{lij}\partial_i u_j)(y)+\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times1}= \\ = F(y)+\mathcal{P}(f)(y)+V(\Psi)(y)-W(\Phi-U^+)(y),\ \ y\in\Omega.$$

Due to Theorems B.1 and B.2 it follows that the right-hand side function in the above equality belongs to the space

$$H^{1,0}(\Omega,\Delta):=\Big\{v\in H^1(\Omega):\ \Delta v\in H^0(\Omega)\Big\},$$

since $U^+\in H^{\frac{1}{2}}(S)$, and therefore the same holds true for the left-hand side function,

$$\frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y)+\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times1}\\ -\mathbb{P}(e_{lij}\partial_i u_j)(y)+\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times1}\in H^{1,0}(\Omega,\Delta). \tag{2.34}$$

Note that

$$\Delta(\partial_x)P(x-y)=[\delta(x-y)+R_\Delta(x-y)]I, \tag{2.35}$$

where

$$R_\Delta(x-y):=-\frac{1}{4\pi}\left\{\frac{\Delta\chi(x-y)}{|x-y|}+2\,\frac{\partial\chi(x-y)}{\partial x_l}\,\frac{\partial}{\partial x_l}\,\frac{1}{|x-y|}\right\}. \tag{2.36}$$

Clearly, $R_\Delta(x-y)=\mathcal{O}(|x-y|^{-2})$ as $x\to y$ and with the help of (2.35) and (2.36) one can prove that for arbitrary scalar function $\phi\in\mathcal{D}(\Omega)$ there holds the relation (see, e.g., [40])

$$\Delta(\partial_y)\mathbb{P}(\phi)(y)=\phi(y)+\mathcal{R}_\Delta(\phi)(y),\ \ y\in\Omega, \tag{2.37}$$

where

$$\mathcal{R}_\Delta(\phi)(y):=\int\limits_\Omega R_\Delta(x-y)\phi(x)\,dx. \tag{2.38}$$

Evidently (2.38) remains true for $\phi\in H^0(\Omega)$, since $\mathcal{D}(\Omega)$ is dense in $H^0(\Omega)$. The operator $\mathcal{R}_\Delta$ has the following mapping property (see [14]):

$$\mathcal{R}_\Delta:H^0(\Omega)\to H^1(\Omega). \tag{2.39}$$

Applying the Laplace operator Δ to the vector-function (2.34) and keeping in mind the relation (2.37), we arrive at the following equation in Ω,

$$\Delta(\partial_y)\,\frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y)+\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times1}\\ -\mathbb{P}(e_{lij}\partial_i u_j)(y)+\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times1}= \\ =\begin{bmatrix}\Big[\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(c_{ijlk}\partial_i u_k)(y)\big)+\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big)\Big]_{3\times1}\\ -\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(e_{lij}\partial_i u_j)(y)\big)+\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\big)\end{bmatrix}=$$

$$= \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_l}\Big(c_{ijlk}(y)\,\dfrac{\partial u_k(y)}{\partial y_i}\Big) + \dfrac{\partial}{\partial y_l}\Big(e_{ikl}(y)\,\dfrac{\partial u_4(y)}{\partial y_i}\Big)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_l}\Big(\Big(e_{lij}(y)\,\dfrac{\partial u_j(y)}{\partial y_i}\Big)\Big) + \dfrac{\partial}{\partial y_l}\Big(\varepsilon_{il}(y)\,\dfrac{\partial u_4(y)}{\partial y_i}\Big) \end{bmatrix} +$$

$$+ \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(c_{ijlk}\partial_i u_k)(y) + \dfrac{\partial}{\partial y_l}\mathcal{R}_\Delta(e_{ikl}\partial_i u_4)(y)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(e_{lij}\partial_i u_j)(y) + \dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(\varepsilon_{il}\partial_i u_4)(y) \end{bmatrix} =$$

$$= A(y,\partial_y)U + \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(c_{ijlk}\partial_i u_k)(y) + \dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(e_{ikl}\partial_i u_4)(y)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(e_{lij}\partial_i u_j)(y) + \dfrac{\partial}{\partial y_l}\mathcal{R}_\Delta(\varepsilon_{il}\partial_i u_4)(y) \end{bmatrix}.$$

Whence the embedding $A(y,\partial_y)U \in H^0(\Omega)$ follows due to (2.34) and (2.39). □

Actually, in the proof of Lemma 2.2 we have shown the following assertion.

Corollary 2.3. *Let* $\chi \in X^3$. *The operator*

$$\mathbf{b} + \mathbf{A} : H^{1,0}(\Omega, A) \to H^{1,0}(\Omega, \Delta)$$

is bounded.

Now, we are in the position to reduce the above formulated Dirichlet boundary value problem to the LBDIEs system equivalently.

2.2. **LBDIE formulation of the Dirichlet problem and the equivalence theorem.** Let $U \in H^{1,0}(\Omega, A)$ be a solution to the Dirichlet BVP (2.6), (2.7) with $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$. As we have derived above, there hold the relations (2.29) and (2.32), which now can be rewritten in the form

$$[\mathbf{b} + \mathbf{A}]\ell_0 U - V(\Psi) = \mathcal{P}(f) - W(\Phi_0) \ \text{ in } \ \Omega, \tag{2.40}$$

$$\mathbf{A}^+ \ell_0 U - \mathcal{V}(\Psi) = \mathcal{P}^+(f) - (\mathbf{b} - \mathbf{d})\Phi_0 - \mathcal{W}(\Phi_0) \ \text{ on } \ S, \tag{2.41}$$

where $\Psi := \mathcal{T}^+ U \in H^{-\frac{1}{2}}(S)$ and $\mathbf{d}$ is defined by (B.3).

One can consider these relations as the LBDIE system with respect to the unknown vector-functions U and Ψ. Now we prove the following equivalence theorem.

Theorem 2.4. *Let* $\chi \in X^3_+$, $\Phi_0 \in H^{\frac{1}{2}}(S)$ *and* $f \in H^0(\Omega)$.

(i) *If a vector-function* $U \in H^{1,0}(\Omega, A)$ *solves the Dirichlet BVP* (2.6), (2.7)*, then it is unique and the pair* $(U, \Psi) \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S)$ *with*

$$\Psi = \mathcal{T}^+ U, \tag{2.42}$$

solves the LBDIE system (2.40), (2.41) *and vice versa.*

(ii) *If a pair* $(U,\Psi)\in H^{1,0}(\Omega,A)\times H^{-\frac12}(S)$ *solves the LBDIE system* (2.40), (2.41)*, then it is unique and the vector-function u solves the Dirichlet BVP* (2.6), (2.7)*, and relation* (2.42) *holds.*

Proof. (i) The first part of the theorem is trivial and directly follows form the relations (2.29), (2.32), (2.42), and Remark 2.1.

(ii) Now, let a pair $(U,\Psi)\in H^{1,0}(\Omega,A)\times H^{-\frac12}(S)$ solve the LBDIE system (2.40), (2.41). Taking the trace of (2.40) on S and comparing with (2.41) we get

$$U^+=\Phi_0 \ \text{ on } \ S. \tag{2.43}$$

Further, since $U\in H^{1,0}(\Omega,A)$, we can write Green's third formula (2.29) which in view of (2.43) can be rewritten as

$$[\mathbf{b}+\mathbf{A}]\ell_0U-V(\mathcal{T}^+U)=\mathcal{P}\big(A(x,\partial_x)U\big)-W(\Phi_0) \ \text{ in } \ \Omega. \tag{2.44}$$

From (2.40) and (2.44) it follows that

$$V(\mathcal{T}^+U-\Psi)+\mathcal{P}\big(A(x,\partial_x)U-f\big)=0 \ \text{ in } \ \Omega.$$

Whence by Lemma 6.3 in [14] we have

$$A(x,\partial_x)U=f \ \text{ in } \ \Omega \ \text{ and } \ \mathcal{T}^+U=\psi \ \text{ on } \ S.$$

Thus U solves the Dirichlet BVP (2.6), (2.7) and equation (2.42) holds.

The uniqueness of solution to the LBDIE system (2.40), (2.41) in the class $H^{1,0}(\Omega,A)\times H^{-\frac12}(S)$ directly follows from the above proved equivalence result and the uniqueness theorem for the Dirichlet problem (2.6), (2.7) (see Remark 2.1). □

3. Invertibility of the Dirichlet LBDIO

From Theorem 2.4 it follows that the LBDIE system (2.40), (2.41) with the special right-hand sides is uniquely solvable in the class $H^{1,0}(\Omega,A)\times H^{-1/2}(S)$. We investigate Fredholm properties of the localized boundary-domain integral operator generated by the left-hand side expressions in (2.40), (2.41) and show the invertibility of the operator in appropriate functional spaces.

The LBDIE system (2.40), (2.41) with an arbitrary right-hand side vector-functions from the space $H^1(\Omega)\times H^{1/2}(S)$ can be written as

$$(\mathbf{b}+\mathbf{A})\ell_0U-V\Psi=F_1 \ \text{ in } \ \Omega, \tag{3.1}$$

$$\mathbf{A}^+\ell_0U-\mathcal{V}\Psi=F_2 \ \text{ on } \ S, \tag{3.2}$$

where $F_1\in H^1(\Omega)$ and $F_2\in H^{1/2}(S)$. Denote

$$\mathbf{B}:=\mathbf{b}+\mathbf{A}. \tag{3.3}$$

Evidently, the principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as (see (2.31))

$$\mathfrak{S}(\mathbf{B})(y,\xi)=|\xi|^{-2}A(y,\xi) \ \text{ for } \ y\in\overline{\Omega}, \ \ \xi\in\mathbb{R}^3\setminus\{0\}. \tag{3.4}$$

It is an even rational homogeneous matrix-function of order 0 in ξ and due to (2.3) it is uniformly strongly elliptic,

$$\operatorname{Re}\mathfrak{S}(\mathbf{B})(y,\xi)\zeta\cdot\zeta \geq c|\zeta|^2 \ \text{ for all } \ y\in\overline{\Omega}, \ \ \xi\in\mathbb{R}^3\setminus\{0\}, \ \ \zeta\in\mathbb{C}^4.$$

Consequently, $\mathbf{B}$ is a strongly elliptic pseudodifferential operator of zero order (i.e., singular integral operator) and the partial indices of factorization of the symbol (3.4) equal to zero (cf. [10, Lemma 1.20]).

In our further analysis we need some auxiliary assertions. To formulate them, let $\widetilde{y}\in\partial\Omega$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B})(\widetilde{y},\xi)\equiv\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$, where $\widetilde{\mathbf{B}}$ denotes the operator $\mathbf{B}$ written in a chosen local coordinate system. Further, let $\widehat{\widetilde{\mathbf{B}}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi',\xi_3) := \mathfrak{S}(\widetilde{\mathbf{B}})\big((1+|\xi'|)\omega,\xi_3\big),$$
$$\omega=\frac{\xi'}{|\xi'|},\ \xi=(\xi',\xi_3),\ \xi'=(\xi_1,\xi_2).$$

The principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$ of the operator $\widehat{\widetilde{\mathbf{B}}}$ can be factorized with respect to the variable ξ_3,

$$\mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = \mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi)\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})(\xi), \tag{3.5}$$

where

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi) = \frac{1}{\Theta^{(\pm)}(\xi',\xi_3)}\,\widetilde{A}^{(\pm)}(\xi',\xi_3),$$

$\Theta^{(\pm)}(\xi',\xi_3):=\xi_3\pm i|\xi'|$ are the "plus" and "minus" factors of the symbol $\Theta(\xi):=|\xi|^2$, and $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in ξ_3 of the polynomial symbol matrix $\widetilde{A}(\xi',\xi_3)\equiv\widetilde{A}(\widetilde{y},\xi',\xi_3)$ (see [22, Theorem 1], [45, Theorem 1.33], [24, Theorem 1.4]), i.e.

$$\widetilde{A}(\xi',\xi_3) = \widetilde{A}^{(-)}(\xi',\xi_3)\widetilde{A}^{(+)}(\xi',\xi_3) \tag{3.6}$$

with $\det\widetilde{A}^{(+)}(\xi',\tau)\neq 0$ for $\operatorname{Im}\tau>0$ and $\det\widetilde{A}^{(-)}(\xi',\tau)\neq 0$ for $\operatorname{Im}\tau<0$. Moreover, the entries of the matrices $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are homogeneous functions in $\xi=(\xi',\xi_3)$ of order 1. Denote by $a^{(\pm)}(\xi')$ the coefficients at ξ_3^4 in the determinants $\det\widetilde{A}^{(\pm)}(\xi',\xi_3)$. Evidently,

$$a^{(-)}(\xi')a^{(+)}(\xi') = \det\widetilde{A}(0,0,1)>0 \quad \text{for} \quad \xi'\neq 0. \tag{3.7}$$

It is easy to see that the factor-matrices $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ have the structure

$$\big[\widetilde{A}^{(\pm)}(\xi',\xi_3)\big]^{-1} = \frac{1}{\det\widetilde{A}^{(\pm)}(\xi',\xi_3)}\,\big[p_{ij}^{(\pm)}(\xi',\xi_3)\big]_{4\times 4},$$

where $p_{ij}^{(\pm)}(\xi',\xi_3)$ is the co-factor corresponding to the element $\widetilde{A}_{ji}^{(\pm)}(\xi',\xi_3)$ of the matrix $\widetilde{A}^{(\pm)}(\xi',\xi_3)$, which can be written in the form

$$p_{ij}^{(\pm)}(\xi',\xi_3) = c_{ij}^{(\pm)}(\xi')\xi_3^3 + b_{ij}^{(\pm)}(\xi')\xi_3^2 + d_{ij}^{(\pm)}(\xi')\xi_3 + e_{ij}^{(\pm)}(\xi'). \tag{3.8}$$

Here $c_{ij}^{(\pm)}$, $b_{ij}^{(\pm)}$, $d_{ij}^{(\pm)}$, and $e_{ij}^{(\pm)}$, $i,j=1,2,3,4$, are homogeneous functions in ξ' of order 0, 1, 2, and 3, respectively. From the above mentioned it follows that the entries of the factor-symbol matrices

$$\mathfrak{B}^{(\pm)}(\omega,r,\xi_3)=\big[\mathfrak{b}_{kj}^{(\pm)}(\omega,r,\xi_3)\big]_{3\times 3}:=\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi',\xi_3)$$

with $\omega=\xi'/|\xi'|$ and $r=|\xi'|$ satisfy the following relations:

$$\frac{\partial^l\mathfrak{b}_{kj}^{(\pm)}(\omega,0,-1)}{\partial r^l}=(-1)^l\,\frac{\partial^l\mathfrak{b}_{kj}^{(\pm)}(\omega,0,+1)}{\partial r^l}\,,\quad l=0,1,2,\dots\,. \tag{3.9}$$

These relations imply that the entries of the matrices $\mathfrak{S}^{\pm}(\widetilde{\mathbf{B}})(\xi',\xi_3)$ belong to the class of symbols D_0 introduced in [23, Ch. III, § 10],

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi',\xi_3)\in D_0. \tag{3.10}$$

Denote by $\Pi^{\pm}$ the Cauchy type integral operators

$$\Pi^{\pm}(h)(\xi):=\pm\frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{h(\xi',\eta_3)}{\xi_3\pm it-\eta_3}\,d\eta_3, \tag{3.11}$$

which are well defined for a bounded smooth function $h(\xi',\cdot)$ satisfying the relation $h(\xi',\eta_3)=\mathcal{O}(1+|\eta_3|)^{-\kappa}$ with some $\kappa>0$.

First we prove the following auxiliary lemma.

Lemma 3.1. *Let $\chi\in X_+^k$ with integer $k\geqslant s+2$ and let ℓ_0 be the extension operator by zero from $\mathbb{R}^3_+$ onto the half-space $\mathbb{R}^3_-$. The operator*

$$r_{\mathbb{R}^3_+}\widehat{\widehat{\mathbf{B}}}\ell_0:H^s(\mathbb{R}^3_+)\to H^s(\mathbb{R}^3_+)$$

is invertible for all $s\ge 0$, where $r_{\mathbb{R}^3_+}$ is the restriction operator to the half-space $\mathbb{R}^3_+$. Moreover, for $f\in H^s(\mathbb{R}^3_+)$ with $s\ge 0$, the unique solution of the equation

$$r_{\mathbb{R}^3_+}\widehat{\widehat{\mathbf{B}}}\ell_0U=f \tag{3.12}$$

can be represented in the form

$$U_+:=\ell_0U=\mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\},$$

where $\ell f\in H^s(\mathbb{R}^3)$ is an arbitrary extension of f onto the whole space $\mathbb{R}^3$.

Proof. Since the right-hand side f of the equation (3.12) belongs to the space $H^s(\mathbb{R}^3_+)$ with $s\ge 0$, it follows that $f\in H^0(\mathbb{R}^3_+)$.

First we show that the equation (3.12) is uniquely solvable in the space $H^0(\mathbb{R}^3_+)$.

Let $U\in H^0(\mathbb{R}^3_+)$ be a solution of the equation (3.12) with $f\in H^0(\mathbb{R}^3_+)$ and let

$$U_-=\ell f-\widehat{\widehat{\mathbf{B}}}U_+, \tag{3.13}$$

where $U_+ = \ell_0 U \in \widetilde{H}^0(\mathbb{R}^3_+)$ and $\ell f \in H^0(\mathbb{R}^3)$ is an arbitrary extension of $f \in H^0(\mathbb{R}^3_+)$ onto $\mathbb{R}^3_+$. We assume that

$$\|\ell f\|_{H^0(\mathbb{R}^3)} \leq 2\|f\|_{H^0(\mathbb{R}^3_+)}.$$

Since $\ell f \in H^0(\mathbb{R}^3)$ and $\widehat{\widetilde{\mathbf{B}}} U_+ \in H^0(\mathbb{R}^3)$, we have $U_- \in H^0(\mathbb{R}^3)$. In addition, $U_- \in \widetilde{H}^0(\mathbb{R}^3_-)$. Here and in what follows we employ the notation

$$\widetilde{H}^s(\Omega) := \Big\{ V \in H^s(\Omega) : \ \operatorname{supp} V \subset \overline{\Omega} \Big\}.$$

The Fourier transform of (3.13) gives the relation

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi)\mathcal{F}(U_+) + \mathcal{F}(U_-)(\xi) = \mathcal{F}(\ell f)(\xi). \tag{3.14}$$

Due to (3.5) we have the factorization

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi', \xi_3) = \widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})(\xi', \xi_3)\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})(\xi', \xi_3), \tag{3.15}$$

where $\widehat{\mathfrak{S}}^{\,(\pm)}(\widetilde{\mathbf{B}})(\xi', \xi_3) = \mathfrak{S}^{\,(\pm)}(\widetilde{\mathbf{B}})((1+|\xi'|)\omega, \xi_3)$ with $\omega = \frac{\xi'}{|\xi'|}$. Substituting (3.15) into (3.14) and multiplying both sides by $[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})]^{-1}$, we get

$$\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})(\xi)\mathcal{F}(U_+)(\xi) + \big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(U_-)(\xi) = \\ = \big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(\ell f)(\xi). \tag{3.16}$$

Introduce the notation

$$v_+(x) = \mathcal{F}^{-1}_{\xi\to x}\Big(\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})(\xi)\mathcal{F}(U_+)(\xi)\Big), \tag{3.17}$$

$$v_-(x) = \mathcal{F}^{-1}_{\xi\to x}\Big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(U_-)(\xi)\Big),$$

$$g(x) = \mathcal{F}^{-1}_{\xi\to x}\Big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(\ell f)(\xi)\Big). \tag{3.18}$$

Then we can conclude that (see [23, Theorem 4.4 and Lemmas 20.2, 20.5])

$$v_+ \in \widetilde{H}^0(\mathbb{R}^3_+), \quad v_- \in \widetilde{H}^0(\mathbb{R}^3_-), \quad g \in H^0(\mathbb{R}^3), \tag{3.19}$$

since the degree of homogeneity of $\mathfrak{S}^{\,(+)}(\widetilde{\mathbf{B}})(\xi)$ and $\mathfrak{S}^{\,(-)}(\widetilde{\mathbf{B}})(\xi)$ equals to 0.

In view of the above notation, the equation (3.16) acquires the form

$$\mathcal{F}(v_+)(\xi) + \mathcal{F}(v_-)(\xi) = \mathcal{F}(g)(\xi). \tag{3.20}$$

In accordance with Lemma 5.4 in [23], we conclude that the representation of the vector-function $\mathcal{F}(g)(\xi)$ in the form (3.20) is unique in view of the inclusions (3.19) which in turn leads to the relations

$$\mathcal{F}(v_+) = \Pi^+\mathcal{F}(g), \quad \mathcal{F}(v_-) = \Pi^-\mathcal{F}(g). \tag{3.21}$$

Now, from (3.17), (3.18), and the first equation in (3.21) it follows that $U_+ \in \widetilde{H}^0(\mathbb{R}^3_+)$ is representable in the form

$$U_+ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\}. \tag{3.22}$$

Evidently, for the solution $U \in H^0(\mathbb{R}^3_+)$ of the equation (3.12) we get the following representation

$$U = r_{\mathbb{R}^3_+} \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\}. \tag{3.23}$$

Note that the representation (3.23) does not depend on the choice of the extension operator ℓ. Indeed, let $\ell_1 f \in H^0(\mathbb{R}^3)$ be another extension of $f \in H^0(\mathbb{R}^3_+)$, i.e., $r_{\mathbb{R}^3_+} \ell_1 f = f$. Since $f_- = \ell f - \ell_1 f \in \widetilde{H}^0(\mathbb{R}^3_-)$, it follows that (see [23, Theorem 4.4, Lemmas 20.2 and 20.5])

$$\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(f_-)\big) \in \widetilde{H}^0(\mathbb{R}^3_-),$$

while

$$\Pi^+\Big\{\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(f_-)\Big\} = \mathcal{F}\Big\{\theta^+ \mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(f_-)\big)\Big\} = 0$$

(see [23, Lemma 5.2]), where θ^+ denotes the multiplication operator by the Heaviside step function $\theta(x_3)$ which equals to 1 for $x_3 > 0$ and vanishes for $x_3 < 0$. Therefore

$$\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big) = \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell_1 f)\big)$$

and the claim follows.

If, in particular, $f = 0$, then we can take $\ell f = 0$, and hence $U = 0$ by virtue of (3.22). Thus the equation (3.12) possesses at most one solution in the space $H^0(\mathbb{R}^3_+)$.

Further, we show that

$$U = r_{\mathbb{R}^3_+} \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\Big)\Big\} \tag{3.24}$$

is a solution of the equation (3.12) for any $f \in H^0(\mathbb{R}^3_+)$.

To this and, let us first note that for the vector-function involved in (3.24) the following embedding holds

$$\mathcal{F}^{-1}\Big\{[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^+\big([\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})]^{-1}\mathcal{F}(\ell f)\big)\Big\} \in \widetilde{H}^0(\mathbb{R}^3_+). \tag{3.25}$$

Indeed, we have

$$\mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\} = \\ = \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}\Big[\theta^+\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big]\Big\}$$

and (3.25) follows from Theorem 4.4, Lemmas 20.2 and 20.5 in [23]. From (3.24) and (3.25) we then get

$$U_+ := \ell_0 U = \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\}. \tag{3.26}$$

With the help of the following relation (see Lemma 5.4 in [23])

$$\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big) = \\ = \big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f) - \Pi^-\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big),$$

from the equality (3.26) we derive

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})\mathcal{F}(U_+) = \widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^+\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big) = \\ = \mathcal{F}(\ell f) - \widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^-\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big).$$

Since

$$F^{-1}\Big\{\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^-\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}F(\ell f)\big)\Big\} \in \widetilde{H}^0(\mathbb{R}^3_-),$$

(see [23, Theorems 4.4, 5.1 and Lemmas 20.2, 20.5]), we easily obtain

$$r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}U_+ = r_{\mathbb{R}^3_+}(\ell f) - r_{\mathbb{R}^3_+}\mathcal{F}^{-1}\Big\{\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^-\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\} = \\ = r_{\mathbb{R}^3_+}(\ell f) = f,$$

i.e., the vector-function (3.24) solves the equation (3.12) and belongs to the space $H^0(\mathbb{R}^3_+)$ for $f \in H^0(\mathbb{R}^3_+)$.

In what follows, we prove that for $f \in H^s(\mathbb{R}^3_+)$ and $\ell f \in H^s(\mathbb{R}^3)$ with

$$\|\ell f\|_{H^s(\mathbb{R}^3)} \leq 2\|f\|_{H^s(\mathbb{R}^3_+)}, \quad s \geq 0, \tag{3.27}$$

the vector-function defined by (3.24) satisfies the inequality

$$\|U\|_{H^s(\mathbb{R}^3_+)} \leq C\|f\|_{H^s(\mathbb{R}^3_+)}, \tag{3.28}$$

and hence belongs to $H^s(\mathbb{R}^3_+)$. Indeed, since (see [23, Lemma 5.2 and Theorem 5.1])

$$\Pi^+(\mathcal{F}g) = \mathcal{F}(\theta^+ g) \ \text{ for all } \ g \in H^0(\mathbb{R}^3),$$

then the representation (3.26) of U_+ can be rewritten as

$$U_+ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}\Big[\theta^+\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big]\Big\}.$$

Therefore, using (3.27) and in view of (3.10), from Theorem 10.1, Lemmas 4.4, 20.2, and 20.5 in [23] we finally derive

$$\|U\|_{H^s(\mathbb{R}^3_+)} \leq c_1\Big\|\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\|_{H^s(\mathbb{R}^3_+)} \leq \\ \leq c_1\Big\|\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\|_{H^s(\mathbb{R}^3)} \leq c\|\ell f\|_{H^s(\mathbb{R}^3)} \leq 2c\|f\|_{H^s(\mathbb{R}^3_+)}$$

with some positive constants c and c_1, whence (3.28) follows. This completes the proof. □

Lemma 3.2. *Let the factor matrix $\widetilde{A}^{(+)}(\xi', \tau)$ be as in (3.6), and $a^{(+)}$ and $c^{(+)}_{ij}$ be as in (3.7) and (3.8), respectively. Then the following equality holds*

$$\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi', \tau)\big]^{-1}\,d\tau = \frac{1}{a^{(+)}(\xi')}\,\big[c^{(+)}_{ij}(\xi')\big]_{4\times 4},$$

and

$$\det\big[c^{(+)}_{ij}(\xi')\big]_{4\times 4} \neq 0 \ \text{ for } \ \xi' \neq 0.$$

Here γ^- is a contour in the lower complex half-plane enclosing all the roots of the polynomial $\det \widetilde{A}^{(+)}(\xi',\tau)$ *with respect to τ.*

Proof. Note that $\det \widetilde{A}^{(+)}(\xi',\tau)$ is a forth order polynomial in τ, while $p_{ij}^{(+)}(\xi',\tau)$ is a third order polynomial in τ defined in (3.8).

Let γ_R be a circle with sufficiently large radius R and centered at the origin. Then by Cauchy theorem we derive

$$\frac{1}{2\pi i}\int\limits_{\gamma^-}\Big\{\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\Big\}_{ij}\,d\tau =$$

$$= \frac{1}{2\pi i}\int\limits_{\gamma^-}\frac{p_{ij}^{(+)}(\xi',\tau)}{\det \widetilde{A}^{(+)}(\xi',\tau)}\,d\tau = \frac{1}{2\pi i}\int\limits_{\gamma_R}\frac{p_{ij}^{(+)}(\xi',\tau)}{\det \widetilde{A}^{(+)}(\xi',\tau)}\,d\tau =$$

$$= \frac{1}{2\pi i}\,\frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')}\int\limits_{\gamma_R}\frac{1}{\tau}\,d\tau + \int\limits_{\gamma_R} Q_{ij}(\xi',\tau)\,d\tau =$$

$$= \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')} + \int\limits_{\gamma_R} Q_{ij}(\xi',\tau)\,d\tau, \tag{3.29}$$

where

$$Q_{ij}(\xi',\tau) = O(|\tau|^{-2}) \;\text{ as }\; |\tau|\to\infty.$$

It is clear that

$$\lim_{R\to\infty}\int\limits_{\gamma_R} Q_{ij}(\xi',\tau)\,d\tau = 0.$$

Therefore by passing to the limit in (3.29) as $R\to\infty$ we obtain

$$\frac{1}{2\pi i}\int\limits_{\gamma^-}\Big\{\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\Big\}_{ij}\,d\tau = \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')}\,.$$

Now we show that $\det[c_{ij}^{(+)}]_{4\times 4}\neq 0$. We introduce the notation

$$P^{(+)}(\xi',\xi_3) = [p_{ij}^{(+)}(\xi',\xi_3)]_{4\times 4} =$$
$$= C^{(+)}(\xi')\xi_3^3 + B^{(+)}(\xi')\xi_3^2 + D^{(+)}(\xi')\xi_3 + E^{(+)}(\xi'),$$

where

$$C^{(+)}(\xi') = \big[c_{ij}^{(+)}(\xi')\big]_{4\times 4},\quad B^{(+)}(\xi') = \big[b_{ij}^{(+)}(\xi')\big]_{4\times 4},$$
$$D^{(+)}(\xi') = \big[d_{ij}^{(+)}(\xi')\big]_{4\times 4},\quad E^{(+)}(\xi') = \big[e_{ij}^{(+)}(\xi')\big]_{4\times 4}.$$

In accordance with the relation $\det[\widetilde{A}^{(+)}(\xi',\xi_3)]^{-1}\neq 0$ for $\xi=(\xi',\xi_3)\neq 0$, we conclude that $\det P^{(+)}(\xi',\xi_3)\neq 0$ for $\xi=(\xi',\xi_3)\neq 0$.

Let us introduce new coordinates $r=|\xi'|$ and $\omega=\xi'/|\xi'|$, and denote

$$\mathcal{P}^{(+)}(\omega,r,\xi_3) := P^{(+)}(\xi',\xi_3) = P^{(+)}(\omega r,\xi_3).$$

Then we have

$$\det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \det P^{(+)}(\xi', \xi_3) =$$
$$= \det\Big(C^{(+)}(\omega)\xi_3^3 + B^{(+)}(\omega)\xi_3^2 r + D^{(+)}(\omega)\xi_3 r^2 + E^{(+)}(\omega) r^3\Big) \neq 0$$
$$\text{for all } \xi_3 \neq 0,$$

whence

$$\lim_{r\to 0} \det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \xi_3^{12} \det C^{(+)}(\omega).$$

Consequently,

$$\det C^{(+)}(\omega) = \det[c_{ij}^{(+)}(\omega)]_{4\times 4} \neq 0$$

and Lemma 3.2 is proved. □

Let us introduce the operator Π' defined as

$$\Pi'(g)(\xi') := \lim_{x_3\to 0+} r_{\mathbb{R}^3_+} \mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi', \xi_3)] =$$
$$= \frac{1}{2\pi} \lim_{x_3\to 0+} \int\limits_{-\infty}^{+\infty} g(\xi', \xi_3) e^{-ix_3\xi_3}\, d\xi_3 =$$
$$= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} g(\xi', \xi_3)\, d\xi_3 \ \text{ for } \ g(\xi', \,\cdot\,) \in L_1(\mathbb{R}).$$

The operator Π' can be extended to the class of functions $g(\xi', \xi_3)$ being rational in ξ_3 with the denominator not vanishing for real non-zero $\xi = (\xi', \xi_3) \in \mathbb{R}^3 \setminus \{0\}$, homogeneous of order $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ in ξ and infinitely differentiable with respect to ξ for $\xi' \neq 0$. Then one can show that (see [20, Appendix C])

$$\Pi'(g)(\xi') = \lim_{x_3\to 0+} r_{\mathbb{R}_+} \mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi', \xi_3)] = -\frac{1}{2\pi} \int\limits_{\gamma^-} g(\xi', \zeta)\, d\zeta,$$

where $r_{\mathbb{R}_+}$ denotes the restriction operator onto $\mathbb{R}_+ = (0, +\infty)$ with respect to x_3, γ^- is a contour in the lower complex half-plane $\operatorname{Im}\zeta < 0$, orientated anticlockwise and enclosing all the poles of the rational function $g(\xi', \cdot)$. It is clear that if $g(\xi', \zeta)$ is holomorphic in ζ in the lower complex half-plane ($\operatorname{Im}\zeta < 0$), then $\Pi'(g)(\xi') = 0$.

Denote by $\mathfrak{D}$ the localized boundary-domain integral operator generated by the left-hand side expressions in LBDIE system (3.1), (3.2),

$$\mathfrak{D} := \begin{bmatrix} r_{\Omega^+}\mathbf{B}\ell_0 & -r_{\Omega^+}V \\ \mathbf{A}^+\ell_0 & -\mathcal{V} \end{bmatrix}.$$

Now we prove the following assertion.

Theorem 3.3. *Let a cut-off function $\chi \in X^\infty_+$ and $r > -\frac{1}{2}$. Then the following operator*

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S) \tag{3.30}$$

is invertible.

Proof. We prove the theorem into four steps where we show that

Step 1. The operator $r_{\Omega^+}\mathbf{B}\ell_0 : H^s(\Omega) \to H^s(\Omega)$ for $s \geq 0$ is Fredholm operator with zero index;

Step 2. The operator $\mathfrak{D}$ given as in (3.30) is Fredholm operator;

Step 3. $\operatorname{Ind}\mathfrak{D} = 0$;

Step 4. The operator $\mathfrak{D}$ is invertible.

Step 1. Since (3.4) is a rational function in ξ, we can apply the theory of pseudodifferential operators with symbol satisfying the transmission conditions (see [2], [3], [23], [44], [45]). Now with the help of the local principle (see, e.g., [1], [23, Lemma 23.9]) and the above Lemma 3.1 we deduce that the operator

$$\mathcal{B} := r_{\Omega+}\mathbf{B}\ell_0 : H^s(\Omega) \to H^s(\Omega)$$

is Fredholm operator for all $s \geq 0$.

To show that $\operatorname{Ind}\mathcal{B} = 0$, we use that the operators $\mathcal{B}$ and

$$\mathcal{B}_t = r_{\Omega^+}(\mathbf{b} + t\mathbf{A})\ell_0,$$

where $t \in [0,1]$, are homotopic. Note that $\mathcal{B} = \mathcal{B}_1$. The principal homogeneous symbol of the operator $\mathcal{B}_t$ has the form

$$\mathfrak{S}(\mathcal{B}_t)(y,\xi) = \mathbf{b}(y) + t\mathfrak{S}(\mathbf{A})(y,\xi) = (1-t)\mathbf{b}(y) + t\mathfrak{S}(\mathbf{B})(y,\xi).$$

It is easy to see that the operator $\mathcal{B}_t$ is uniformly strongly elliptic,

$$\operatorname{Re}\mathfrak{S}(\mathcal{B}_t)(y,\xi)\zeta\cdot\zeta = (1-t)\operatorname{Re}\mathbf{b}(y)\zeta\cdot\zeta + t\operatorname{Re}\mathfrak{S}(\mathbf{B})(y,\xi)\zeta\cdot\zeta \geq c|\zeta|^2$$

for all $y \in \overline{\Omega}$, $\xi \neq 0$, $\zeta \in \mathbb{C}^4$ and $t \in [0,1]$, where c is some positive number.

Since $\mathfrak{S}(\mathcal{B}_t)(y,\xi)$ is rational, even, and homogeneous of order zero in ξ, as above we conclude that the operator

$$\mathcal{B}_t : H^s(\Omega) \to H^s(\Omega)$$

is Fredholm operator for all $s \geq 0$ and for all $t \in [0,1]$. Therefore $\operatorname{Ind}\mathcal{B}_t$ is the same for all $t \in [0,1]$. On the other hand, due to the equality $\mathcal{B}_0 = r_{\Omega^+}I$, we get

$$\operatorname{Ind}\mathcal{B} = \operatorname{Ind}\mathcal{B}_1 = \operatorname{Ind}\mathcal{B}_t = \operatorname{Ind}\mathcal{B}_0 = 0.$$

Step 2. To investigate Fredholm properties of the operator $\mathfrak{D}$ we apply the local principle (cf. e.g., [1], [23, §§ 19, 22]). Due to this principle, we have to check that the so-called generalized *Šapiro–Lopatinskiĭ condition* for the operator $\mathfrak{D}$ holds at an arbitrary "frozen" point $\widetilde{y} \in S$. To obtain the explicit form of this condition we proceed as follows. Let $\widetilde{\mathcal{U}}$ be a neighbourhood of a fixed point $\widetilde{y} \in \overline{\Omega}$ and let $\widetilde{\psi}_0, \widetilde{\varphi}_0 \in \mathcal{D}(\widetilde{\mathcal{U}})$ such that

$$\operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0 \neq \varnothing, \quad \widetilde{y} \in \operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0,$$

and consider the operator $\widetilde{\psi}_0\mathfrak{D}\widetilde{\varphi}_0$. We separate the two possible cases 1) $\widetilde{y} \in \Omega$ and 2) $\widetilde{y} \in S$.

Case 1). If $\widetilde{y} \in \Omega$, then we can choose a neighbourhood $\overline{\widetilde{\mathcal{U}}}$ of the point $\widetilde{y}$ such that $\overline{\widetilde{\mathcal{U}}} \subset \Omega$. Then

$$\widetilde{\psi}_0 \mathfrak{D} \widetilde{\varphi}_0 = \widetilde{\psi}_0 \mathbf{B} \widetilde{\varphi}_0$$

where $\mathbf{B}$ is the operator defined by (3.3). As we have already shown above (see Step 1) this operator is Fredholm operator with zero index.

Case 2). If $\widetilde{y} \in S$, then at this point we have to "froze" the operator $\widetilde{\psi}_0 \mathfrak{D} \widetilde{\varphi}_0$, which means that we can choose a neighbourhood $\overline{\widetilde{\mathcal{U}}}$ of the point $\widetilde{y}$ sufficiently small such that at the local coordinate system with the origin at the point $\widetilde{y}$ and the third axis coinciding with the normal vector at the point $\widetilde{y} \in S$, the following decomposition holds

$$\widetilde{\psi}_0 \mathfrak{D} \widetilde{\varphi}_0 = \widetilde{\psi}_0 \Big(\widehat{\widehat{\mathfrak{D}}} + \widetilde{\mathbf{K}} + \widetilde{\mathbf{T}} \Big) \widetilde{\varphi}_0, \tag{3.31}$$

where $\widetilde{\mathbf{K}}$ is a bounded operator with small norm

$$\widetilde{\mathbf{K}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2),$$

while $\widetilde{\mathbf{T}}$ is a bounded operator

$$\widetilde{\mathbf{T}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+2}(\mathbb{R}^3_+) \times H^{r+3/2}(\mathbb{R}^2).$$

The operator $\widehat{\widehat{\mathfrak{D}}}$ is defined in the upper half-space $\mathbb{R}^3_+$ as follows

$$\widehat{\widehat{\mathfrak{D}}} := \begin{bmatrix} r_{\mathbb{R}^3_+} \widehat{\widehat{\mathbf{B}}} \ell_0 & -r_{\mathbb{R}^3_+} \widehat{\widehat{V}} \\ \widehat{\widehat{\mathbf{A}}}^{+} \ell_0 & -\widehat{\widehat{\mathcal{V}}} \end{bmatrix}$$

and possesses the following mapping property

$$\widehat{\widehat{\mathfrak{D}}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2). \tag{3.32}$$

The operators involved in the expression of $\widehat{\widehat{\mathfrak{D}}}$ are defined as follows: for the operator $\widetilde{M}$, $\widehat{\widehat{M}}$ denotes the operator in $\mathbb{R}^n$ $(n = 2, 3)$ constructed by the symbol

$$\widehat{\mathfrak{S}}(\widetilde{M})(\xi) = \mathfrak{S}(\widetilde{M})\big((1 + |\xi'|)\omega, \xi_3\big) \ \text{ if } \ n = 3,$$

and

$$\widehat{\mathfrak{S}}(\widetilde{M})(\xi) = \mathfrak{S}(\widetilde{M})\big((1 + |\xi'|)\omega\big) \ \text{ if } \ n = 2,$$

where $\omega = \frac{\xi'}{|\xi'|}$, $\xi = (\xi', \xi_n)$, $\xi' = (\xi_1, \dots, \xi_{n-1})$.

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of the operator (3.32). Indeed, let us write the system corresponding to the operator $\widehat{\widehat{\mathfrak{D}}}$:

$$r_{\mathbb{R}^3_+} \widehat{\widehat{\mathbf{B}}} \ell_0 \widetilde{U} - r_{\mathbb{R}^3_+} \widehat{\widehat{V}} \widetilde{\Psi} = \widetilde{F}_1 \ \text{ in } \ \mathbb{R}^3_+, \tag{3.33}$$

$$\widehat{\widehat{\mathbf{A}}}^{+} \ell_0 \widetilde{U} - \widehat{\widehat{\mathcal{V}}} \widetilde{\Psi} = \widetilde{F}_2 \ \text{ on } \ \mathbb{R}^2, \tag{3.34}$$

where $\widetilde{F}_1 \in H^1(\mathbb{R}^3_+)$, $\widetilde{F}_2 \in H^{1/2}(\mathbb{R}^2)$.

Note that the operator $r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}\ell_0$ is a singular integral operator with even rational elliptic principal homogeneous symbol. Then due to Lemma 3.1 the operator

$$r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}\ell_0 : H^{r+1}(\mathbb{R}^3_+) \to H^{r+1}(\mathbb{R}^3_+)$$

is invertible. Therefore we can define $\widetilde{U}$ from equation (3.33)

$$\begin{aligned}\ell_0\widetilde{U} = \big[r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{f} =&\\ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell\widetilde{f})\big)\Big\},\quad (3.35)\end{aligned}$$

where $\widetilde{f} = \widetilde{F}_1 + r_{\mathbb{R}^3_+}\widehat{\widetilde{V}}\widetilde{\Psi}$, ℓ is an extension operator from $\mathbb{R}^3_+$ to $\mathbb{R}^3$ preserving the function space, while ℓ_0 is an extension operator $\mathbb{R}^3_+$ to $\mathbb{R}^3_-$ by zero; here $\widehat{\mathfrak{S}}^{\,(\pm)}(M)$ denote the so-called "plus" and "minus" factors in the factorization of the symbol $\widehat{\mathfrak{S}}(M)$ with respect to the variable ξ_3. The operator Π^+ involved in (3.35) is the Cauchy type integral (see (3.11)). Note that the function $\ell_0\widetilde{U}$ in (3.35) does not depend on the extension operator ℓ.

Substituting (3.35) into (3.34) leads to the following pseudodifferential equation with respect to the unknown function $\widetilde{\Psi}$:

$$\widehat{\widetilde{\mathbf{A}}}^{+}\mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widetilde{V}}\widetilde{\Psi})\big)\Big\} - \widehat{\widetilde{\mathcal{V}}}\widetilde{\Psi} = \widetilde{F} \ \text{ on } \mathbb{R}^2, \quad (3.36)$$

where

$$\widetilde{F} = \widetilde{F}_2 - \widehat{\widetilde{\mathbf{A}}}^{+}\ell_0\big[r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{F}_1.$$

It is easy to see that

$$\begin{aligned}\widetilde{\mathbf{A}}^{+}v(\widetilde{y}') = \Big[\mathcal{F}^{-1}_{\xi\to\widetilde{y}}\big[\mathfrak{S}(\widetilde{\mathbf{A}})(\xi)\mathcal{F}(v)(\xi)\big]\Big]_{\widetilde{y}_3=0+} =&\\ = \mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\Big[\Pi'\big[(\mathfrak{S}\widetilde{\mathbf{A}})(\xi)\mathcal{F}(v)(\xi)\big]\Big],&\end{aligned}$$

and in view of the relation

$$V(\Psi) = -\mathbf{P}(\Psi\otimes\delta)$$

with $\delta = \delta(x_3)$ being the Dirac distribution, we arrive at the equality

$$\begin{aligned}\widehat{\widetilde{\mathbf{A}}}^{+}\mathcal{F}^{-1}_{\xi\to\widetilde{x}}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\mathbf{B})(\xi)\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widetilde{V}}\Psi)\big)(\xi)\Big\}(\widetilde{y}') =&\\ = -\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\Big\{\Pi'\Big[\widehat{\mathfrak{S}}(\widetilde{\mathbf{A}})\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\widehat{\mathfrak{S}}(\widetilde{\mathbf{P}})\big)\Big](\xi')\mathcal{F}_{\widetilde{x}'\to\xi'}\widetilde{\Psi}\Big\}.&\end{aligned}$$

With the help of these relations equation (3.36) can be rewritten in the following form

$$\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\big[\widehat{e}(\xi')\mathcal{F}(\widetilde{\psi})(\xi')\big] = \widetilde{F}(\widetilde{y}') \ \text{ on } \mathbb{R}^2, \quad (3.37)$$

where

$$\widehat{e}(\xi') = e\big((1+|\xi'|)\omega\big), \ \ \omega = \frac{\xi'}{|\xi'|}\,,$$

with e being a homogeneous function of order -1 given by the equality

$$e(\xi') = -\Pi'\Big\{\mathfrak{S}(\widetilde{\mathbf{A}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi')- \\ -\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') \ \ \forall\,\xi' \neq 0. \tag{3.38}$$

If the function $\det e(\xi')$ is different from zero for all $\xi' \neq 0$, then $\det \widehat{e}(\xi') \neq 0$ for all $\xi' \in \mathbb{R}^2$, and the corresponding pseudodifferential operator

$$\widehat{\mathbf{E}} : H^s(\mathbb{R}^2) \to H^{s+1}(\mathbb{R}^2) \ \text{ for all } \ s \in \mathbb{R}$$

generated by the left-hand side expression in (3.37) is invertible. In particular, it follows that the system of equation (3.33), (3.34) is uniquely solvable with respect to $(\widetilde{U}, \widetilde{\Psi})$ in the space $H^1(\mathbb{R}^3_+) \times H^{-1/2}(\mathbb{R}^2)$ for arbitrary right-hand sides $(\widetilde{F}_1, \widetilde{F}_2) \in H^1(\mathbb{R}^3_+) \times H^{1/2}(\mathbb{R}^2)$. Consequently, the operator $\widehat{\widetilde{\mathfrak{D}}}$ in (3.32) is invertible, which implies that the operator (3.31) possesses a left and right regularizer. In turn, this yields that the operator (3.30) possesses a left and right regularizer as well. Thus the operator (3.30) is Fredholm operator if

$$\det e(\xi') \neq 0 \ \ \forall\,\xi' \neq 0. \tag{3.39}$$

This condition is called the *Šapiro–Lopatinskiĭ condition* (cf. [23, Theorems 12.2 and 23.1 and also formulas (12.27), (12.25)]). Let us show that in our case the Šapiro–Lopatinskiĭ condition holds. To this end, let us note that the principal homogeneous symbols $\mathfrak{S}(\widetilde{\mathbf{A}})$, $\mathfrak{S}(\widetilde{\mathbf{B}})$, $\mathfrak{S}(\widetilde{\mathbf{P}})$, and $\mathfrak{S}(\widetilde{\mathcal{V}})$ of the operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$, and $\mathcal{V}$ in the chosen local coordinate system involved in the formula (3.39) read as:

$$\begin{aligned}
&\mathfrak{S}(\widetilde{\mathbf{A}})(\xi) = |\xi|^{-2}\widetilde{A}(\xi) - \widetilde{\mathbf{b}}, \\
&\mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = |\xi|^{-2}\widetilde{A}(\xi), \\
&\mathfrak{S}(\widetilde{\mathbf{P}})(\xi) = -|\xi|^{-2} I, \\
&\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\, I, \\
&\xi = (\xi', \xi_3), \ \ \xi' = (\xi_1, \xi_2),
\end{aligned}$$

where $\widetilde{\mathbf{b}}$ denotes the matrix $\mathbf{b}$ written in the chosen local co-ordinate system. Further, $\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})$ and $\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})$ are the so-called "plus" and "minus" factors in the factorization of the symbol $\mathfrak{S}(\widetilde{\mathbf{B}})$ with respect to the variable ξ_3, i.e.

$$\mathfrak{S}(\widetilde{\mathbf{B}}) = \mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}),$$

where

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi) = \frac{1}{\Theta^{(\pm)}(\xi)}\widetilde{A}^{(\pm)}(\xi)$$

due to (3.4). Rewrite (3.38) in the form

$$e(\xi') = -\Pi'\Big\{\big(\mathfrak{S}(\widetilde{\mathbf{B}}) - \widetilde{\mathbf{b}}\big)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^{+}\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') - \mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = e_1(\xi') + e_2(\xi') - \mathfrak{S}(\widetilde{\mathcal{V}})(\xi'), \quad (3.40)$$

where

$$e_1(\xi') = -\Pi'\Big\{\mathfrak{S}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^{+}\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi'), \quad (3.41)$$

$$e_2(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^{+}\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi'), \quad (3.42)$$

$$\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\, I. \quad (3.43)$$

Direct calculations give

$$\begin{aligned}
&\Pi^{+}\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)(\xi') = \\
&= \frac{i}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}\frac{([\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}}))(\xi',\eta_3)}{\xi_3 + it - \eta_3}\, d\eta_3 = \\
&= -\frac{i}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\eta_3)}{(\xi_3 + it - \eta_3)(|\xi'|^2 + \eta_3^2)}\, d\eta_3 = \\
&= \frac{i}{2\pi}\lim_{t\to 0+}\int_{\gamma^-}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)}{(\xi_3 + it - \tau)(|\xi'|^2 + \tau^2)}\, d\tau = \\
&= \frac{i}{2\pi}\lim_{t\to 0+}\frac{2\pi i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{(\xi_3 + it + i|\xi'|)2(-i|\xi'|)} = \\
&= -\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|\Theta^{(+)}(\xi',\xi_3)}\,. \qquad (3.44)
\end{aligned}$$

Now from (3.41) with the help of (3.44) we derive

$$\begin{aligned}
&e_1(\xi') = \\
&= -\Pi'\Big\{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^{+}\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\
&= -\Pi'\Big\{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\Pi^{+}\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\
&= \Pi'\Big\{\frac{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})}{\Theta^{(+)}}\Big\}(\xi')\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) = \\
&= -\frac{1}{2\pi}\int_{\gamma^-}\frac{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\tau)}{\tau + i|\xi'|}\, d\tau\,\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) = \\
&= -i\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big(\xi',-i|\xi'|\big)\,\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|} = \frac{1}{2|\xi'|}\, I. \qquad (3.45)
\end{aligned}$$

Quite similarly, from (3.42) with the help of (3.44) we get

$$e_2(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') =$$

$$= -\widetilde{\mathbf{b}}\Pi'\Big\{\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}}{\Theta^{(+)}}\Big\}(\xi')\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) =$$

$$= -\frac{i\widetilde{\mathbf{b}}}{2|\xi'|}\bigg(-\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)}{\tau+i|\xi'|}\,d\tau\bigg)\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\big(\xi',-i|\xi'|\big) =$$

$$= \frac{i\widetilde{\mathbf{b}}}{4\pi|\xi'|}\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau(-2i|\xi'|)\,\big[\widetilde{A}^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1} =$$

$$= i\widetilde{\mathbf{b}}\bigg\{\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\bigg\}\big[\widetilde{A}^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1}.$$

Therefore, due to Lemma 3.2, we have

$$e_2(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij}^{(+)}(\xi')]_{4\times4}}{a^{(+)}(\xi')}\,\big[\widetilde{A}^{(-)}(\xi',-i|\xi'|)\big]^{-1}. \tag{3.46}$$

In view of (3.40), (3.43), (3.45), and (3.46) we finally obtain

$$e(\xi') = e_2(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij}^{(+)}(\xi')]_{4\times4}}{a^{(+)}(\xi')}\,\big[\widetilde{A}^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1},$$

where

$$\det\widetilde{\mathbf{b}}\neq0,\quad \det[c_{ij}^{(+)}]_{4\times4}\neq0$$

(see Lemma 3.2), and $\det\widetilde{A}^{(-)}(\xi',-i|\xi'|)\neq0$ for all $\xi'\neq0$.

Then it is clear that for all $\xi'\neq0$ we have

$$\det e(\xi') = \frac{1}{(a^{(+)}(\xi'))^4}\,\det\widetilde{\mathbf{b}}\det[c_{ij}^{(+)}]_{4\times4}\det\big[\widetilde{A}^{-}\big(\xi',-i|\xi'|\big)\big]^{-1}\neq0.$$

Thus, we have obtained that for the operator $\mathfrak{D}$ the Šapiro–Lopatinskiĭ condition holds. Therefore, the operator

$$\mathfrak{D}: H^{r+1}(\Omega)\times H^{r-1/2}(S)\to H^{r+1}(\Omega)\times H^{r+1/2}(S)$$

is Fredholm operator for $r>-\frac{1}{2}$.

Step 3. Here we will show that $\operatorname{Ind}\mathfrak{D}=0$. To this end, for $t\in[0,1]$ let us consider the operator

$$\mathfrak{D}_t := \begin{bmatrix} r_{\Omega^+}\mathbf{B}_t\ell_0 & -r_{\Omega^+}V \\ t\mathbf{A}^+\ell_0 & -\mathcal{V} \end{bmatrix}$$

with $\mathbf{B}_t=\mathbf{b}+t\mathbf{A}$ and establish that it is homotopic to the operator $\mathfrak{D}=\mathfrak{D}_1$. We have to check that for the operator $\mathfrak{D}_t$ the Šapiro–Lopatinskiĭ condition is satisfied for all $t \in [0,1]$. Indeed, in this case the Šapiro–Lopatinskiĭ condition reads as (cf. (3.39))

$$\det e_t(\xi') \neq 0 \ \ \forall \xi' \neq 0,$$

where

$$e_t(\xi') = -\Pi'\Big\{\big(\mathfrak{S}(\widetilde{\mathbf{B}}_t) - \widetilde{\mathbf{b}}\big)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') - \\ -\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = e_t^{(1)}(\xi') + e_t^{(2)}(\xi') - \mathfrak{S}(\widetilde{\mathcal{V}})(\xi'), \tag{3.47}$$

$$e_t^{(1)}(\xi') = -\Pi'\Big\{\mathfrak{S}(\widetilde{\mathbf{B}}_t)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\ = \frac{1}{2|\xi'|}\, I, \tag{3.48}$$

$$e_t^{(2)}(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi'), \\ \mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\, I. \tag{3.49}$$

By direct calculations we get

$$e_t^{(2)}(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\ = -\widetilde{\mathbf{b}}\Pi'\Big\{\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}}{\Theta^{(+)}}\Big\}(\xi')\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi', -i|\xi'|)}{2|\xi'|}\Big) = \\ = -\frac{i\widetilde{\mathbf{b}}}{2|\xi'|}\bigg(-\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi', \tau)}{\tau + i|\xi'|}\,d\tau\bigg)\big[\mathfrak{S}^{-}(\widetilde{\mathbf{B}}_t)\big]^{-1}(\xi', -i|\xi'|) = \\ = \frac{i\widetilde{\mathbf{b}}}{4\pi|\xi'|}\int\limits_{\gamma^-}\big[\widetilde{A}_t^{(+)}(\xi', \tau)\big]^{-1}\,d\tau\,(-2i|\xi'|)\big[\widetilde{A}_t^{(-)}\big(\xi', -i|\xi'|\big)\big]^{-1} = \\ = i\widetilde{\mathbf{b}}\bigg\{\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[\widetilde{A}_t^{(+)}(\xi', \tau)\big]^{-1}\,d\tau\bigg\}\big[\widetilde{A}_t^{(-)}\big(\xi', -i|\xi'|\big)\big]^{-1}, \tag{3.50}$$

where $\widetilde{A}_t(\xi) = (1-t)|\xi|^2\widetilde{\mathbf{b}} + t\widetilde{A}(\xi)$ and $\widetilde{A}_t(\xi', \xi_3) = \widetilde{A}_t^{(-)}(\xi', \xi_3)\widetilde{A}_t^{(+)}(\xi', \xi_3)$, $\widetilde{A}_t^{(\pm)}(\xi', \xi_3)$ are the "plus" and "minus" polynomial matrix factors in ξ_3 of the polynomial symbol matrix $\widetilde{A}_t(\xi', \xi_3)$.

Due to Lemma 3.2 and the equality (3.50) we have

$$e_t^{(2)}(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij,t}^{(+)}(\xi')]_{4\times 4}}{a_t^{(+)}(\xi')}\,\big[\widetilde{A}_t^{(-)}\big(\xi', -i|\xi'|\big)\big]^{-1}, \tag{3.51}$$

where $c_{ij,t}^{(+)}$, $i,j = \overline{1,4}$, are the main coefficients of the co-factors $p_{ij,t}^{(+)}(\xi', \tau)$ of the polynomial matrix $\widetilde{A}_t^{(+)}(\xi', \tau)$ and $a^{(+)}$ is the coefficient at τ^4 in the determinant $\det \widetilde{A}_t^{(+)}(\xi', \tau)$.

In view of (3.47), (3.48), (3.49), and (3.51), we finally obtain

$$e_t(\xi') = e_t^{(2)}(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij,t}^{(+)}(\xi')]_{4\times 4}}{a_t^{(+)}(\xi')}\,\big[\widetilde{A}_t^{(-)}(\xi', -i|\xi'|)\big]^{-1},$$

where $\det \widetilde{\mathbf{b}} \neq 0$, $\det[c_{ij,t}^{(+)}]_{4\times 4} \neq 0$ (see Lemma 3.2), and $\det \widetilde{A}_t^{(-)}(\xi', -i|\xi'|) \neq 0$ for all $\xi' \neq 0$ and $t \in [0, 1]$.

Then it follows that

$$\det e_t(\xi') = \frac{1}{[a_t^{(+)}(\xi')]^4}\,\det \mathbf{b} \det \big[c_{ij,t}^{(+)}(\xi')\big]_{4\times 4} \det \big[\widetilde{A}_t^{(-)}(\xi', -i|\xi'|)\big]^{-1} \neq 0$$
$$\text{for all } \ \xi' \neq 0 \ \text{ and for all } \ t \in [0, 1],$$

which implies that for the operator $\mathfrak{D}_t$ the Šapiro–Lopatinskiĭ condition is satisfied.

Therefore the operator

$$\mathfrak{D}_t : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm operator for all $r > -\frac{1}{2}$ and $t \in [0, 1]$. Consequently,

$$\operatorname{Ind} \mathfrak{D} = \operatorname{Ind} \mathfrak{D}_1 = \operatorname{Ind} \mathfrak{D}_t = \operatorname{Ind} \mathfrak{D}_0 = 0.$$

Step 4. Since the operator $\mathfrak{D}$ is Fredholm operator with zero index, its injectivity implies the invertibility. Thus it remains to prove that the null space of the operator $\mathfrak{D}$ is trivial for $r > -\frac{1}{2}$. Assume that $\mathcal{U} = (U, \Psi)^\top \in H^{r+1}(\Omega) \times H^{r-1/2}(S)$ is a solution to the homogeneous equation

$$\mathfrak{D}\mathcal{U} = 0. \tag{3.52}$$

The operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm operator with index zero for all $r > -\frac{1}{2}$. It is well known that then there exists a left regularizer $\mathfrak{L}$ of the operator $\mathfrak{D}$,

$$\mathfrak{L} : H^{r+1}(\Omega) \times H^{r+1/2}(S) \to H^{r+1}(\Omega) \times H^{r-1/2}(S),$$

such that

$$\mathfrak{L}\mathfrak{D} = I + \mathfrak{T},$$

where $\mathfrak{T}$ is the operator of order -1 (cf. [23, Proofs of Theorems 22.1 and 23.1]), i.e.,

$$\mathfrak{T} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{3.53}$$

Therefore, from (3.52) we have

$$\mathfrak{L}\mathfrak{D}\mathcal{U} = U + \mathfrak{T}\mathcal{U} = 0. \tag{3.54}$$

In view of (3.53) we see that

$$\mathfrak{T}\mathcal{U} \in H^{r+2}(\Omega) \times H^{r+1/2}(S).$$

Consequently, in view of (3.54),

$$\mathcal{U} = (U, \Psi)^\top \in H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{3.55}$$

If $r \geq 0$, this implies $U \in H^{1,0}(\Omega, A)$. If $-\frac{1}{2} < r < 0$, we iterate the above reasoning for U satisfying (3.55) to obtain

$$\mathcal{U} = (U, \Psi)^\top \in H^{r+3}(\Omega) \times H^{r+3/2}(S)$$

which again implies $U \in H^{1,0}(\Omega, A)$. Then we can apply the equivalence Theorem 2.4 to conclude that a solution $\mathcal{U} = (U, \Psi)^\top$ to the homogeneous equation (3.52) is zero vector, i.e.,

$$U = 0 \ \text{ in } \ \Omega, \quad \Psi = 0 \ \text{ on } \ S.$$

Thus, $\operatorname{Ker} \mathfrak{D} = \{0\}$ in the class $H^{r+1}(\Omega) \times H^{r-1/2}(S)$ and therefore the operator

$$\mathfrak{D} \ : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is invertible for all $r > -\frac{1}{2}$. □

For localizing function χ of finite smoothness we have the following result.

Corollary 3.4. *Let a cut-off function $\chi \in X^3_+$. Then the operator*

$$\mathfrak{D} : H^1(\Omega) \times H^{-1/2}(S) \to H^1(\Omega) \times H^{1/2}(S)$$

is invertible.

Proof. We have to use mapping properties of the localized potentials with a localizing cut-off function of finite smoothness (see Appendix B) and repeat word for word the arguments of the above proof of Theorem 3.3 for $r = 0$. □

From Corollaries 2.3, 3.4, and Lemma 2.2 the following result follows directly.

Corollary 3.5. *Let a cut-off function $\chi \in X^3_+$. Then the operator*

$$\mathfrak{D} : H^{1,0}(\Omega, A) \times H^{-1/2}(S) \to H^{1,0}(\Omega, \Delta) \times H^{1/2}(S)$$

is invertible.

Appendix A: Classes of Cut-off Functions

Here we present the classes of localizing functions used in the main text (for details see the reference [14]).

Definition A.1. We say $\chi \in X^k$ for integer $k \geq 0$, if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W^k_1(0,\infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0,\infty)$. We say $\chi \in X^k_+$ for integer $k \geq 1$, if

$\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\widehat{\chi}_s(\omega)}{\omega} > 0 & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho \breve{\chi}(\varrho)\, d\varrho & \text{for } \omega = 0, \end{cases}$$

and $\widehat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$

$$\widehat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho) \sin(\varrho\omega)\, d\varrho.$$

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$ and $X_+^{k_1} \subset X_+^{k_2}$ for $k_1 > k_2$. The class X_+^k is defined in terms of the sine-transform. The following lemma provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class (for details see [14]).

Lemma A.2. *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0, \infty)$, and $\breve{\chi}$ is a non-increasing function on $[0, +\infty)$, then $\chi \in X_+^k$.*

The following examples for χ are presented in [14],

$$\chi_{1k}(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon. \end{cases}$$

One can observe that $\chi_{1k} \in X_+^k$, while $\chi_2 \in X_+^\infty$ due to Lemma A.2.

Appendix B: Properties of Localized Potentials

Here we collect some theorems describing mapping properties of the localized layered and volume potentials defined by the relations (2.23)–(2.24). The proofs can be found in [14] (see also [25], Chapter 8 and the references therein).

Let us introduce the boundary operators generated by the localized layer potentials associated with the localized parametrix $P(x-y) \equiv P_\chi(x-y)$

$$\mathcal{V}g(y) := -\int_S P(x-y) g(x)\, dS_x, \quad y \in S, \tag{B.1}$$

$$\mathcal{W}g(y) := -\int_S \left[\widetilde{T}(x, \partial_x) P(x-y)\right]^\top g(x)\, dS_x, \quad y \in S, \tag{B.2}$$

$$\mathcal{W}'g(y) := -\int\limits_S \big[\mathcal{T}(y,\partial_y)P(x-y)\big]g(x)\,dS_x, \ \ y \in S,$$

$$\mathcal{L}^{\pm}g(y) := \big[\mathcal{T}(y,\partial_y)Wg(y)\big]^{\pm}, \ \ y \in S.$$

Theorem B.1. *The following operators are continuous*

$$\begin{aligned}
\mathcal{P} : \widetilde{H}^s(\Omega) &\to H^{s+2,s}(\Omega;\Delta), \ \ -\frac{1}{2} < s < \frac{1}{2}\,, \ \ \chi \in X^1,\\
: H^s(\Omega) &\to H^{s+2,s}(\Omega;\Delta), \ \ -\frac{1}{2} < s < \frac{1}{2}\,, \ \ \chi \in X^1,\\
: H^s(\Omega) &\to H^{\frac{5}{2}-\varepsilon,\frac{1}{2}-\varepsilon}(\Omega;\Delta), \ \ \frac{1}{2} \le s < \frac{3}{2}\,, \ \ \forall\,\varepsilon \in (0,1), \ \ \chi \in X^2,
\end{aligned}$$

where $\mathcal{P}$ is the volume localized potential defined in (2.24) *and Δ is the Laplace operator.*

Theorem B.2. *The following localized single and double layer operators are continuous*

$$\begin{aligned}
V : H^{s-\frac{3}{2}}(S) &\to H^s(\mathbb{R}^3), \ \ s < \frac{3}{2}\,, \ \ \text{if } \chi \in X^1,\\
: H^{s-\frac{3}{2}}(S) &\to H^{s,s-1}(\Omega^{\pm};\Delta), \ \ \frac{1}{2} < s < \frac{3}{2}\,, \ \ \text{if } \chi \in X^2,\\
W : H^{s-\frac{1}{2}}(S) &\to H^s(\Omega^{\pm}), \ \ s < \frac{3}{2}\,, \ \ \text{if } \chi \in X^2,\\
: H^{s-\frac{1}{2}}(S) &\to H^{s,s-1}(\Omega^{\pm};\Delta), \ \ \frac{1}{2} < s < \frac{3}{2}\,, \ \ \text{if } \chi \in X^3.
\end{aligned}$$

Theorem B.3. *If $\chi \in X^k$ has a compact support and $-\frac{1}{2} \le s \le \frac{1}{2}$, then the following localized operators are continuous:*

$$\begin{aligned}
V : H^s(S) &\to H^{s+\frac{3}{2}}(\Omega^{\pm}) \ \ \text{for } \ k = 2,\\
W : H^{s+1}(S) &\to H^{s+\frac{3}{2}}(\Omega^{\pm}) \ \ \text{for } \ k = 3.
\end{aligned}$$

Theorem B.4. *Let $\psi \in H^{-\frac{1}{2}}(S)$ and $\varphi \in H^{\frac{1}{2}}(S)$. Then the following jump relations hold on S:*

$$\begin{aligned}
V^+\psi = V^-\psi &= \mathcal{V}\psi, \ \ \chi \in X^1,\\
W^{\pm}\varphi &= \mp\mathbf{d}\varphi + \mathcal{W}\varphi, \ \ \chi \in X^2,\\
\mathcal{T}^{\pm}V\psi &= \pm\mathbf{d}\psi + \mathcal{W}'\psi, \ \ \chi \in X^2,
\end{aligned}$$

where

$$\mathbf{d}(y) := \frac{1}{2}\begin{bmatrix} [c_{ijlk}(y)n_in_l]_{3\times 3} & [e_{lij}(y)n_in_l]_{3\times 1}\\ [-e_{ikl}(y)n_in_l]_{1\times 3} & \varepsilon_{il}(y)n_in_l \end{bmatrix}_{4\times 4}, \ \ y \in S, \qquad \text{(B.3)}$$

and $\mathbf{d}(y)$ is strongly elliptic due to (2.3).

Theorem B.5. *Let* $-\frac{1}{2} \le s \le \frac{1}{2}$ *. The following operators*

$$\mathcal{V} : H^s(S) \to H^{s+1}(S), \ \ \chi \in X^2,$$
$$\mathcal{W} : H^{s+1}(S) \to H^{s+1}(S), \ \ \chi \in X^3,$$
$$\mathcal{W}' : H^s(S) \to H^s(S), \ \ \chi \in X^3,$$
$$\mathcal{L}^{\pm} : H^{s+1}(S) \to H^s(S), \ \ \chi \in X^3,$$

are continuous.

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