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Behaviors and symbols of rational matrices

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ABSTRACT

Recently, we have defined rational differential equations, and derived conditions when two such equations have the same solution set. In this article, we study the same question, but present a different approach based on the new notion of symbols. Also we try to describe the relationship with the work "Behaviors defined by rational functions" of Willems and Yamamoto.

1. Introduction

As is known, in behavioral system theory (see Polderman and Willems [1], Willems [2]), dynamical systems are viewed as collections of time trajectories. In most cases they are described via equations. "These are often differential or difference equations, sometimes integral equations" [2]. Pursuing this ("behavioral") point of view, it is quite natural to consider differential/integral equations and study the corresponding dynamical systems.

In our recent article [3], we introduced linear constant coefficient differential/integral equations. They are associated to rational matrices and naturally generalize linear constant coefficient differential equations. The inspiration came from Willems and Yamamoto [4,5], where rational matrices have been considered as new representations for linear differential systems.

For the convenience of the reader, we briefly recall the main point from the above-mentioned article.

Let $\mathbb{F}$ be the field of real or complex numbers and s an indeterminate, and let $\mathcal{U}$ be the space of infinitely differentiable $\mathbb{F}$-valued functions defined on some interval of the real time axis on which an initial time is fixed. Let ∂ denote the differentiation operator. The initial time allows us to introduce the indefinite integral operator

$$\int : \mathcal{U} \to \mathcal{U},$$

and we define ∂^{-1} by setting

$$\partial^{-1} = \int .$$

(The relation $\partial \circ \int = id$ justifies this definition.) For $n \geq 0$, we set

$$\partial^{-n} = (\partial^{-1})^n = \int^n .$$

Given now a rational matrix G, we define the operator $G(\partial)$ to be

$$G_{-n}\partial^n + \cdots + G_{-1}\partial + G_0 I + G_1\partial^{-1} + G_2\partial^{-2} + \cdots,$$

where

$$G_{-n}, \ldots, G_{-1}, G_0, G_1, G_2, \ldots$$

are the coefficients in the Laurent expansion of G at infinity. This operator is a linear differential/integral operator with constant coefficients, and it gives rise to the equation

$$G(\partial)w = 0.$$

The solution set of this equation has been called, in [3], the behavior of G, and it is denoted by $Bh(G)$.

This interpretation of the equation $G(\partial)w = 0$ is different from that offered by Willems and Yamamoto [4,5]. We remind readers that, according to these works, a function w is said to be a solution of this equation if it satisfies the equation $Q(\partial)w = 0$, where Q is the numerator in a left coprime factorization of G.

The following simple example illustrates well the difference between the two approaches.

Example 1. The behavior of $s^{-1}(s^2 - 1) = s - s^{-1}$ in the sense of [3] is the solution set of the equation

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$$y' - \int y = 0,$$

which is the set

$$\{C(e^x + e^{-x}) \mid C \in \mathbb{R}\}.$$

The behavior in the sense of Willems and Yamamoto is the solution set of the equation

$$y'' - y = 0,$$

that is, the set

$$\{C_1e^x + C_2e^{-x} \mid C_1, C_2 \in \mathbb{R}\}.$$

Any set representable as the behavior of a rational matrix is called a linear differential/integral system. Of course, the class of linear differential/integral systems is larger than the class of linear differential systems. As shown in [3], linear differential systems are precisely those linear differential/integral systems that are differentiation invariant.

In [3], we have derived conditions under which two rational matrices give rise to the same linear differential/integral system. In the present article, we shall deal with the same problem. However, we shall offer a very different approach to it.

It turns out that a relevant notion is the notion of a symbol. By a symbol, we understand a pair (μ, A), where μ is a finite sequence of nonnegative integers and A is a right invertible proper rational matrix such that the length of μ is equal to the row number of A. These simple objects permit us to represent all linear differential/integral systems. Symbols can be regarded as the most "economic" representations; they contain the minimum of data that is necessary to define a linear differential/integral system. There is a natural partial order on the set of symbols, which allows us to say when one symbol is more powerful than another symbol. The main result of the article is Theorem 3, stating that, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two linear differential/integral systems, then $\mathcal{B}_1 \subseteq \mathcal{B}_2$ if and only if the symbol representing $\mathcal{B}_1$ is more powerful than the symbol representing $\mathcal{B}_2$. It should be pointed out that the proof of this theorem is straightforward and easy. The equivalence theorem is an immediate consequence of this result.

In this article, we also establish, in a special but interesting case, a connection with the Willems–Yamamoto definition of rational behaviors. Namely, we show that if a rational matrix has positive Wiener–Hopf indices, then the differential closure of its behavior in the sense of [3] coincides with that in the sense of Willems and Yamamoto.

The notations $\mathbb{F}$, s, and $\mathcal{U}$ will remain in force. We let O denote the ring of proper rational functions, and put $t = s^{-1}$. The symbol $\hbar$ will stand for the function that is identically 1 on the interval.

We can define a natural composition law between proper rational functions and C^∞-functions, so that $\mathcal{U}$ turns out to be a module over O (see [6,3]). For any $g \in O$, and for any $w \in \mathcal{U}$, the product gw is defined by the formula

$$gw = b_0w + b_1\int w + b_2\int^2 w + \cdots + b_n\int^n w + \cdots,$$

where $b_0, b_1, b_2, \ldots, b_n, \ldots$ are the coefficients in the expansion of g at infinity. (The reader can easily prove that the series above converges uniformly on every compact neighborhood of the initial time.)

Example 2. For $n \geq 1$ and $\lambda \in \mathbb{F}$, we have

$$\frac{1}{(s-\lambda)^n}u = \frac{x^{n-1}}{(n-1)!}e^{\lambda x} * u,$$

where $*$ denotes the convolution operation (see [3]).

Define the L-transform $L : O \to \mathcal{U}$ by the formula

$$L(g) = g\hbar.$$

It is clear that $L(1) = \hbar$ and $L(t^n) = x^n/n!$ for $n \geq 1$. More generally, we have the following.

Example 3. For $n \geq 1$ and $\lambda \in \mathbb{F}$,

$$L\left(\frac{s}{(s-\lambda)^n}\right) = \frac{x^{n-1}}{(n-1)!}e^{\lambda x}$$

(see [3]).

The two examples above together with the partial fraction expansion theorem enable us to compute all products gu and all L-transforms $L(g)$. We can see that the functions $L(g)$ are precisely Bohl functions, which play a very important role in the theory of linear systems (see, for example, Polderman and Willems [1]).

We remind readers that a square polynomial matrix U is called unimodular if it has a polynomial inverse; a square proper rational matrix A is called biproper if it has a proper rational inverse. A polynomial matrix U will be said to be left unimodular if it has a left polynomial inverse; likewise, a proper rational matrix A will be said to be right biproper if it has a right proper rational inverse.

For every integer n, we shall write $\mathbb{F}[s]_{\leq n}$ to denote the space of polynomials (in s) having degree $\leq n$; likewise, $\mathbb{F}[t]_{\leq n}$ will denote the space of polynomials (in t) having degree $\leq n$ (*Reminder*: The degree of the zero polynomial is $-\infty$).

If $\lambda = (l_1, \ldots, l_r)$ is a sequence of integers, we let s^λ (respectively, t^λ) be the diagonal matrix with s^{l_i} (respectively, t^{l_i}) on the diagonal. We set

$$\mathbb{F}[t]^r_\lambda = \bigoplus_i \mathbb{F}[t]_{\leq(l_i-1)}.$$

If μ and ν are two finite integer sequences, we shall write $\mu \simeq \nu$ when they are equal up to order.

We shall use the concept of (short) exact sequences. The sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0,$$

where M_1, M, M_2 are modules and f, g are homomorphisms, is said to be exact if f is a monomorphism, g is an epimorphism, and $Im(f) = Ker(g)$. One says that the exact sequence splits if it satisfies the following equivalent conditions.

(a) There is a homomorphism $u : M \to M_1$ such that $u \circ f$ is the identity on M_1.
(b) There is a homomorphism $v : M_2 \to M$ such that $g \circ v$ is the identity on M_2.

If these conditions are satisfied, then

$$M = Im(f) \oplus Ker(u) \quad \text{and} \quad M = Ker(g) \oplus Im(v).$$

2. Some preliminaries

This section deals with matrices with entries in $\mathbb{F}[t]$; they are interpreted as homomorphisms between matrices of the form s^μ. At the end, we recall the Wiener–Hopf factorization theorem, which will play a key role in what follows.

Let

$$\mu = (m_1, \ldots, m_p) \quad \text{and} \quad \nu = (n_1, \ldots, n_r)$$

be two integer sequences. By a homomorphism from s^μ to s^ν, we mean a proper rational matrix $(x_{ij}) \in O^{r\times p}$ such that

$$x_{ij} \in \mathbb{F}[t]_{\leq(n_i-m_j)}$$

for all i, j. Let $Hom(s^\mu, s^\nu)$ denote the set of all homomorphisms from s^μ to s^ν. Note that

$Hom(s^{\mu}, s^{\nu}) = \{X \in O^{r\times p} \mid s^{\nu} X t^{\mu} \in \mathbb{F}[s]^{r\times p}\}$.

Note also that, if k is large enough (say, such that $k + m_j \geq 0$ for all j), then

$$Hom(s^{\mu}, s^{\nu}) = \left\{ X \in O^{r\times p} \,\middle|\, X\left(\bigoplus_j \mathbb{F}[t]_{\leq(k+m_j)}\right) \subseteq \bigoplus_i \mathbb{F}[t]_{\leq(k+n_i)} \right\}.$$

Example 4.

$$Hom\left(\begin{bmatrix}1 & 0\\ 0 & s^2\end{bmatrix}, s^4\right) = \left\{\begin{bmatrix} g_1 & g_2\end{bmatrix} \mid g_1 \in \mathbb{F}[t]_{\leq 4}, g_2 \in \mathbb{F}[t]_{\leq 2}\right\}.$$

It is easily seen that the composition of two homomorphisms is a homomorphism; that is, if λ, μ, and ν are integer sequences, then

$$X \in Hom(s^{\lambda}, s^{\mu}), \quad Y \in Hom(s^{\mu}, s^{\nu}) \Rightarrow YX \in Hom(s^{\lambda}, s^{\nu}).$$

We can speak therefore about isomorphisms. If μ and ν are integer sequences, then a homomorphism $X : s^{\mu} \to s^{\nu}$ is an isomorphism if there exists a homomorphism $Y : s^{\nu} \to s^{\mu}$ such that $XY = I$ and $YX = I$.

The following lemma gives a characterization of isomorphisms.

Lemma 1. *Let μ and ν be integer sequences, and let $X \in Hom(s^{\mu}, s^{\nu})$. The following conditions are equivalent.*

(a) *X is an isomorphism.*
(b) *$\mu \simeq \nu$ and X is biproper.*

Proof. Let

$$\mu = (m_1, \ldots, m_p) \quad \text{and} \quad \nu = (n_1, \ldots, n_r).$$

(a) $\Rightarrow$ (b): That X is biproper and that $p = r$ is clear.

We certainly may assume that the terms in μ and ν are arranged in increasing order, and we have to show then that $\mu = \nu$.

Assume that $\mu \neq \nu$, and let i be the biggest number for which $m_i \neq n_i$. Say that $m_i > n_i$. Consider the following two spaces:

$$\mathbb{F}[t]_{\leq(m_1-m_i)} \oplus \cdots \oplus \mathbb{F}[t]_{\leq(m_i-m_i)} \oplus \mathbb{F}[t]_{\leq(m_{i+1}-m_i)} \oplus \cdots \oplus \mathbb{F}[t]_{\leq(m_p-m_i)}$$

and

$$\mathbb{F}[t]_{\leq(n_1-m_i)} \oplus \cdots \oplus \mathbb{F}[t]_{\leq(n_i-m_i)} \oplus \mathbb{F}[t]_{\leq(n_{i+1}-m_i)} \oplus \cdots \oplus \mathbb{F}[t]_{\leq(n_p-m_i)}.$$

The matrix X induces a linear map from the first one to the second. Because X is biproper, this map is injective. On the other hand, the summands $\mathbb{F}[t]_{\leq(n_1-m_i)}, \ldots, \mathbb{F}[t]_{\leq(n_i-m_i)}$ are zero, and consequently the first space has greater dimension than the second one.

This contradiction proves what we want.

(a) $\Leftarrow$ (b): Certainly, $p = r$. Take sufficiently large k (so that $k + m_i \geq 0$ for each i). Then X determines a linear map

$$\mathbb{F}[t]_{\leq(k+m_1)} \oplus \cdots \oplus \mathbb{F}[t]_{\leq(k+m_p)} \to \mathbb{F}[t]_{\leq(k+n_1)} \oplus \cdots \oplus \mathbb{F}[t]_{\leq(k+n_p)}.$$

This linear map is injective. Because the spaces have equal dimensions, it must be bijective. The inverse linear map is induced by the matrix X^{-1}. We thus have

$$X^{-1}\left(\bigoplus_i \mathbb{F}[t]_{\leq(k+n_i)}\right) \subseteq \bigoplus_i \mathbb{F}[t]_{\leq(k+m_i)},$$

and hence X^{-1} is a homomorphism of s^{ν} into s^{μ}.

The proof is complete. $\square$

It follows from this lemma (and from the fact that the composition of two homomorphisms is a homomorphism) that the biproper rational matrices in $Hom(s^{\mu}, s^{\mu})$ form a group. This is denoted by $Aut(s^{\mu})$. Notice that this is precisely the group of Brunovsky μ-transformations. (We remind readers that a Brunovsky μ-transformation is a biproper rational matrix $X \in O^{p\times p}$ such that $s^{\mu} X t^{\mu} \in \mathbb{F}[s]^{p\times p}$ is a unimodular polynomial matrix.)

Example 5.

$$Aut\left(\begin{bmatrix}1 & 0\\ 0 & s^2\end{bmatrix}\right) = \left\{\begin{bmatrix} a & 0\\ b_0t^2 + b_1t + b_2 & c\end{bmatrix} \,\middle|\, a, b_0, b_1, b_2, c \in \mathbb{F}, a, c \neq 0\right\}.$$

Closing the section, we formulate the Wiener–Hopf factorization theorem in the form we shall need. No doubt, this is the deepest fact about rational matrices.

Lemma 2 (*Wiener–Hopf Factorization Theorem*)**.** *Let G be a rational matrix of size $p \times q$ and rank r.*

(a) There exists a factorization

$$G = Us^{\mu}A,$$

where μ is a sequence of integers of length r, U is a left unimodular polynomial matrix of size $p\times r$, and A is a right biproper rational matrix of size $r \times q$.

(b) If

$$G = U_1 s^{\mu_1} A_1 \quad \text{and} \quad G = U_2 s^{\mu_2} A_2$$

are two such factorizations, then there exists an isomorphism $X \in Hom(s^{\mu_1}, s^{\mu_2})$ such that

$$A_2 = XA_1 \quad \text{and} \quad U_1 = U_2(s^{\mu_1} X t^{\mu_2}).$$

3. Symbols

In this section, we introduce our main objects, which are called symbols.

We start with the remark that, because $\partial^{-1} \circ \partial \neq id$, one should not expect the equality

$$(G_1G_2)(\partial) = G_1(\partial) \circ G_2(\partial)$$

in general. However, as the following lemma says, this equality does hold to be true in two important cases.

Lemma 3. *(a) Let P be a polynomial matrix and G a rational matrix (such that the column number of P is equal to the row number of G). Then*

$$(PG)(\partial) = P(\partial) \circ G(\partial).$$

(b) Let G be a rational matrix and A a proper rational matrix (such that the column number of G is equal to the row number of A). Then

$$(GA)(\partial) = G(\partial) \circ A(\partial).$$

Proof. Using $\partial \circ \partial^{-1} = id$, one can easily verify that the relation

$$\partial^n \circ \partial^m = \partial^{m+n}$$

is true in the following two special cases.

Case 1: $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}$.

Case 2: $n \in \mathbb{Z}$, $m \in \mathbb{Z}_-$.

By linearity, statement (a) follows from Case 1 and statement (b) follows from Case 2.

The proof is complete. $\square$

Lemma 3(a) has the following important consequence.

Corollary 1. *If G_1 and G_2 are rational matrices such that $G_2 = PG_1$ for some polynomial matrix P, then*

$$Bh(G_1) \subseteq Bh(G_2).$$

Two rational matrices G_1 and G_2 (with the same column number) are said to be strongly equivalent if there exist polynomial matrices P and Q such that $G_2 = PG_1$ and $G_1 = QG_2$. It immediately follows from the above corollary that strongly equivalent rational matrices have the same behavior. The converse is not true, of course.

Important invariants of a rational matrix are the transfer function and the initial condition space (see [3]). If G is a rational matrix of size $p \times q$, then its transfer function is defined as the set

$$\{g \in O^q \mid Gg = 0\};$$

the initial condition space is defined to be the space

$$\mathbb{F}[s]^p \cap tGO^q.$$

The dimension of this latter is called the McMillan degree.

It is worth noting that, if A is a proper rational matrix, say, of size $p \times q$, then $A(\partial)$ is nothing but the operator

$$\mathcal{U}^q \xrightarrow{A} \mathcal{U}^p, \qquad w \mapsto Aw.$$

Lemma 4. *Let A be a right invertible proper rational matrix of size $p \times q$. Then the operator*

$$\mathcal{U}^q \xrightarrow{A} \mathcal{U}^p, \qquad w \mapsto Aw$$

is surjective.

Proof. To see this, it suffices to tensor by $\mathcal{U}$ the surjective map

$$O^q \xrightarrow{A} O^p,$$

and use the general well-known fact that tensoring a surjective homomorphism by a module yields another surjective homomorphism.

An elementary proof is possible. Indeed, let A_1 be a right proper rational inverse of A. Then the composition

$$O^p \xrightarrow{A_1} O^q \xrightarrow{A} O^p$$

is identical. Consequently, the composition

$$\mathcal{U}^p \xrightarrow{A_1} \mathcal{U}^q \xrightarrow{A} \mathcal{U}^p$$

is identical as well. □

Given a sequence $\lambda = (l_1, \dots, l_r)$ of arbitrary integers, we define

$$\lambda^+ = (\max\{l_1, 0\}, \dots, \max\{l_r, 0\}) \quad \text{and} \quad |\lambda| = l_1 + \dots + l_r.$$

Let G be a rational matrix of size $p \times q$ and rank r, and let $G = Us^\mu A$ be its Wiener–Hopf factorization. We remind readers that this means that μ is a sequence of r integers, U is a left unimodular polynomial matrix of size $p \times r$, and A is a right biproper rational matrix of size $r \times q$.

Theorem 1. *We have*

$$Bh(G) = \{w \in \mathcal{U}^q \mid Aw \in L(\mathbb{F}[t]^r_{\mu+})\}.$$

Proof. Multiplication from the left of $G = Us^\mu A$ by a left polynomial inverse U_1 of U gives $s^\mu A = U_1 G$. Hence, the matrices G and $s^\mu A$ are strongly equivalent, and therefore

$$Bh(G) = Bh(s^\mu A).$$

Further, because A is proper, by Lemma 3(b), the operator $(s^\mu A)(\partial)$ is equal to the composition

$$\mathcal{U}^q \xrightarrow{A} g\mathcal{U}^r \xrightarrow{\partial^\mu} \mathcal{U}^r.$$

Hence,

$$Bh(G) = \{w \in \mathcal{U}^q | \partial^\mu(Aw) = 0\} = \{w \in \mathcal{U}^q | \, Aw \in Ker(\partial^\mu)\}.$$

It is clear that

$$Ker(\partial^\mu) = L(\mathbb{F}[t]^r_\mu).$$

The trivial observation that $\mathbb{F}[t]^r_\mu = \mathbb{F}[t]^r_{\mu+}$ completes the proof. □

As we saw in Lemma 4, the map

$$\mathcal{U}^q \xrightarrow{A} \mathcal{U}^r, \qquad w \mapsto Aw$$

is surjective. The theorem above tells us that $Bh(G)$ is the preimage of

$$L(\mathbb{F}[t]^r_{\mu+}) \subseteq \mathcal{U}^r$$

under this map.

Theorem 1 leads to the notion of what we call symbols.

Definition. A symbol is a pair (μ, A), where μ is a sequence of nonnegative integers and A is a right biproper rational matrix such that the length of μ is equal to the row number of A.

Let (μ, A) be a symbol. The following definitions are obvious. The signal number of (μ, A) is the column number of A; the output number is the row number of A, and the input number is the signal number minus the output number. If the signal number of (μ, A) is equal to q and the output number to p, then its behavior is

$$Bh(\mu, A) = \{w \in \mathcal{U}^q \mid Aw \in L(\mathbb{F}[t]^p_\mu)\};$$

the transfer function is

$$\{g \in O^q \mid Ag = 0\};$$

the initial condition space is

$$\mathbb{F}[t]^p_\mu;$$

and the McMillan degree is $|\mu|$.

The following gives a complete description of the behavior of a symbol.

Theorem 2. *Let (μ, A) be a symbol with signal number q, output number p, and input number m. Let A_1 be a right proper rational inverse of A, and let A_2 be a maximal (proper rational) right annihilator of A. Then*

$$Bh(\mu, A) = A_2\mathcal{U}^m \oplus L(A_1\mathbb{F}[t]^p_\mu).$$

Proof. We have an exact sequence

$$0 \to O^m \xrightarrow{A_2} O^q \xrightarrow{A} O^p \to 0,$$

which splits because $AA_1 = I$. Tensoring this by $\mathcal{U}$, we get therefore a split exact sequence

$$0 \to \mathcal{U}^m \xrightarrow{A_2} \mathcal{U}^q \xrightarrow{A} \mathcal{U}^p \to 0.$$

By definition, the linear map

$$Bh(G) \xrightarrow{A} L(\mathbb{F}[t]^p_\mu)$$

is surjective. By the exact sequence above, the kernel of this map is equal to $A_2\mathcal{U}^m$. The theorem follows now from the decomposition

$$\mathcal{U}^q = A_2\mathcal{U}^m \oplus A_1\mathcal{U}^p.$$

The proof is complete. □

Example 6. Consider the symbol (μ, A), where

$$\mu = (0, 2) \quad \text{and} \quad A = \begin{bmatrix} \dfrac{4s^2-1}{s^2} & -1 & 0 \\ \dfrac{1}{(s-1)^2} & \dfrac{s}{(s+2)^2} & \dfrac{s}{s-5} \end{bmatrix}.$$

The matrix

$$A_1 = \begin{bmatrix} \dfrac{s^2}{4s^2-1} & 0 \\ 0 & 0 \\ \dfrac{-(s-5)s}{(s-1)^2(4s^2-1)} & \dfrac{s-5}{s} \end{bmatrix}$$

is a right proper rational inverse of A, and the matrix

$$A_2 = \begin{bmatrix} \dfrac{-s}{s-5} \\ \dfrac{1-4s^2}{(s-5)s} \\ \dfrac{4s^2-1}{s(s+2)^2} + \dfrac{1}{(s-1)^2} \end{bmatrix}$$

is a maximal (proper rational) right annihilator of A.

By Theorem 2, every trajectory w can be written in a unique way as

$$\begin{aligned} w &= A_2 u + A_1 L\left(\begin{bmatrix} 0 \\ C_1 + C_2 t \end{bmatrix}\right) \\ &= A_2 u + \begin{bmatrix} 0 \\ 0 \\ C_1 L(1-5t) + C_2 L(t-5t^2) \end{bmatrix}, \end{aligned}$$

where $u \in \mathcal{U}$ and C_1, C_2 are constants. Using the partial fraction expansions

$$\frac{-s}{s-5} = -1 + \frac{-5}{s-5},$$
$$\frac{1-4s^2}{(s-5)s} = -4 - \frac{99/5}{s-5} - \frac{1/5}{s},$$
$$\frac{4s^2-1}{s(s+2)^2} + \frac{1}{(s-1)^2} = \frac{-1/4}{s} + \frac{17/4}{s+2} + \frac{-15/2}{(s+2)^2} + \frac{1}{(s-1)^2}$$

and Example 3, we find

$$w = \begin{bmatrix} -u - 5e^{5x} * u \\ -4u - \dfrac{99}{5}e^{5x} * u - \dfrac{1}{5}\displaystyle\int u \\ -\dfrac{1}{4}\displaystyle\int u + \left(xe^x + \dfrac{17}{4}e^{-2x} - \dfrac{15}{2}xe^{-2x}\right) * u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ C_1(1-5x) + C_2\left(x - \dfrac{5x^2}{2}\right) \end{bmatrix}.$$

4. More powerfulness and equivalence

The goal of this section is to define a partial order on symbols, and then an equivalence between symbols.

We need to modify $Hom(s^\mu, s^\nu)$ for nonnegative integer sequences μ and ν.

Given two sequences of nonnegative integers

$$\mu = (m_1, \ldots, m_p) \quad \text{and} \quad \nu = (n_1, \ldots, n_r),$$

we call a *relaxed* homomorphism from s^μ to s^ν any proper rational matrix (x_{ij}) of size $r \times p$ such that

$$x_{ij} \in \mathbb{F}[t]_{\le(n_i - m_j)} \quad \text{if } m_j \ge 1.$$

(Notice that the entries $x_{1j}, \ldots, x_{rj}$ are unrestricted when $m_j = 0$.) The set of all relaxed homomorphisms from s^μ to s^ν will be denoted by $\overline{Hom}(s^\mu, s^\nu)$. Obviously,

$$Hom(s^\mu, s^\nu) \subseteq \overline{Hom}(s^\mu, s^\nu).$$

In case, when all $m_j \ge 1$, we certainly have

$$Hom(s^\mu, s^\nu) = \overline{Hom}(s^\mu, s^\nu).$$

Note that

$$\overline{Hom}(s^\mu, s^\nu) = \{X \in O^{p\times q} \mid X\mathbb{F}[t]^p_\mu \subseteq \mathbb{F}[t]^r_\nu\}.$$

Example 7.

$$\overline{Hom}\left(\begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix}, s^4\right) = \left\{\begin{bmatrix} g_1 & g_2 \end{bmatrix} \mid g_1 \in O,\ g_2 \in \mathbb{F}[t]_{\le 2}\right\}.$$

It is clear that the composition of two relaxed homomorphisms is a relaxed homomorphism. More precisely, if λ, μ, and ν are sequences of nonnegative integers, then

$$\begin{aligned} &X \in \overline{Hom}(s^\lambda, s^\mu), \\ &Y \in \overline{Hom}(s^\mu, s^\nu) \ \Rightarrow\ YX \in \overline{Hom}(s^\lambda, s^\nu). \end{aligned}$$

This permits us to speak about relaxed isomorphisms. If μ and ν are sequences of non-negative integers, we say that $X \in \overline{Hom}(s^\mu, s^\nu)$ is a relaxed isomorphism if there exists a $Y \in \overline{Hom}(s^\nu, s^\mu)$ such that $XY = I$ and $YX = I$.

The following lemma is similar to Lemma 1; it characterizes relaxed isomorphisms.

Lemma 5. *Let μ and ν be sequences of nonnegative integers, and let $X \in \overline{Hom}(s^\mu, s^\nu)$. The following conditions are equivalent.*
(a) X is a relaxed isomorphism.
(b) $\mu \simeq \nu$ and X is biproper.

Proof. We may assume that both $\mu = (m_1, \ldots, m_p)$ and $\nu = (n_1, \ldots, n_r)$ are increasing sequences.

(a) $\Rightarrow$ (b): Clearly, X is biproper and $p = r$. Let a be the number of zeros in μ and b the number of zeros in ν, and suppose that $a \le b$. Since $n_1 = \cdots = n_a = 0$ and $m_{a+1} = \cdots = m_p \ge 1$, we have

$$\mathbb{F}[t]_{\le(n_i - m_j)} = \mathbb{F}[t]_{\le(-m_j)} = \{0\}$$

for $i = 1, \ldots, a$ and $j = a+1, \ldots, p$. It follows from this that X has the form

$$\begin{bmatrix} X_1 & 0 \\ \star & X_2 \end{bmatrix},$$

where X_1 and X_2 are square biproper rational matrices of size a and $p-a$, respectively. If $a < b$, we would obtain that the first row in X_2 is zero, which is impossible, since X_2 is biproper. Hence, $a = b$.

Let $\tilde\mu = (m_{a+1}, \ldots, m_p)$ and $\tilde\nu = (n_{a+1}, \ldots, n_p)$. Because all the entries in $\tilde\mu$ are greater than or equal to 1,

$$\overline{Hom}(s^{\tilde\mu}, s^{\tilde\nu}) = Hom(s^{\tilde\mu}, s^{\tilde\nu}).$$

It follows that $X_2 \in Hom(s^{\tilde\mu}, s^{\tilde\nu})$. It is clear that X_2 is an isomorphism of $s^{\tilde\mu}$ onto $s^{\tilde\nu}$, and, applying Lemma 1, we get that $\tilde\mu = \tilde\nu$. Hence, $\mu = \nu$.

(a) $\Leftarrow$ (b): In view of our assumption, we have $\mu = \nu$. Let again a be the number of zeros in μ, and let $\tilde\mu = (m_{a+1}, \ldots, m_p)$. Write X as

$$X = \begin{bmatrix} X_1 & 0 \\ Z & X_2 \end{bmatrix},$$

where X_1 and X_2 are square biproper rational matrices of size a and $p-a$, respectively, and Z is a proper rational matrix of size $(p-a) \times a$. We have

$$Z \in \overline{Hom}(I_a, s^{\tilde\mu}) \quad \text{and} \quad X_2 \in \overline{Hom}(s^{\tilde\mu}, s^{\tilde\mu}).$$

(Here and in what follows, I_a denotes the identity matrix of size a). Clearly,

$$X^{-1} = \begin{bmatrix} X_1^{-1} & 0 \\ -X_2^{-1}ZX_1^{-1} & X_2^{-1} \end{bmatrix}.$$

From Lemma 1, and from the fact that $\overline{Hom}(s^{\tilde{\mu}}, s^{\tilde{\mu}}) = Hom(s^{\tilde{\mu}}, s^{\tilde{\mu}})$, we obtain that $X_2^{-1} \in \overline{Hom}(s^{\tilde{\mu}}, s^{\tilde{\mu}})$. Next, it is easily seen that $X_2^{-1}ZX_1^{-1} \in \overline{Hom}(I_a, s^{\tilde{\mu}})$. It follows that

$$X^{-1} \in \overline{Hom}(s^{\mu}, s^{\mu}).$$

Hence, X is an isomorphism.

The proof is complete. □

By the above lemma, the biproper rational matrices in $\overline{Hom}(s^{\mu}, s^{\mu})$ form a group; we denote it by $\overline{Aut}(s^{\mu})$.

Example 8.

$$\overline{Aut}\left(\begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix}\right) = \left\{ \begin{bmatrix} u & 0 \\ v & c \end{bmatrix} \middle| u, v \in O, u \notin tO, c \in \mathbb{F}, c \neq 0 \right\}.$$

The following says that the "plus-map" preserves homomorphisms.

Lemma 6. *Let*

$$\mu = (m_1, \dots, m_p) \quad \textit{and} \quad \nu = (n_1, \dots, n_r),$$

be two sequences of arbitrary integers. Then

$$Hom(s^{\mu}, s^{\nu}) \subseteq \overline{Hom}(s^{\mu^+}, s^{\nu^+}).$$

Proof. Take any i and j, and consider two cases.

Case 1: $m_j \leq 0$. The j-th component of μ^+ is 0, and therefore the entries at the (i, j)-th position of matrices in $\overline{Hom}(s^{\mu^+}, s^{\nu^+})$ are unrestricted.

Case 2: $m_j \geq 1$. We have $n_i - m_j \leq \max\{n_i, 0\} - m_j$.

The proof is complete. □

Note that, if A and B are right biproper rational matrices, then there may exist only one proper rational matrix X such that $B = XA$. (If such a matrix exists, then it is equal to BA_1, where A_1 is a right proper rational inverse of A.)

We are now finally in a position to give the following definition.

Definition. Let (μ, A) and (ν, B) be symbols with the same signal number. We shall say that (μ, A) is more powerful than (ν, B) and write $(\mu, A) \succeq (\nu, B)$, if

$$\exists X \in \overline{Hom}(s^{\mu}, s^{\nu}) \quad \text{such that } B = XA.$$

Example 9. The symbol

$$\left((0, 2), \begin{bmatrix} \dfrac{4s^2 - 1}{s^2} & -1 & 0 \\ \dfrac{1}{(s-1)^2} & \dfrac{s}{(s+2)^2} & \dfrac{s}{s-5} \end{bmatrix}\right)$$

is more powerful than the symbol

$$\left(4, \begin{bmatrix} \dfrac{2s^3 - 3s^2 + 4}{s(s-1)^2} & \dfrac{-s^3 - 4s^2 + 2s - 3}{(s-2)^2(2s-1)} & \dfrac{3}{s-5} \end{bmatrix}\right).$$

Indeed, the matrix of the second symbol is obtained from that of the first one by left multiplication by

$$\begin{bmatrix} \dfrac{s}{2s-1} & \dfrac{3}{s} \end{bmatrix},$$

which belongs to $\overline{Hom}\left(\begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix}, s^4\right)$.

The definition of equivalence of symbols is obvious: two symbols (with the same signal number) are equivalent if and only if each of them is more powerful than the other.

Lemma 7. *Let (μ, A) and (ν, B) be two symbols with the same signal number. They are equivalent if and only if there is an isomorphism $X \in \overline{Hom}(s^{\mu}, s^{\nu})$ such that $B = XA$.*

Proof. The "If" part is obvious. To show the "Only if" part, assume that $(\mu, A) \succeq (\nu, B)$ and $(\nu, B) \succeq (\mu, A)$. Then

$$B = XA \quad \text{and} \quad A = YB,$$

where $X \in \overline{Hom}(s^{\mu}, s^{\nu})$ and $Y \in \overline{Hom}(s^{\nu}, s^{\mu})$. We have

$$B = XYB \quad \text{and} \quad A = YXA;$$

hence, $XY = I$ and $YX = I$. This means that X is an isomorphism.

The proof is complete. □

It is clear that equivalent symbols have the same input and output numbers, the same transfer function, the same initial condition space, and the same McMillan degree. In the next section, we shall see that they have the same behavior as well.

5. Inclusion lemma and equivalence theorem

In this section, we state and prove our main results.

We need the following lemma. It says that the transfer function of a symbol can be defined in terms of its behavior.

Lemma 8. *Let (μ, A) be a symbol with signal number q, and let $\mathcal{B}$ be its behavior and T its transfer function. Then*

$$T = \{g \in O^q \mid g\mathcal{U} \subseteq \mathcal{B}\}.$$

Proof. Let p denote the output number. If $g \in T$, then $Ag = 0$, and therefore, for every $u \in \mathcal{U}$, we have

$$A(gu) = (Ag)u = 0u = 0 \in \mathbb{F}[t]^p_{\mu}.$$

Hence, $gu \in \mathcal{B}$. Conversely, suppose that $g \in O^q$ is such that $g \notin T$, i.e., $Ag \neq 0$. It is clear that, if we take n to be sufficiently large, then

$$t^n Ag \notin \mathbb{F}[t]^p_{\mu}.$$

Consequently, $g(t^n \hbar)$ is not a trajectory of $\mathcal{B}$.

The proof is complete. □

Theorem 3 (*Inclusion Lemma*)**.** *Let (μ, A) and (ν, B) be two symbols (with the same signal number q). Then*

$$Bh(\mu, A) \subseteq Bh(\nu, B) \quad \Leftrightarrow \quad (\mu, A) \succeq (\nu, B).$$

Proof. "⇒": By the previous lemma, the transfer function of (μ, A) is contained in that of (ν, B). Let

$$T_A = \{g \in O^q \mid Ag = 0\} \quad \text{and} \quad T_B = \{g \in O^q \mid Bg = 0\},$$

and let $T_A \to T_B$ be the inclusion map. Letting p and r denote the output numbers of (μ, A) and (ν, B), respectively, we have a diagram

$$\begin{array}{ccccccccc} 0 \to & T_A & \to & O^q & \xrightarrow{A} & O^p & \to 0 \\ & \downarrow & & \| & & & \\ 0 \to & T_B & \to & O^q & \xrightarrow{B} & O^r & \to 0, \end{array}$$

where the rows are exact and the square is commutative. It immediately follows from this that $B = XA$ for some proper rational matrix X.

Next, the hypothesis implies that

$$B(Bh(\mu, A)) \subseteq B(Bh(\nu, B)).$$

Because

$$B(Bh(\mu,A)) = XA(Bh(\mu,A)) = XL(\mathbb{F}[t]^r_\mu) \quad \text{and}$$
$$B(Bh(\nu,B)) = L(\mathbb{F}[t]^r_\nu),$$

we get

$$XL(\mathbb{F}[t]^p_\mu) \subseteq L(\mathbb{F}[t]^r_\nu).$$

From this, we certainly have

$$X\mathbb{F}[t]^p_\mu \subseteq \mathbb{F}[t]^r_\nu.$$

This means that X is a relaxed homomorphism from s^μ to s^ν.

"$\Leftarrow$": This is easy, and is left to the reader.

The proof is complete. □

The following is an immediate consequence of the above theorem.

Corollary 2 (*Equivalence Theorem*)**.** *Two symbols determine the same behavior if and only if they are equivalent.*

Let G be a rational matrix, and let $G = Us^\mu A$ be its Wiener–Hopf factorization. We call (μ^+, A) a symbol of G.

A symbol of a rational matrix is not uniquely determined, of course. But, as the following lemma says, it is uniquely determined up to equivalence.

Lemma 9. *If $G = U_1 s^{\mu_1} A_1$ and $G = U_2 s^{\mu_2} A_2$ are two Wiener–Hopf factorizations of G, then (μ_1^+, A_1) and (μ_2^+, A_2) are equivalent.*

Proof. By Lemma 2(b), there exists an isomorphism $X \in Hom(s^{\mu_1}, s^{\mu_2})$ such that $A_2 = XA_1$. In view of Lemma 6, $X \in \overline{Hom}(s^{\mu_1^+}, s^{\mu_2^+})$. Because X is biproper, the lemma follows. □

If G is a rational matrix and if (μ, A) is its symbol, then, by Theorem 1,

$$Bh(G) = Bh(\mu, A).$$

Lemma 9 permits us to extend the notion of more powerfulness to rational matrices. If G_1 and G_2 are two rational matrices, we say that G_1 is more powerful than G_2 and write $G_1 \succeq G_2$, if a symbol of G_1 is more powerful than a symbol of G_2. One can define in an obvious way the equivalence between rational matrices.

Example 10. Let

$$G_1 = \begin{bmatrix} \dfrac{4s^2-1}{s^3} & \dfrac{-1}{s} & 0 \\ \dfrac{-8s^3+3s^2+2s-1}{s^2(s-1)^2} & \dfrac{-4s^3-s^2-4s-4}{(s+2)^2} & \dfrac{-4s^3}{s-5} \end{bmatrix}.$$

The Wiener–Hopf factorization of this matrix is

$$G_1 = \begin{bmatrix} 1 & 0 \\ s & -4 \end{bmatrix} \begin{bmatrix} s^{-1} & 0 \\ 0 & s^2 \end{bmatrix} \begin{bmatrix} \dfrac{4s^2-1}{s^2} & -1 & 0 \\ \dfrac{1}{(s-1)^2} & \dfrac{s}{(s+2)^2} & \dfrac{s}{s-5} \end{bmatrix},$$

and therefore it is more powerful than

$$G_2 = s^4 \begin{bmatrix} \dfrac{2s^3-3s^2+4}{s(s-1)^2} & \dfrac{-s^3-4s^2+2s-3}{(s-2)^2(2s-1)} & \dfrac{3}{s-5} \end{bmatrix}$$

(see Example 9).

We leave to the reader the obvious reformulation of Inclusion Lemma and Equivalence Theorem in terms of rational matrices.

We shall now try to derive from Theorem 3 the following well-known result, which, to our knowledge, was first proved by Schumacher [7].

Corollary 3. *Let Q and R be two polynomial matrices with the same column number. Then*

$$Bh(Q) \subseteq Bh(R) \quad \Leftrightarrow \quad \exists P \in \mathbb{F}[s]^{\bullet\times\bullet} \text{ such that } R = PQ.$$

Proof. "$\Rightarrow$": Let

$$Q = Us^\mu A \quad \text{and} \quad R = Vs^\nu B$$

be Wiener–Hopf factorizations of our matrices. Then (μ, A) and (ν, B) are symbols of Q and R, respectively. If $Bh(Q) \subseteq Bh(R)$, then, by Inclusion Lemma, $(\mu, A) \succeq (\nu, B)$. Hence, $B = XA$ for some $X \in \overline{Hom}(s^\mu, s^\nu)$.

We need to prove that

$$X \in Hom(s^\mu, s^\nu).$$

If we can prove this, it will follow that

$$R = Vs^\nu B = Vs^\nu XA = V(s^\nu X t^\mu)s^\mu A = V(s^\nu X t^\mu)KQ,$$

where K is a left polynomial inverse of U, and this will finish the proof.

Let a be the number of the zero components in μ. Let 0_a denote the sequence of a zeros. Reordering (if necessary) the entries in μ, we can write $\mu = (0_a, \lambda)$. The matrix A can be written in the form

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

Because $s^\mu A$ is a polynomial matrix, we can see from

$$\begin{bmatrix} I_a & 0 \\ 0 & s^\lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = s^\mu A$$

that A_1 is a polynomial matrix. On the other hand, this is a proper rational matrix. Hence, A_1 is a scalar matrix.

We have $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, where $X_1 \in \overline{Hom}(I_a, s^\nu)$ and $X_2 \in \overline{Hom}(s^\lambda, s^\nu)$. Because all the components of λ are positive, $\overline{Hom}(s^\lambda, s^\nu) = Hom(s^\lambda, s^\nu)$, and therefore the matrix $s^\nu X_2 t^\lambda$ is a polynomial matrix in s.

By the hypothesis, $B = X_1 A_1 + X_2 A_2$. From this, we have

$$s^\nu B = s^\nu X_1 A_1 + (s^\nu X_2 t^\lambda) s^\lambda A_2.$$

Because A_1 has a right scalar inverse, we can see from this equality that $s^\nu X_1$ is a polynomial matrix in s. It follows that

$$s^\nu X t^\mu = \begin{bmatrix} s^\nu X_1 & s^\nu X_2 \end{bmatrix} \begin{bmatrix} I_a & 0 \\ 0 & t^\lambda \end{bmatrix} = \begin{bmatrix} s^\nu X_1 \\ s^\nu X_2 t^\nu \end{bmatrix}$$

is a polynomial matrix in s.

"$\Leftarrow$": This is obvious.

The proof is complete. □

6. Conclusion

Given an integer sequence $\lambda = (l_1, \ldots, l_r)$, let

$$s^\lambda = \operatorname{diag}(s^{l_1}, \ldots, s^{l_r}) \quad \text{and} \quad \lambda^+ = (\max\{l_1, 0\}, \ldots, \max\{l_r, 0\}).$$

If G is a rational matrix, then, according to the Wiener–Hopf factorization theorem, it admits a factorization $G = Us^\lambda A$, where λ is a sequence of integers, U is a left unimodular polynomial matrix, and A is a right biproper rational matrix. It turns out that the pair (λ^+, A) is exactly what is needed for the definition of the behavior of G. (The matrix U and the negative entries in λ are irrelevant.) It is reasonable therefore to focus attention on pairs (μ, A), where μ is a finite sequence of nonnegative integers and A is a right invertible proper rational matrix such that the length of μ is equal to the row number of A. Such pairs have been called symbols. For every symbol (μ, A), we have defined the behavior $Bh(\mu, A)$. There is a natural partial order on the set of symbols, denoted by $\succeq$. And we have shown that if (μ, A) and (ν, B) are two symbols with the same signal number, then

$$Bh(\mu, A) \subseteq Bh(\nu, B) \quad \Leftrightarrow \quad (\mu, A) \succeq (\nu, B).$$

This, in turn, leads to a condition for the equality of behaviors.

Appendix. Connection with the Willems–Yamamoto approach

Given a rational matrix G with left coprime factorization $G = P^{-1}Q$, Willems and Yamamoto (see [4,5]) define the behavior of G as the solution set of the differential equation

$$Q(\partial)w = 0.$$

In this Appendix, we address the following question: Does there exists a connection between the two definitions of the behavior of a rational matrix?

Obviously,

$$Bh(G) \subseteq Bh(Q).$$

We are going to show that, in the case when the Wiener-Hopf indices of G are positive, $Bh(Q)$ is the differential closure of $Bh(G)$, that is, the smallest differentiation-invariant subspace containing $Bh(G)$.

The following example tells us that one should not expect the existence of any relationship in the general case.

Example 11. The behavior of the rational function $(s-1)^{-1}s$ is $\{0\}$; the behavior of s, in the sense of Willems–Yamamoto, is the set of all constant functions.

The following lemma implies that, for every rational matrix G, we can find a full row rank rational matrix G_1 that is strongly equivalent to G and has a numerator that is equivalent to a numerator of G.

Lemma 10. *Let G be a rational matrix of size $p \times q$ and rank r, and let $G = P^{-1}Q$ be its left coprime factorization. Then, there exist a nonsingular polynomial matrix P_1 of size $r \times r$, a full row rank polynomial matrix Q_1 of size $r \times q$, and left unimodular polynomial matrices U, V of size $p \times r$ such that*

- *P_1 and Q_1 are left coprime;*
- *$Q = UQ_1$;*
- *$G = VG_1$, where $G_1 = P_1^{-1}Q_1$.*

Proof. We can find a left unimodular polynomial matrix U and a full row rank polynomial matrix Q_1 such that $Q = UQ_1$. By hypothesis, there exist polynomial matrices X and Y such that $PX + QY = I$. We then have $PX + UQ_1Y = I$. This can be written as

$$\begin{bmatrix} U & -P \end{bmatrix} \begin{bmatrix} Q_1Y \\ -X \end{bmatrix} = I.$$

From this it follows that there are P_1 and V such that the sequence

$$0 \to \mathbb{F}[s]^r \xrightarrow{\begin{bmatrix} P_1 \\ V \end{bmatrix}} \mathbb{F}[s]^r \oplus \mathbb{F}[s]^p \xrightarrow{\begin{bmatrix} U & -P \end{bmatrix}} \mathbb{F}[s]^p \to 0,$$

is an exact sequence. It follows also that this is a split exact sequence, and that $\begin{bmatrix} Q_1Y \\ -X \end{bmatrix}$ splits $\begin{bmatrix} U & -P \end{bmatrix}$.

We therefore have an isomorphism

$$\begin{bmatrix} P_1 & Q_1Y \\ V & -X \end{bmatrix} : \mathbb{F}[s]^r \oplus \mathbb{F}[s]^p \to \mathbb{F}[s]^r \oplus \mathbb{F}[s]^p.$$

It follows that there exist polynomial matrices A and B such that

$$\begin{bmatrix} P_1 & Q_1Y \\ V & -X \end{bmatrix} \begin{bmatrix} A & B \\ U & -P \end{bmatrix} = I \quad \text{and}$$

$$\begin{bmatrix} A & B \\ U & -P \end{bmatrix} \begin{bmatrix} P_1 & Q_1Y \\ V & -X \end{bmatrix} = I.$$

We have, in particular, the following relations:

$$P_1A + Q_1YU = I, \qquad AP_1 + BV = I, \qquad UP_1 - PV = 0.$$

The first relation tells us that P_1 and Q_1 are left coprime. Choosing a left polynomial inverse K of U, from the two other relations, we get that $(AKP + B)V = I$; hence, V is left unimodular. It is clear from $UP_1 = PV$ that P_1 is nonsingular. Finally, we have $G = P^{-1}Q = P^{-1}UQ_1 = VP_1^{-1}Q_1$.

The proof is complete. □

We want to present the following consequence (although we shall not need it).

Corollary 4. *Let G_1 and G_2 be two rational matrices (with the same column number), and let $G_1 = P_2^{-1}Q_1$ and $G_2 = P_2^{-1}Q_2$ be their left coprime factorizations. If G_1 and G_2 are strongly equivalent, then Q_1 and Q_2 are equivalent.*

Proof. The lemma above allows us to reduce the proof to the case when both G_1 and G_2 have full row rank. In this case, if G_1 and G_2 are strongly equivalent, then $G_2 = UG_1$ for some unimodular polynomial matrix U. We then have

$$G_2 = UP_1^{-1}Q_1 = (P_1U^{-1})^{-1}Q_1.$$

This means that $(P_1U^{-1})^{-1}Q_1$ is a left coprime factorization of G_2.

One easily completes the proof. □

Remark. The corollary implies that, if G_1 and G_2 are strongly equivalent, then they have the same behavior in the sense of Willems and Yamamoto. The converse is not true, of course. In Gottimukkala et al. [8], the reader can find a necessary and sufficient condition for these behaviors to be equal.

We shall need also the following lemma, which is of some independent interest.

Lemma 11. *Let G and H be two rational matrices with the same column number, and suppose that the Wiener–Hopf indices of G are positive. Then*

$$Bh(G) \subseteq Bh(H) \quad \Leftrightarrow \quad \exists P \in \mathbb{F}[s]^{\bullet\times\bullet} \text{ such that } H = PG.$$

Proof. Let

$$G = Us^{\mu}A \quad \text{and} \quad H = Vs^{\nu}B$$

be Wiener–Hopf factorizations.

If $Bh(G) \subseteq Bh(H)$, then $(\mu, A) \succeq (\nu, B)$, and consequently there is $X \in \overline{Hom}(\mu, \nu)$ such that $B = XA$. The hypothesis about the Wiener–Hopf indices of G implies that $\overline{Hom}(\mu, \nu) = Hom(\mu, \nu)$. So, $s^{\nu}Xt^{\mu}$ is a polynomial matrix (in s). We have

$$H = Vs^{\nu}B = Vs^{\nu}XA = V(s^{\nu}Xt^{\mu})s^{\mu}A.$$

Selecting a left polynomial inverse K of U, we get $s^{\mu}A = KG$. Therefore,

$$H = V(s^{\nu}Xt^{\mu})KG,$$

and this proves "⇒".

The implication "⇐" is obvious.

The proof is complete. □

Before proceeding further, we recall, from [6], the definition of the relative dimension of a linear subspace in a "universum". Let $\mathcal{X} \subseteq \mathcal{U}^q$ be an $\mathbb{F}$-linear subspace, and let

$$T = \{g \in O^q \mid g\mathcal{U} \subseteq \mathcal{X}\}.$$

This is a submodule of O^q. Letting $T\mathcal{U}$ denote the set of all finite sums Σgu with $g \in T$ and $u \in \mathcal{U}$, we clearly have

$$T\mathcal{U} \subseteq \mathcal{X}.$$

The dimension of $\mathcal{X}/T\mathcal{U}$ as an $\mathbb{F}$-linear space is called the relative dimension of $\mathcal{X}$.

If G is a rational matrix, and if T is its transfer function, then it can be easily shown that

$$\{g \in O^q \mid g\mathcal{U} \subseteq Bh(G)\} = T.$$

So the relative dimension of $Bh(G)$ is equal to the McMillan degree of G (see Theorem 3 in [3]).

Theorem 4. *Let G be a rational matrix with positive Wiener-Hopf indices, and let $G = P^{-1}Q$ be its left coprime factorization. Then, the differential closure of $Bh(G)$ coincides with $Bh(Q)$.*

Proof. Lemma 10 allows us to reduce the proof to the full row rank case.

We therefore shall assume that our matrix G has full row rank.

Let q be the column number of G, and let $\mathcal{B}$ denote the differential closure of $Bh(G)$. Obviously,

$$\{f \in O^q | Gf = 0\} = \{f \in O^q | Qf = 0\}.$$

Hence, G and Q have the same transfer function. Denote this common transfer function by T. From

$$Bh(G) \subseteq \mathcal{B} \subseteq Bh(Q),$$

it is clear that

$$\{g \in O^q \mid g\mathcal{U} \subseteq \mathcal{B}\} = T.$$

It is obvious that

$$\dim(\mathcal{B}/T\mathcal{U}) \leq \dim(Bh(Q)/T\mathcal{U}),$$

and thus $\mathcal{B}$ has finite relative dimension. Because $\mathcal{B}$ is differentiation invariant, by Theorem 3 in [6], there is a full row rank polynomial matrix Q_1 such that

$$\mathcal{B} = Bh(Q_1).$$

Because $Bh(G) \subseteq Bh(Q_1)$, by Lemma 11, we can find a polynomial matrix P such that $Q_1 = P_1G$. By the rank assumption, the matrix P_1 must be square nonsingular. Hence,

$$G = P_1^{-1}Q_1.$$

We clearly have $Bh(Q_1) \subseteq Bh(Q)$; hence $Q = DQ_1$ for some square polynomial matrix D. We have

$$P_1^{-1}Q_1 = P^{-1}Q \Rightarrow P_1^{-1}Q_1 = P^{-1}DQ_1 \Rightarrow P_1^{-1} = P^{-1}D \Rightarrow P = DP_1.$$

We see that D is a common left divisor of P and Q. Because P and Q are left coprime, D must be unimodular. So,

$$\mathcal{B} = Bh(Q_1) = Bh(Q).$$

The proof is complete. □

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