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Brief paper

Axiomatic characterization of linear differential systems (and operators) ☆

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Abstract

In his IEEE Trans. Aut. Contr. paper in 1991, Willems posed the following question: given a set of smooth trajectories, when does there exist a linear constant coefficient differential operator whose kernel is precisely the given set? We show that the properties that are necessary and sufficient for a set of smooth trajectories to admit a "kernel" representation are: linearity, time-invariance, jet-closedness and jet-determinedness. (It is interesting to note that the properties of jet-closedness and jet-determinedness were introduced by Willems himself in his Automatica paper in 1986.)

Introduction

In the series of papers (Willems, 1979, Willems, 1986a, Willems, 1986b, Willems, 1987, Willems, 1989, Willems, 1991), Jan Willems developed a new approach to systems and control theory, according to which a dynamical system is viewed as a collection of trajectories lying in a space of signal functions (called an universum). A very common class of dynamical systems is the class consisting of linear time-invariant differential (LTID) systems, which are described by means of linear constant coefficient differential equations. A natural question that arises in this context is: What are the properties that characterize LTID systems among all (continuous-time) dynamical systems? Three properties are standard; these are linearity, time-invariance and closedness. A crucial property that LTID systems possess is also, the so-called, locally specifiedness. (A dynamical system $\mathscr{B}$ is said to be locally specified if for every $\varepsilon > 0$, $w \in \mathscr{B} \Leftrightarrow \forall t, \quad w_{|[t-\varepsilon,t+\varepsilon]} \in \mathscr{B}_{|[t-\varepsilon,t+\varepsilon]}$; see Polderman and Willems (1998), Willems, 1986a, Willems, 1991). Willems conjectured that these four properties together are sufficient for a dynamical system to be an LTID system. However, as is known, this conjecture was disproved by Hörmander (see Soethoudt (1993), Willems (1991) for the counterexample).

It should be pointed out that the question was already studied in a number of works (see Delvenne and Ivanov (2009), Lomadze, 2007, Lomadze, 2010, Lomadze, in press-a, Oberst (1990), Soethoudt (1993)), where different approaches and solutions were provided.

In this article, we are going to show that LTID systems are exactly dynamical systems that are linear, time-invariant, jet-closed and jet-determined. We were led to consider the properties of jet-closedness and jet-determinedness based on our note (Lomadze, in press-b). However, these properties have already been known by Willems for quite a long time (see Willems (1986a)). The definitions will be recalled, of course. Here we only note that "jet-closedness" is close to "closedness" and "jet-determinedness" is close to "locally specifiedness". So, the characterization that we offer is close to the characterization that was conjectured by Willems.

We shall give in this article a characterization of linear differential operators also, which, we suppose, is not of less interest.

Throughout, $\mathscr{U}$ will denote $C^{\infty}(\mathbb{R},\mathbb{R})$. The differentiation operator in $\mathscr{U}$ will be denoted by ∂ (and sometimes by $'$ also); the shift operator by a time t will be denoted by S^t. There are two interesting Hausdorff locally convex topologies on $\mathscr{U}$: the standard C^{∞}-topology, which is well-known, and the point-wise convergence topology (PWC-topology), which was introduced by Delvenne and Ivanov (2009). The latter is defined as follows. For every triple (t,n,ε), where t is a time, n a nonnegative integer and ε a positive real number, let $\mathscr{U}_t^{n,\varepsilon} = \{w \in \mathscr{U} \mid \forall i \le n, |w^{(i)}(t)| < \varepsilon\}$.

A fundamental system of neighborhoods of 0 in the PWC-topology is given by finite intersections of the sets $\mathscr{U}_t^{n,\varepsilon}$.

We shall write $C(\mathbb{Z}_+,\mathbb{R})$ to denote the space of all sequences $(x_0,x_1,x_2,\ldots)$ with real entries. We shall consider on this space the product topology. (The field of real numbers is assumed to be equipped with the ordinary topology defined by the absolute value $||$.) Remark that $C(\mathbb{Z}_+,\mathbb{R})$

together with this topology is a Fréchet space. The shift operator $(x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots)$ will be denoted by σ.

Section snippets

Preliminaries on jets

Given a time $t \in \mathbb{R}$ and a function $w \in \mathscr{U}$, the jet of w at t is defined to be $J_t(w) = (w(t), w'(t), w''(t), \ldots)$. For every time t, we thus have a linear map $J_t : \mathscr{U} \to C(\mathbb{Z}_+, \mathbb{R})$. One easily checks that $J_t \circ \partial = \sigma \circ J_t$.

The following remarkable result is known as Borel's theorem, after Émile Borel. It tells us that every sequence of real numbers can appear as the jet of a C^∞-function.

Lemma 1 Borel's Theorem

The maps $J_t, t \in \mathbb{R}$ are surjective.…

Proof

See Borel (1895) and Section 26 in Meise and Vogt (1997). □…

For every time t, we define the t-adic topology on $\mathscr{U}$…

Characterization of LTID systems

In this section, q denotes an arbitrary positive integer. It will serve as the signal number; in other words, $\mathscr{U}^q$ will be our universum.

Let $\mathscr{B}$ be a subset of $\mathscr{U}^q$, a dynamical system with signal number q. Following Willems (1986a), we say: $\mathscr{B}$ is jet-closed if the jet-space $J_t(\mathscr{B})$ is a closed subset of $C(\mathbb{Z}_+, \mathbb{R})^q$ for every t; $\mathscr{B}$ is jet-determined if it satisfies the following condition $w \in \mathscr{B} \Leftrightarrow \forall t, \quad J_t(w) \in J_t(\mathscr{B})$; For every time t, let us set $\mathscr{B}_t = J_t(\mathscr{B})$ and call it the localization (or the local behavior) of $\mathscr{B}$ at t.…

Characterization of linear differential operators

As it is known, of importance are not only objects, but morphisms between these objects as well. In the context of linear systems theory the importance of morphisms was emphasized by Fuhrmann (2001), and Pillai, Wood, and Rogers (2002).

By a linear differential operator we shall mean a homomorphism as defined in Pillai et al. (2002). So that a linear differential operator from one LTID system $\mathscr{B}_1 \subseteq \mathscr{U}^{q_1}$ to another LTID system $\mathscr{B}_2 \subseteq \mathscr{U}^{q_2}$ is a map $\alpha : \mathscr{B}_1 \to \mathscr{B}_2$ that is induced by an operator of the form $A(\partial) : \mathscr{U}^{q_1}$ ...

Conclusions

It was shown in Willems (1986a) that a discrete-time dynamical system can be represented by a linear constant coefficient difference equation if and only if it is linear, shift-invariant and closed.

In this article we have demonstrated that the continuous-time analog of "linear, shift-invariant and closed" is "linear, time-invariant, jet-closed and jet-determined".

Linearity and time-invariance are standard properties. Jet-closedness and jet-determinedness were introduced by Willems (1986a). It...

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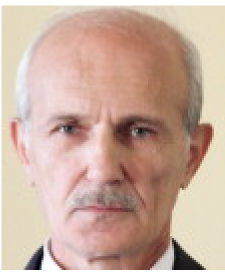

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