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# Lifting discrete trajectories

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ABSTRACT

It is shown that every discrete trajectory of a polynomial matrix can be lifted to a continuous one.



## 1. Introduction

The starting point of this note has been E. Borel's theorem, which states that every formal series can be represented as the Taylor expansion of some $C^{\infty}$-function. This beautiful fact (which we found in [1, Section 26]) gives rise to the following natural question: Can any discrete trajectory of a polynomial matrix be represented as the Taylor expansion of a continuous one?

More precisely, let $s$ be an indeterminate and $t = s^{-1}$. Let $I$ be an interval containing 0, $\partial : C^{\infty}(I) \to C^{\infty}(I)$ the differentiation operator, $\sigma : \mathbb{R}[[t]] \to \mathbb{R}[[t]]$ the backward shift operator. ($\mathbb{R}[[t]]$ denotes the ring of formal series in $t$.)

Define the operator $T : C^{\infty}(I) \to \mathbb{R}[[t]]$ by the formula

$$T(w) = w(0) + w'(0)t + w''(0)t^2 + \cdots .$$

This is surjective by Borel's theorem. Remark that $T \circ \partial = \sigma \circ T$.

Let now $p$ and $q$ be positive integers, and let $R \in \mathbb{R}[s]^{p\times q}$. In view of the above remark, we clearly have $T \circ R(\partial) = R(\sigma) \circ T$. It is immediate from this that $T$ induces a map

$$\operatorname{Ker} R(\partial) \to \operatorname{Ker} R(\sigma).$$

In other words, $T$ transforms continuous trajectories of $R$ into discrete trajectories of $R$. The question is whether this map is surjective.

In this note we shall prove that the map is surjective; we shall find also its kernel.

Let $O$ denote the ring of proper rational functions in $s$. (It is worth recalling that $O$ coincides with $\mathbb{R}(s) \cap \mathbb{R}[[t]]$.) Let $\mathbb{R}((t))$ be the field of Laurent formal series, and let $\Pi_- : \mathbb{R}((t)) \to \mathbb{R}[s]$ be the canonical projection ("taking the polynomial part"), which is determined by the decomposition $\mathbb{R}((t)) = \mathbb{R}[s] \oplus t\mathbb{R}[[t]]$.

Let $r$ be the rank of $R$, and put $m = q - r$. Choose once for all a proper rational matrix $G$ such that

$$0 \to O^m \xrightarrow{G} O^q \xrightarrow{R} \mathbb{R}(s)^p \qquad (1)$$

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is an exact sequence. Define the initial condition space $X$ of $R$ by setting

$$X = \mathbb{R}[s]^p \cap tRO^q.$$

The space $X$ will permit us to parameterize the trajectories of $R$ (see the two lemmas below) and this justifies the terminology. Clearly, this is a finite-dimensional linear space over $\mathbb{R}$.

## 2. The differential operator $R(\partial)$

For every $w \in C^\infty(I)$, define its indefinite integral $\int w$ by the formula

$$\left(\int w\right)(x) = \int_0^x w(\alpha)d\alpha \quad (x \in I).$$

There is a natural composition law between proper rational functions and $C^\infty$-functions. If $g \in O$ and $w \in C^\infty(I)$, then the product $gw$ is defined by the formula

$$gw = b_0 w + b_1 \int w + b_2 \int^2 w + \cdots + b_n \int^n w + \cdots,$$

where $b_i$ are the coefficients in the expansion of $g$ at infinity. (The reader can easily see that the series above converges uniformly on every compact neighborhood of 0.) This makes $C^\infty(I)$ a module over $O$. Because the integration operator is injective, this module is without torsion. Let us denote its fraction space by $\mathcal{M}$. Elements in $\mathcal{M}$ will be referred to as Mikusinski functions. The canonical map $w \mapsto w/1$ is injective, and this permits us to identify $C^\infty(I)$ with a subset in $\mathcal{M}$. Every Mikusinski function can be represented as $s^n w$, where $w \in C^\infty(I)$ and $n \geq 0$.

Let $\hbar$ denote the function that is identically one on $I$, and put $\delta = s\hbar$, which is an analog of Dirac's delta-function.

The Newton–Leibniz formula for $w \in C^\infty(I)$ can be rewritten in the form

$$sw = w' + w(0)\delta.$$

This, by induction, yields a more general formula

$$s^n w = w^{(n)} + (s^{n-1}w(0) + \cdots + w^{(n-1)}(0))\delta. \tag{2}$$

One can see that $C^\infty(I) \cap \mathbb{R}[s]\delta = \{0\}$, and thus we have

$$\mathcal{M} = C^\infty(I) \oplus \Delta,$$

where $\Delta = \mathbb{R}[s]\delta$. Functions in $\Delta$ should be interpreted as purely impulsive functions.

Using (2) (and linearity), one can easily see that

$$Rw = R(\partial)w + \Pi_-((R_0 s^{n-1} + \cdots + R_{n-1})T(w))\delta. \tag{3}$$

It follows from this that

$$\operatorname{Ker} R(\partial) = \{w \in C^\infty(I)^q \mid Rw \in \Delta^p\}. \tag{4}$$

As we have already remarked, $C^\infty(I)$ is a torsion free module, and hence flat. Therefore, tensoring (1) by $C^\infty(I)$, we obtain an exact sequence

$$0 \to C^\infty(I)^m \stackrel{G}{\to} C^\infty(I)^q \stackrel{R}{\to} \mathcal{M}^p.$$

In view of (4), this yields an exact sequence

$$0 \to C^\infty(I)^m \stackrel{G}{\to} \operatorname{Ker} R(\partial) \stackrel{R}{\to} \Delta^p.$$

Let us compute the image of $\operatorname{Ker} R(\partial) \stackrel{R}{\to} \Delta^p$, i.e., the set $\Delta^p \cap RC^\infty(I)^q$.

Choose a full column rank rational matrix $D$ such that $RO^q = DO^r$. We then have

$$\Delta^p \cap RC^\infty(I)^q = \mathbb{R}[s]^p\delta \cap DC^\infty(I)^r = \mathbb{R}[s]^p\delta \cap \mathbb{R}(s)^p\hbar \cap DC^\infty(I)^r.$$

We claim that $\mathbb{R}(s)^p\hbar \cap DC^\infty(I)^r = DO^r\hbar$. To show this, take a left inverse $C$ of $D$. If $w \in C^\infty(I)^r$ is such that $Dw \in \mathbb{R}(s)^p\hbar$, then $w = CDw \in \mathbb{R}(s)^r\hbar$. Because $C^\infty(I)^r \cap \mathbb{R}(s)^r\hbar = O^r\hbar$, it follows that $w \in O^r\hbar$. The claim is proved, and thus our image is equal to $\mathbb{R}[s]^p\delta \cap DO^r\hbar$. Further, we have

$$\mathbb{R}[s]^p\delta \cap DO^r\hbar = (s\mathbb{R}[s]^p \cap DO^r)\hbar = (\mathbb{R}[s]^p \cap tRO^q)\delta = X\delta.$$

So, the image, in which we are interested, is $X\delta$. There is an evident bijective map of $X\delta$ onto $X$. Composing $\operatorname{Ker} R(\partial) \to X\delta$ with this map, we get a canonical $\mathbb{R}$-linear surjective map

$$\operatorname{Ker} R(\partial) \to X.$$

(If $w$ is a trajectory of $R$, then its image under this map is called the initial condition of $w$.)

We have proved the following.

**Lemma 1.** *There is a canonical exact sequence*

$$0 \to C^\infty(I)^m \xrightarrow{G} \operatorname{Ker} R(\partial) \to X \to 0.$$

(The interested reader is referred to [2], where a little more about the material of this section can be found.)

## 3. The difference operator $R(\sigma)$

Difference operators can be treated in a similar but easier manner.

Instead of $C^\infty(I)$ we have to consider $\mathbb{R}[[t]]$, which certainly is a module over $O$. The role of the Mikusinski function space is played by $\mathbb{R}((t))$.

One can easily verify that

$$Rg = R(\sigma)g + \Pi_-((R_0 s^{n-1} + \cdots + R_{n-1})g)s. \tag{5}$$

Consequently,

$$\operatorname{Ker} R(\sigma) = \{w \in \mathbb{R}[[t]]^q \mid Rw \in s\mathbb{R}[s]^p\}.$$

As above, we have the following.

**Lemma 2.** *There is a canonical exact sequence*

$$0 \to \mathbb{R}[[t]]^m \xrightarrow{G} \operatorname{Ker} R(\sigma) \to X \to 0.$$

## 4. Lifting theorem

To begin with, remark that

$$T : C^\infty(I) \to \mathbb{R}[[t]]$$

is an $O$-homomorphism. Indeed, it is easily verified that if $g \in O$ and $w \in C^\infty(I)$, then

$$(gw)^{(n)}(0) = b_0 w^{(n)}(0) + b_1 w^{(n-1)}(0) + \cdots + b_n w(0),$$

where $b_i$ are the coefficients in the expansion of $g$ at infinity. It follows that

$$\sum_{n\geq 0} (gw)^{(n)}(0)t^n = \left(\sum_{i\geq 0} b_i t^i\right)\left(\sum_{j\geq 0} w^{(j)}(0)t^j\right).$$

Hence,

$$T(gw) = g(Tw).$$

Let $C^\infty_{fl}(I)$ be the space of flat functions at 0, i.e., $C^\infty$-functions having zero Taylor expansion at 0.

**Theorem 1.** *There is a short exact sequence*

$$0 \to C^\infty_{fl}(I)^m \xrightarrow{G} \operatorname{Ker} R(\partial) \xrightarrow{T} \operatorname{Ker} R(\sigma) \to 0.$$

**Proof.** Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \to & C^\infty(I)^m & \to & \operatorname{Ker}\ R(\partial) & \to & X & \to & 0 \\ & & T\downarrow & & T\downarrow & & || & & \\ 0 & \to & \mathbb{R}[[t]]^m & \to & \operatorname{Ker}\ R(\sigma) & \to & X & \to & 0 \end{array}.$$

One easily verifies that the first square in this diagram commutes. (Indeed, for any $u \in C^\infty(I)^m$, $T(Gu) = GT(u)$.) In view of (3), the map $\operatorname{Ker} R(\partial) \to X$ sends $w \in \operatorname{Ker} R(\partial)$ to

$$\Pi_-((R_0 s^{n-1} + \cdots + R_{n-1})T(w));$$

similarly, in view of (5), the map $\operatorname{Ker} R(\sigma) \to X$ sends $g \in \operatorname{Ker} R(\sigma)$ to

$$\Pi_-((R_0 s^{n-1} + \cdots + R_{n-1})g).$$

We see that the second square also commutes. The rows are exact by the lemmas above. The left downward arrow is surjective by Borel's theorem, and its kernel is equal to $C^\infty_{fl}(I)^m$.

It remains to use the snake lemma (see Proposition 2.10 in [3]). □

By a linear time-invariant differential (resp. difference) system one understands a set that can be represented as the kernel of a linear differential (resp. difference) operator with constant coefficients (see [4]). One knows that there is a bijective correspondence between the two classes of linear systems. The following is an explicit formulation of this fact.

**Corollary 1.** *The mapping*

$$\mathcal{B} \mapsto T(\mathcal{B})$$

*establishes a bijective correspondence between linear time-invariant differential systems and linear time-invariant difference systems.*

**Proof.** The surjectivity is immediate by Theorem 1. Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are two linear time-invariant differential systems such that $T(\mathcal{B}_1) = T(\mathcal{B}_2)$. Let $R_1$ and $R_2$ be their "kernel" representations. By Theorem 1, $\operatorname{Ker} R_1(\sigma) = \operatorname{Ker} R_2(\sigma)$. By the discrete-time version of Equivalence Theorem (see [5]),

$$R_2 = AR_1 \quad \text{and} \quad R_1 = BR_2$$

for some polynomial matrices $A$ and $B$. It immediately follows from this that $\operatorname{Ker} R_1(\partial) = \operatorname{Ker} R_2(\partial)$. □

## 5. Application

Lefschetz [6] introduced the notion of linearly compact vector spaces and extended the ordinary duality for finite-dimensional vector spaces to a duality between all vector spaces and all linearly compact vector spaces (see also [7]). Various results about linear time-invariant difference systems can be very easily deduced from the Lefschetz theory. In our opinion, Theorem 1 may serve as an effective tool in extending these results to linear time-invariant differential systems. To demonstrate how it works, let us prove Duality Theorem, which is fundamental in the "behavioral" systems theory of Willems.

Consider the canonical pairing $\mathbb{R}[s]^q \times \mathbb{R}[[t]]^q \to \mathbb{R}$ defined by

$$\langle f, g\rangle = \text{the free coefficient of } f^{\text{tr}}(\sigma)g.$$

(The superscript "tr" stands for the transpose.) For any subset $V$ in $\mathbb{R}[s]^q$ or $\mathbb{R}[[t]]^q$, let $V^\perp$ denote the orthogonal complement of $V$ with respect to this pairing.

One can easily check that

$$(R^{\text{tr}}\mathbb{R}[s]^p)^\perp = \operatorname{Ker} R(\sigma).$$

By the Lefschetz duality, we get

$$\operatorname{Ker} R(\sigma)^\perp = R^{\text{tr}}\mathbb{R}[s]^p. \tag{6}$$

(See also Section 3 in [8].)

Recall that the annihilator of any dynamical system $\mathcal{B} \subseteq C^\infty(I)^q$ is defined to be

$$\operatorname{Ann}(\mathcal{B}) = \{f \in \mathbb{R}[s]^q \mid f^{\text{tr}}(\partial)w = 0 \text{ for all } w \in \mathcal{B}\}.$$

**Theorem 2** (*Duality Theorem*)**.** *There holds*

$$\operatorname{Ann}(\operatorname{Ker}(\partial)) = R^{\text{tr}}\mathbb{R}[s]^p.$$

**Proof.** The inclusion "$\supseteq$" is obvious. (Indeed, for every $f \in \mathbb{R}[s]^p$ and $w \in \mathcal{B}$, we have

$$(R^{\text{tr}}f)^{\text{tr}}(\partial)w = f^{\text{tr}}(\partial)R(\partial)w = 0.)$$

The hard part is to prove "$\subseteq$". For this, take any $f \in \operatorname{Ann}(\operatorname{Ker}(\partial))$. In view of (6), to show that $f \in R^{\text{tr}}\mathbb{R}[s]^p$, it suffices to show that $f \in \operatorname{Ker} R(\sigma)^\perp$. If $g \in \operatorname{Ker}(\sigma)$, then (by Theorem 1) it can be written as $g = Tw$ with $w \in \operatorname{Ker} R(\partial)$. We therefore have

$$f^{\text{tr}}(\sigma)g = f^{\text{tr}}(\sigma)Tw = Tf^{\text{tr}}(\partial)w = T0 = 0;$$

whence, $\langle f, g\rangle = 0$. □

An immediate consequence of Duality Theorem is the following corollary.

**Corollary 2** (*Inclusion Lemma*)**.** *Let $R_1$ and $R_2$ be two polynomial matrices with the same column number and with row numbers $p_1$ and $p_2$, respectively. The following conditions are equivalent:*

(a) $\operatorname{Ker} R_1(\partial) \subseteq \operatorname{Ker} R_2(\partial)$*;*
(b) $R_2^{\text{tr}}\mathbb{R}[s]^{p_2} \subseteq R_1^{\text{tr}}\mathbb{R}[s]^{p_1}$*;*
(c) $R_2 = AR_1$ *for some polynomial matrix A.*

Inclusion Lemma implies in turn the following important corollary.

**Corollary 3** (*Equivalence Theorem*)**.** *Let $R_1$, $R_2$, $p_1$ and $p_2$ be as above. The following conditions are equivalent:*

(a) $\operatorname{Ker} R_1(\partial) = \operatorname{Ker} R_2(\partial)$;
(b) $R_2^{tr}\mathbb{R}[s]^{p_2} = R_1^{tr}\mathbb{R}[s]^{p_1}$;
(c) $R_2 = AR_1$ *and* $R_1 = BR_2$ *for some polynomial matrices A and B.*

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