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A note on Ehrenpreis' fundamental principle

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ABSTRACT

Ehrenpreis' fundamental principle is an existence theorem for inhomogeneous linear partial differential equations, and has great system-theoretic significance. Usually it is formulated for the equations defined on open convex sets. However, openness of domains of definition is somewhat restrictive in systems theory. In this article, we show that the principle is valid in the case of sets that are the unions of increasing sequences of convex compact sets. Moreover, we offer a simplified proof.

1. Introduction

Let $s_1, \ldots, s_n$ be indeterminates, and let $\partial_1, \ldots, \partial_n$ be partial differentiation operators. Put

$$s = (s_1, \ldots, s_n) \quad \text{and} \quad \partial = (\partial_1, \ldots, \partial_n).$$

Given a convex open set $\Omega \subseteq \mathbb{R}^n$, a polynomial matrix $P \in \mathbb{C}[s]^{r\times p}$ and $f \in C^\infty(\Omega)^r$, one has a linear PDE

$$P(\partial)w = f, \quad w \in C^\infty(\Omega)^p.$$

There is an obvious compatibility condition for this equation to have a solution. Indeed, choose a maximal left annihilator $Q \in \mathbb{C}[s]^{q\times r}$ of the matrix P (that is, a matrix for which the sequence

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$\mathbb{C}[s]^q \xrightarrow{Q^{tr}} \mathbb{C}[s]^r \xrightarrow{P^{tr}} \mathbb{C}[s]^p$ is exact). Then $Q(\partial) \circ P(\partial) = 0$, and consequently, if the equation is solvable,

$$Q(\partial)f = 0$$

necessarily. The Ehrenpreis Fundamental Principle states that this condition is sufficient as well.

Needless to say that this is a very important result. It is due to Ehrenpreis, Malgrange and Palamodov. A proof can be found in Hörmander [2] (see Theorem 7.6.13).

The Fundamental Principle is of great importance in linear systems theory as it yields the elimination theorem (see [6,8–11]).

Unfortunately, the principle, in the form above, does not apply for linear PDE's defined on sets that are not open but are interesting for applications (say, $\mathbb{R}^n_+$, for example). One wants therefore to relax the requirement that the domains of definition are open.

In this article, we shall extend the Fundamental Principle to linear PDE's defined on convex F_σ-sets with nonempty interior.

We remind that a subset of $\mathbb{R}^n$ is said to be an F_σ-set if it is the union of a countable number of closed sets (see Kechris [4] about F_σ-sets.) Every closed set trivially is an F_σ-set; it is not difficult to see that every open set is an F_σ-set. Certainly, the union of countably many F_σ-sets is an F_σ-set, and the intersection of finitely many F_σ-sets is an F_σ-set. Notice that an F_σ-set can be written as the union of an increasing sequence of bounded closed sets.

Remark. From the topological point of view, the simplest sets of $\mathbb{R}^n$ are the open sets and the closed sets. Next come the G_δ-sets (= countable intersections of open sets) and the F_σ-sets (= countable unions of closed sets). Continuing this way, one builds up the family of Borel sets, which include most of the sets that arise naturally in mathematical practice.

In what follows, $\Omega \subseteq \mathbb{R}^n$ is a fixed convex F_σ-set with nonempty interior.

By a compact set, let us mean a subset of $\mathbb{R}^n$ that is the closure of a bounded open subset in $\mathbb{R}^n$. It is a standard fact that the convex hull of a bounded closed set is bounded and closed. Using this, one can easily show that Ω is the union of an increasing sequence

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$$

of convex compact sets.

Given a compact subset $K \subseteq \mathbb{R}^n$, for every $k \in \mathbb{Z}_+$, let $C^k(K)$ denote the space of complex valued functions f on K such that f has continuous derivatives up to order k in the interior of K and all these derivatives extend continuously to K. Then $C^k(K)$ has the natural structure of a Banach space (see Example 5.16(6) in Meise and Vogt [5]). One defines in an obvious way C^∞-functions on K. Following the tradition, we denote the space of such functions by $\mathcal{E}(K)$. This is the projective limit of the system

$$C^0(K) \leftarrow C^1(K) \leftarrow C^2(K) \leftarrow C^3(K) \leftarrow \cdots,$$

and consequently this is a Fréchet space.

A function $f : \Omega \to \mathbb{C}$ is said to be of class C^∞ if

$$f|_K \in \mathcal{E}(K)$$

for every compact K in Ω. The space of all such functions will be denoted by $\mathcal{E}(\Omega)$. One defines in an obvious way the derivatives of a function $f \in \mathcal{E}(\Omega)$. For every compact $K \subseteq \Omega$ and for every integer $k \in \mathbb{Z}_+$, one defines the seminorm $|| \, ||_{K,k}$ by the formula

$$||f||_{K,k} = \sup\{|\partial^i f(t)| : \ t \in K, \ |i| \leqslant k\}.$$

(Here $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n_+$, and $\partial^i = (\partial_1^{i_1} \cdots \partial_n^{i_n})$ and $|i| = i_1 + \cdots i_n$. These seminorms make $\mathcal{E}(\Omega)$ into a Fréchet space. Indeed, one can see that $\mathcal{E}(\Omega)$ is the projective limit of

$$\mathcal{E}(K_0) \leftarrow \mathcal{E}(K_1) \leftarrow \mathcal{E}(K_2) \leftarrow \mathcal{E}(K_3) \leftarrow \cdots.$$

(Notice that it can be viewed also as the projective limit of

$$C^0(K_0) \leftarrow C^1(K_1) \leftarrow C^2(K_2) \leftarrow C^3(K_3) \leftarrow \cdots .)$$

The strong dual $\mathcal{V}'$ of a locally convex space $\mathcal{V}$ is the space of all continuous linear functionals on $\mathcal{V}$, equipped with the strong dual topology, i.e., the topology of uniform convergence on bounded subsets of $\mathcal{V}$.

For every compact set K, we let $\mathcal{E}'(K)$ denote the strong dual of $\mathcal{E}(K)$. This is precisely the space of distributions of $\mathbb{R}^n$ having support in K. Similarly, $\mathcal{E}'(\Omega)$ will stand for the strong dual of $\mathcal{E}(\Omega)$. Remark that

$$\mathcal{E}'(\Omega) = \bigcup_K \mathcal{E}'(K),$$

where K ranges over compact subsets of Ω.

One defines an exponential function as a function of the form

$$f(x)e^{\lambda \cdot x}, \ x \in \mathbb{R}^n,$$

where f is a polynomial and $\lambda \in \mathbb{C}^n$. (With obvious notation, $\lambda \cdot x = \lambda_1 x_1 + \cdots + \lambda_n x_n$.)

The following result will be our starting point.

Lemma 1. *Let K be a convex compact set, and let R be a polynomial matrix, say, of size $p \times q$. Then the space of continuous linear functionals on $\mathcal{E}(K)^q$ vanishing on the exponential solutions of the equation*

$$R(\partial)w = 0, \ \ w \in \mathcal{E}(K)^q \tag{1}$$

is equal to $R^{tr}(-\partial)\mathcal{E}'(K)^p$.

The proof of this difficult result can be found in the proof of Theorem 7.6.14 in Hörmander [2]. (The proof there is for the case when a linear PDE is defined on a convex open set. But it is easily seen (and Hörmander himself remarks) that it applies also when a linear PDE is defined on a convex compact set. The proof applies since, as remarked already, for compact K, $\mathcal{E}'(K)$ is the space of distributions of $\mathbb{R}^n$ with support in K.)

We shall see that very little beyond the above lemma is needed to prove the Fundamental Principle.

2. Preliminaries

Let R be a polynomial matrix of size $p \times q$, and let $\mathcal{B}$ be the solution set of the equation

$$R(\partial)w = 0, \ \ w \in \mathcal{E}(\Omega)^q. \tag{2}$$

We have a canonical duality

$$\mathcal{E}'(\Omega)^q \times \mathcal{E}(\Omega)^q \to \mathbb{C}.$$

For $X \subseteq \mathcal{E}(\Omega)^q$ and $Y \subseteq \mathcal{E}'(\Omega)^q$, one defines in an obvious way the orthogonal sets $X^\perp$ and $Y^\perp$. It is clear that both these sets are closed linear subspaces. For later use, we remark that if $X \subseteq \mathcal{E}(\Omega)^q$ is a linear subspace, then

$$\overline{X} = X^{\perp\perp}.$$

This follows from the bipolar theorem (see Theorem 22.13 in Meise and Vogt [5]). If one wants, one can easily deduce it directly from the Hahn–Banach theorem.

Let E denote the space of the linear combinations of exponential functions. The following lemma is a generalization of Lemma 1.

Lemma 2. *There holds*

$$(\mathcal{B} \cap E^q)^\perp = R^{tr}(-\partial)\mathcal{E}'(\Omega)^p.$$

Proof. The inclusion "$\subseteq$" is immediate from Lemma 1. Indeed, let $f \in \mathcal{E}'(\Omega)^q$. Then $f \in \mathcal{E}'(K)^q$ for some convex compact subset $K \subseteq \Omega$. Clearly, the exponential solutions of (2) are the same as the exponential solutions of (1). Consequently, if $f \in (\mathcal{B} \cap E^q)^\perp$, then, by Lemma 1,

$$f \in R^{tr}(-\partial)\mathcal{E}'(K)^p \subseteq R^{tr}(-\partial)\mathcal{E}'(\Omega)^p.$$

The reverse inclusion "$\supseteq$" is elementary. Indeed, if $g \in \mathcal{E}'(\Omega)^p$, then, for every $w \in \mathcal{B}$, we have

$$\langle R^{tr}(-\partial)g, w\rangle = \langle g, R(\partial)w\rangle = \langle g, 0\rangle = 0;$$

hence, $R^{tr}(-\partial)g \in (\mathcal{B} \cap E^q)^\perp$.

The proof is complete. □

Corollary 1 (Malgrange's approximation theorem)**.**

$$\overline{\mathcal{B} \cap E^q} = \mathcal{B}.$$

Proof. As already said, for every $g \in \mathcal{E}'(\Omega)^p$ and every $w \in \mathcal{E}(\Omega)^q$, we have

$$\langle R^{tr}(-\partial)g, w\rangle = \langle g, R(\partial)w\rangle.$$

Using this, we can easily see that

$$(R^{tr}(-\partial)\mathcal{E}'(\Omega)^p)^\perp = \mathcal{B}.$$

From this (and from the lemma), we get

$$\overline{\mathcal{B} \cap E^q} = (\mathcal{B} \cap E^q)^{\perp\perp} = (R^{tr}(-\partial)\mathcal{E}'(\Omega)^p)^\perp = \mathcal{B}.$$

The proof is complete. □

A very important consequence of Lemma 2 is also the following

Corollary 2. $R^{tr}(-\partial)\mathcal{E}'(\Omega)^p$ *is closed in* $\mathcal{E}'(\Omega)^q$.

Proof. Any linear subspace that is orthogonal to some set is closed. □

We shall recall now a bit of "Local Duality" (see Grothendieck [3, Ch. III]).

For every $\lambda \in \mathbb{C}^n$, let M_λ be the maximal ideal of $\mathbb{C}[s]$ generated by $s_1 - \lambda_1, \ldots, s_n - \lambda_n$. It is worth noting that the k-th power M_λ^k of the ideal M_λ is generated by the "λ-monomials"

$$(s_1 - \lambda_1)^{i_1} \ldots (s_n - \lambda_n)^{i_n}, \quad i_1 + \cdots + i_n = k.$$

Let H_λ denote the space of linear functionals on $\mathbb{C}[s]$ that are continuous with respect to the M_λ-adic topology; that is,

$$H_\lambda = \{\varphi \in Hom_{\mathbb{C}}(\mathbb{C}[s], \mathbb{C}) : \varphi \text{ vanishes on some power of } M_\lambda\}.$$

We shall view H_λ as a module over $\mathbb{C}[s]$. (The module structure is defined by the formula $(f\varphi)(g) = \varphi(fg)$.)

Lemma 3 (Local duality theorem)**.** *Let A be a finitely generated $\mathbb{C}[s]$-module such that $M_\lambda^k A = 0$ for some $k \geqslant 0$. There is a canonical isomorphism*

$$Hom_{\mathbb{C}[s]}(A, H_\lambda) \simeq Hom_{\mathbb{C}}(A, \mathbb{C}).$$

Proof. For $u \in Hom_{\mathbb{C}[s]}(A, H_\lambda)$, let $\tilde{u}$ denote the linear map

$$a \mapsto (u(a))(1), \quad a \in A.$$

Next, for $\chi \in Hom_{\mathbb{C}}(A, \mathbb{C})$ and $a \in A$, define $\chi_a : \mathbb{C}[s] \to \mathbb{C}$ to be

$$f \mapsto \chi(fa), \quad f \in \mathbb{C}[s].$$

It is easily seen that χ_a vanishes on M_λ^k, and hence belongs to H_λ. We leave to the reader to check that the mappings

$$u \to \tilde{u} \text{ and } \chi \mapsto [a \mapsto \chi_a]$$

are inverse to each other.

The proof is complete. □

Lemma 4. *For every λ, H_λ is an injective module.*

Proof. Suppose that I is an ideal of $\mathbb{C}[s]$, and let $u : I \to H_\lambda$ be an arbitrary homomorphism. Since I is finitely generated, we can find k such that $M_\lambda^k u(I) = 0$. Then u vanishes on the ideal $M_\lambda^k I$. Further, by the Artin–Rees lemma (see Bourbaki [1, Ch. III, §3]), there exists l such that

$$M_\lambda^l \cap I \subseteq M_\lambda^k I.$$

It follows that u vanishes on $M_\lambda^l \cap I$, and therefore induces a canonical homomorphism from $I/(M_\lambda^l \cap I)$ to H_λ. We have a canonical embedding $I/(M_\lambda^l \cap I) \to \mathbb{C}[s]/M_\lambda^l$. Applying the exact functor $Hom_{\mathbb{C}}(-, \mathbb{C})$ to it, we get a surjective homomorphism

$$Hom_{\mathbb{C}}(\mathbb{C}[s]/M_\lambda^l, \mathbb{C}) \to Hom_{\mathbb{C}}(I/(M_\lambda^l \cap I), \mathbb{C}).$$

In view of the previous lemma, this yields a surjective homomorphism

$$Hom_{\mathbb{C}[s]}(\mathbb{C}[s]/M_\lambda^l, H_\lambda) \to Hom_{\mathbb{C}[s]}(I/(M_\lambda^l \cap I), H_\lambda).$$

Thus, the homomorphism $I/(M_\lambda^l \cap I) \to H_\lambda$ can be extended to some homomorphism $\mathbb{C}[s]/M_\lambda^l \to H_\lambda$. The composition $\mathbb{C}[s] \to \mathbb{C}[s]/M_\lambda^l \to H_\lambda$ is a homomorphism that extends the given homomorphism u.

The proof is complete. □

3. Ehrenpreis' fundamental principle

Given $\lambda \in \mathbb{C}^n$ and $k \in \mathbb{Z}_+$, we set

$$E_{\lambda,k} = \{f(x)e^{\lambda \cdot x} : f \text{ is a polynomial of degree } \leqslant k-1\}.$$

Remark that $E_{\lambda,k}$ is the solution set of the equations

$$(\partial_1 - \lambda_1)^{i_1} \ldots (\partial_n - \lambda_n)^{i_n} y = 0, \qquad i_1 + \cdots + i_n = k.$$

The following result can be found in Oberst [7, Section 6].

Lemma 5. *There is a canonical isomorphism*

$$E_{\lambda,k} \simeq Hom_{\mathbb{C}}(\mathbb{C}[s]/M_\lambda^k, \mathbb{C}).$$

Proof. Consider the canonical pairing

$$\mathbb{C}[s] \times E_{\lambda,k} \to \mathbb{C}$$

defined by

$$(f, g) \mapsto (f(\partial)g)(0).$$

This certainly is nondegenerate from the right. From the remark above, it follows that M_λ^k annihilates $E_{\lambda,k}$. So that we have a pairing

$$\mathbb{C}[s]/M_\lambda^k \times E_{\lambda,k} \to \mathbb{C}.$$

Again, this is nondegenerate from the right. As both spaces have the same dimension, the pairing must be nondegenerate from the left as well.

This completes the proof. □

For $\lambda \in \mathbb{C}^n$, let E_λ denote the set of all exponential functions of "type λ", that is, the union of all $E_{\lambda,k}$, $k \geqslant 0$. As it is differentiation invariant, it can be viewed as a module over $\mathbb{C}[s]$.

Lemma 5 yields an isomorphism

$$E_\lambda \simeq H_\lambda.$$

Hence, by Lemma 4, the modules E_λ are injective. We conclude that

$$E = \oplus E_\lambda$$

is an injective module.

Before proceeding let us recall the following useful fact about continuous linear maps between Fréchet spaces.

Lemma 6 (Closed range theorem). *Let $A : \mathcal{V} \to \mathcal{W}$ be a continuous linear operator of Fréchet spaces. Then*

$$A(\mathcal{V}) \text{ is closed in } \mathcal{W} \quad \Leftrightarrow \quad A'(\mathcal{W}') \text{ is closed in } \mathcal{V}'.$$

Proof. See Theorem 26.3 in Meise and Vogt [5]. □

We are now ready to prove the Ehrenpreis Fundamental Principle.

Theorem 1. *Let $P \in \mathbb{C}[s]^{r\times p}$ and $Q \in \mathbb{C}[s]^{q\times r}$ be two polynomial matrices such that the sequence*

$$\mathbb{C}[s]^q \stackrel{Q^{tr}}{\to} \mathbb{C}[s]^r \stackrel{P^{tr}}{\to} \mathbb{C}[s]^p \tag{3}$$

is exact. Then the sequence

$$\mathcal{E}(\Omega)^p \stackrel{P(\partial)}{\to} \mathcal{E}(\Omega)^r \stackrel{Q(\partial)}{\to} \mathcal{E}(\Omega)^q$$

is exact.

Proof. By Corollary 2, the image of

$$\mathcal{E}'(\Omega)^r \stackrel{P^{tr}(-\partial)}{\to} \mathcal{E}(\Omega)'^p$$

is closed. Using Lemma 6, we obtain from this that

$$\mathcal{E}(\Omega)^p \stackrel{P(\partial)}{\to} \mathcal{E}(\Omega)^r$$

has closed image. Obviously, this image is contained in $KerQ(\partial)$, and therefore we only need to show that $P(\partial)\mathcal{E}(\Omega)^p$ is dense in $KerQ(\partial)$.

Because E is an injective module, (3) gives rise to the exact sequence

$$E^p \xrightarrow{P(\partial)} E^r \xrightarrow{Q(\partial)} E^q.$$

In other words, we have that

$$P(\partial)E^p = KerQ(\partial) \cap E^r.$$

Using Corollary 1, we conclude

$$\overline{P(\partial)E^p} = KerQ(\partial).$$

The proof is complete. □

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References

[1] N. Bourbaki, Commutative Algebra, Springer-Verlag, New-York, 1989.
[2] L. Hörmander, An Introduction to Complex Analysis in Several Variables, Elsevier, Amsterdam, 1990.
[3] A. Grothendieck, Local Cohomology, Springer-Verlag, New-York, 1967.
[4] A.S. Kechris, Classical Descriptive Set Theory, Springer-Verlag, New York, 1995.
[5] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxford University Press, New York, 1997.
[6] U. Oberst, Multidimensional Constant Linear Systems, Acta Appl. Math. 29 (1990) 1–175.
[7] U. Oberst, Variations on the fundamental principle for linear systems of partial differential and difference equations with constant coefficients, Appl. Algebra Eng. Commun. Comput. 6 (1995) 211–243.
[8] H. Pillai, S. Shankar, A behavioral approach to control of distributed systems, SIAM J. Contr. Optim. 37 (1999) 388–408.
[9] J.W. Polderman, J.C. Willems, Introduction to Mathematical Systems Theory, Springer-Verlag, New York, 1998.
[10] S. Shankar, Geometric completeness of distribution spaces, Acta Appl. Math. 77 (2003) 163–180.
[11] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Autom. Contr. 36 (1991) 259–294.