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Duality in the behavioral systems theory☆

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ABSTRACT

In this note, Willems' behavioral framework is slightly generalized in order to define the dual of a linear system and make the concepts of controllability and observability dual to each other.

1. Introduction

For a linear (differential) system $\mathcal{B}$ defined by the AR-equation $R(\partial)w = 0$, let $\mathcal{B}^*$ denote the linear system defined by the MA-equation $w = R^{\mathrm{tr}}(\partial)l$. (Notice that $\mathcal{B}^*$ is controllable.) It is shown in Kuijper (1994) (see Lemma 3.23) that the map $\mathcal{B} \mapsto \mathcal{B}^*$ gives a duality for controllable linear systems; in other words, this map, when applied twice to a controllable linear system, gives that system back again.

Our purpose is to extend this duality to arbitrary (not necessarily controllable) linear systems. For this, we need to introduce a more general notion of a linear system.

The idea is very simple. We consider pairs of the form $(\mathcal{B}, \mu)$, where $\mathcal{B}$ is the solution set of a linear constant coefficient differential equation and μ is a quasi-injective differential operator from this set (the set of "internal" trajectories) to a fixed "universum", i.e., a fixed space of external variables. (A differential operator is said to be quasi-injective if its kernel is "small".) We call such pairs linear models. One may think of a linear model as a black box that converts "internal" trajectories into external ones. There is an evident equivalence relation on the set of all linear models, and we define linear systems to be equivalence classes. These linear systems are natural generalizations of linear systems as defined by Polderman and Willems (1998) and Willems (1991). Willems' linear systems correspond precisely to observable linear systems. (A linear system is called observable if it is represented by a linear model with an injective differential operator.)

This more general framework will permit us to define the dual of an arbitrary linear system. Moreover, we shall see that observability as controllability is an intrinsic property of a linear system and that the two properties are dual to each other.

The development is based on the correspondence between linear systems and ARMA-models. It should be stressed that our treatment of ARMA-models is somewhat different from the traditional one. Usually, one eliminates the latent variable from the internal behavior of an ARMA-model and what one gets after this elimination is the external behavior, which is a linear system in the sense of Willems. However, performing the elimination procedure, one loses some information. The point of view that we pursue is that one should consider the behavioral invariant of an ARMA-model to be the equivalence class of the linear model consisting of the internal behavior and the canonical map from this behavior to the universum.

In one sense or another, any linear system admits many different representations, and the question of great interest is: When two representations represent the same linear system? In the behavioral setting, this question was studied in Fuhrmann (2001, 2002), Kuijper (1994), Pillai, Wood, and Rogers (2002), Polderman and Willems (1998), Schumacher (1998).

We shall see that every linear system (in the sense we offer) can be represented via an ARMA-model, and that two ARMA-models represent the same linear system if and only if they are homotopy

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equivalent. There is an obvious duality for ARMA-models, which is consistent with homotopy equivalence. And this duality leads naturally to a duality between linear systems.

A word about "homotopy". What we mean by homotopy equivalence is, in principle, Fuhrmann's strict system equivalence (Fuhrmann, 2001, 2002). We believe that this is the right term. In Lomadze (2011) we have demonstrated that these equivalences are "relaxed" isomorphisms, and in Mathematics isomorphisms of this kind are called homotopy equivalences.

Throughout, $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, s an indeterminate, and q a fixed positive integer. The latter will serve as the signal number of ARMA-models and linear systems. We let $\mathcal{U}$ be the space of C^∞-functions or distributions defined on a fixed time interval.

2. Linear (differential) systems

By a linear behavior in $\mathcal{U}^r$, one means any subset of $\mathcal{U}^r$ that can be represented as the solution set of an equation of the form

$$R(\partial)w = 0 \quad (w \in \mathcal{U}^r),$$

where R is a polynomial matrix with r columns. Such a polynomial matrix is called a kernel representation.

If $\mathcal{B}_1 \subseteq \mathcal{U}^{r_1}$ and $\mathcal{B}_2 \subseteq \mathcal{U}^{r_2}$ are two linear behaviors, then a differential operator between them is a map $\alpha : \mathcal{B}_1 \to \mathcal{B}_2$ such that

$$\forall w \in \mathcal{B}_1, \quad \alpha(w) = A(\partial)w$$

for some polynomial matrix $A \in \mathbb{F}[s]^{r_2 \times r_1}$.

Differential operators are the same as homomorphisms in Fuhrmann (2001, 2002) and Pillai et al. (2002). It is clear that the composition of two differential operators is again a differential operator. Consequently, we can speak about isomorphisms between linear behaviors. A differential operator $\alpha : \mathcal{B}_1 \to \mathcal{B}_2$ is an isomorphism, if there exists a differential operator $\beta : \mathcal{B}_2 \to \mathcal{B}_1$ such that $\beta \circ \alpha = id$ and $\alpha \circ \beta = id$.

The following two facts are fundamental (see Lemma 2.4 in Schumacher (1998) and Theorem 6.2.6 in Polderman and Willems (1998), respectively).

Lemma 1 (*Inclusion Lemma*)**.** *Let R_1 and R_2 be polynomial matrices with the same column number. Then*

$$Ker\ R_1(\partial) \subseteq Ker\ R_2(\partial)$$

if and only if $R_2 = AR_1$ for some polynomial matrix A.

Lemma 2 (*Elimination Theorem*)**.** *Let R and M be polynomial matrices with the same column number. Then*

$$M(\partial)(Ker\ R(\partial))$$

is a linear behavior in $\mathcal{U}^n$, where n is the number of rows of M.

Inclusion Lemma implies the equivalence theorem, which states that two polynomial matrices R_1 and R_2, having the same column number, determine the same linear behavior if and only if $R_2 = AR_1$ and $R_1 = BR_2$ for some polynomial matrices A and B. It immediately follows from this that every linear behavior has a minimal (i.e., full row rank) kernel representation, and if R_1 and R_2 are two such representations of a linear behavior, then there exists a unimodular polynomial matrix U such that $R_2 = UR_1$.

Let $\mathcal{B}$ be a linear behavior, and let R be any of its minimal kernel representation. The rank of $\mathcal{B}$ is defined to be the column number of R minus the row number. One says that $\mathcal{B}$ is controllable if R is left prime. From what we said above, it is clear that the notions of rank and controllability do not depend on the choice of a minimal representation.

A differential operator is said to be quasi-injective if its kernel is small in the sense that it has finite dimension as a linear space (over $\mathbb{F}$).

The following lemma is immediate from the Smith factorization theorem (see Appendix B in Polderman & Willems, 1998).

Lemma 3. *Let R be a polynomial matrix of size $l \times r$, say. Then:*

(a) *$R(\partial) : \mathcal{U}^r \to \mathcal{U}^l$ is quasi-injective if and only if R is of full column rank;*
(b) *$R(\partial) : \mathcal{U}^r \to \mathcal{U}^l$ is injective if and only if R is right prime.*

We now introduce the notion of linear differential systems. (We remind that q is a fixed positive integer.)

Call a linear (differential) model (with signal number q) any pair $(\mathcal{B}, \mu)$, where $\mathcal{B}$ is a linear behavior and $\mu : \mathcal{B} \to \mathcal{U}^q$ a quasi-injective differential operator. Say that two linear models $(\mathcal{B}_1, \mu_1)$ and $(\mathcal{B}_2, \mu_2)$ are equivalent if there is an isomorphism $\alpha : \mathcal{B}_1 \simeq \mathcal{B}_2$ such that $\mu_1 = \mu_2 \circ \alpha$.

Definition. A linear (differential) system (with signal number q) is an equivalence class of linear models.

Let Σ be a linear system, and let $(\mathcal{B}, \mu)$ be any of its representative. The input number of Σ is defined to be the rank of $\mathcal{B}$; the output number is defined as q minus the input number. We say that Σ is controllable if $\mathcal{B}$ is controllable; we say that Σ is observable if μ is injective. (All these concepts are well-defined.) By Lemma 2, the image of $\mathcal{B}$ under μ is a linear behavior in $\mathcal{U}^q$. Following Willems, we call it the external (or manifest) behavior of Σ and denote by $Bh^{ext}(\Sigma)$. Clearly, the external behavior does not depend on the choice of a representative.

It is easily seen that the map $\Sigma \mapsto Bh^{ext}(\Sigma)$ induces a bijection between observable linear systems and linear behaviors in $\mathcal{U}^q$. This bijection permits us to identify observable linear systems with linear systems in the sense of Willems.

A simple example of a linear system that is not observable and therefore cannot be identified with Willems' linear system is the class of the linear model $(\mathcal{U}, \partial : \mathcal{U} \to \mathcal{U})$. (The "$q$" here is equal to 1.)

3. ARMA-models and their homotopy

In this section, we briefly revisit the theory of "strict systems equivalence" as developed in Fuhrmann (2001, 2002), Kuijper (1994), Pillai et al. (2002), and Schumacher (1998). (The interested reader may also consult Lomadze (2011), where the theory is presented in the purely algebraic manner.)

There are two types of ARMA-models. To distinguish, we shall call them left and right ARMA-models.

A left ARMA-model (with signal number q) is an equation of the form

$$\begin{cases} R(\partial)y = 0 \\ w = M(\partial)y \end{cases}, \tag{1}$$

where $R \in \mathbb{F}[s]^{\bullet\times\bullet}$ and $M \in \mathbb{F}[s]^{q\times\bullet}$ satisfy the following minimality conditions:

(a) R is of full row rank;
(b) $\begin{bmatrix} R \\ M \end{bmatrix}$ is of full column rank.

The number of columns of R minus the number of rows is called the input number; it is easily seen that this number $\le q$. The output number is q minus the input number. The model is called controllable if R is left prime; the model is called observable if $\begin{bmatrix} R \\ M \end{bmatrix}$ is right prime.

Example 1. An MA-model, i.e., an equation of the form $w = M(\partial)y$, where M is a polynomial matrix of full column rank, can be viewed in an obvious way as a left ARMA-model. As such it is controllable.

A right ARMA-model (with signal number q) is an equation of the form

$$M(\partial)z = R(\partial)w, \tag{2}$$

where $M \in \mathbb{F}[s]^{\bullet\times\bullet}$ and $R \in \mathbb{F}[s]^{\bullet\times q}$ satisfy the following minimality conditions:

(a) M is of full column rank;
(b) $\begin{bmatrix} M & R \end{bmatrix}$ is of full row rank.

The number of rows of M minus the number of columns is called the output number; it is easily seen that this number $\leq q$. The input number is q minus the output number. The model is called controllable if $\begin{bmatrix} M & R \end{bmatrix}$ is left prime; the model is called observable if M is right prime.

Example 2. An AR-model, i.e., an equation of the form $0 = R(\partial)w$, where R is a polynomial matrix of full row rank, can be viewed in an obvious way as a right ARMA-model. As such it is observable.

An ARMA-model of one type can be easily transformed into an ARMA-model of the other type. Indeed, (1) can be rewritten as

$$\begin{bmatrix} R(\partial) \\ M(\partial) \end{bmatrix} y = \begin{bmatrix} 0 \\ I \end{bmatrix} w,$$

and we call this the adjoint of (1). Likewise, (2) can be rewritten as

$$\begin{cases} \begin{bmatrix} M(\partial) & -R(\partial) \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = 0 \\ w = \begin{bmatrix} 0 & I \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \end{cases},$$

and we call this the adjoint of (2).

The adjoint of an ARMA-model $\mathfrak{A}$ will be denoted by $\mathfrak{A}^{ad}$.

Assume we have (1). Set $\mathcal{B} = Ker\, R(\partial)$. The operator $M(\partial)$ induces a homomorphism $\mu : \mathcal{B} \to \mathcal{U}^q$, which is quasi-injective. Indeed, its kernel clearly is equal to $Ker \begin{bmatrix} R(\partial) \\ M(\partial) \end{bmatrix}$, and the latter is finite-dimensional (by Lemma 3(a)). Hence, the pair $(\mathcal{B}, \mu)$ is a linear model.

Assume now we have (2). Set $\mathcal{B} = Ker \begin{bmatrix} M(\partial) & -R(\partial) \end{bmatrix}$, and let π denote the projection operator from $\mathcal{B}$ into $\mathcal{U}^q$. Clearly, $Ker\, \pi = Ker\, M(\partial)$. In view of Lemma 3(a), $Ker\, M(\partial)$ is finite-dimensional. It follows that π is quasi-injective, and therefore the pair $(\mathcal{B}, \pi)$ is a linear model.

Thus, associated with every ARMA-model $\mathfrak{A}$ there is a canonical linear model; the linear system determined by this linear model will be denoted by $Sigma(\mathfrak{A})$. We obviously have

$$Sigma(\mathfrak{A}^{ad}) = Sigma(\mathfrak{A}).$$

The importance of ARMA-models is due to the following theorem, the proof of which is left to the reader.

Theorem 1. *Every linear system has an ARMA-representation.*

It is clear that the input and output numbers of an ARMA-model $\mathfrak{A}$ are equal respectively to the input and output numbers of $Sigma(\mathfrak{A})$. Next, one can show easily that $\mathfrak{A}$ is controllable (observable) if and only if $Sigma(\mathfrak{A})$ is controllable (observable). (The assertion concerning observability follows from Lemma 3(b).)

Two left ARMA-models

$$\begin{cases} R_1(\partial)y_1 = 0 \\ w = M_1(\partial)y_1 \end{cases} \quad \text{and} \quad \begin{cases} R_2(\partial)y_2 = 0 \\ w = M_2(\partial)y_2 \end{cases}$$

are called homotopy equivalent if they have equal input numbers and if there exists a triple of polynomial matrices (U_1, U_2, K) satisfying the following conditions (Fuhrmann's conditions):

(1) $\begin{bmatrix} U_2 & 0 \\ K & I \end{bmatrix} \begin{bmatrix} R_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} R_2 \\ M_2 \end{bmatrix} U_1$;

(2) $\begin{bmatrix} U_1 \\ R_1 \end{bmatrix}$ is right prime and $\begin{bmatrix} U_2 & R_2 \end{bmatrix}$ is left prime.

Example 3. (a) Suppose that (1) is controllable. Let U be a maximal right annihilator of R, and put $M_1 = MU$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ M_1 \end{bmatrix} = \begin{bmatrix} R \\ M \end{bmatrix} U.$$

We can see that (1) is homotopy equivalent to the MA-model $w = M_1(\partial)y$.

(b) Two MA-models $w = M_1(\partial)y_1$ and $w = M_2(\partial)y_2$ are homotopy equivalent if and only if $M_2 = M_1 U$ for some unimodular matrix U.

Two right ARMA-models

$$M_1(\partial)z_1 = R_1(\partial)w \quad \text{and} \quad M_2(\partial)z_2 = R_2(\partial)w$$

are called homotopy equivalent if they have equal output numbers and if there exists a triple of polynomial matrices (U_1, U_2, K) satisfying the following conditions (Fuhrmann's conditions):

(1) $\begin{bmatrix} M_1 & R_1 \end{bmatrix} \begin{bmatrix} U_2 & K \\ 0 & I \end{bmatrix} = U_1 \begin{bmatrix} M_2 & R_2 \end{bmatrix}$;

(2) $\begin{bmatrix} U_1 & M_1 \end{bmatrix}$ is left prime and $\begin{bmatrix} U_2 \\ M_2 \end{bmatrix}$ is right prime.

Example 4. (a) Suppose that (2) is observable. Let U be a maximal left annihilator of M, and put $R_1 = UR$. Then

$$\begin{bmatrix} 0 & R_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = U \begin{bmatrix} M & R \end{bmatrix}.$$

We can see that (2) is homotopy equivalent to the AR-model $R_1(\partial)w = 0$.

(b) Two AR-models $R_1(\partial)w = 0$ and $R_2(\partial)w = 0$ are homotopy equivalent if and only if $R_2 = UR_1$ for some unimodular matrix U.

One shows that being homotopy equivalent is an equivalence relation indeed.

As we said, every linear system can be represented via an ARMA-model. The following theorem is the main result of the theory of "strict system equivalence". It gives an answer to the question: When two ARMA-models represent the same linear system?

Theorem 2. *Two ARMA-models (of the same type) determine the same linear system if and only if they are homotopy equivalent.*

Proof. It suffices to consider, say, the case of left ARMA-models. (The other case will follow immediately by transposition, which will be introduced in the next section.)

Assume that given are two left ARMA-models

$$\begin{cases} R_1(\partial)y = 0 \\ w = M_1(\partial)y \end{cases} \quad \text{and} \quad \begin{cases} R_2(\partial)y = 0 \\ w = M_2(\partial)y \end{cases}.$$

"If" Let (U_1, U_2, K) be a triple that determines the homotopy equivalence. We then have $U_2R_1 = R_2U_1$ and $KR_1 + M_1 = M_2U_1$. It follows that

$$\forall y \in Ker\, R_1(\partial), \quad R_2(\partial)U_1(\partial)y = U_2(\partial)R_1(\partial)y = 0.$$

Hence, $U_1(\partial)$ induces a differential operator from $Ker\, R_1(\partial)$ into $Ker\, R_2(\partial)$. Next, for every $y \in Ker\, R_1(\partial)$,

$$M_2(\partial)U_1(\partial)y = M_1(\partial)y + K(\partial)R_1(\partial)y = M_1(\partial)y.$$

Thus, we have a commutative diagram

$$\begin{array}{ccc} Ker\, R_1(\partial) & \xrightarrow{U_1(\partial)} & Ker\, R_2(\partial) \\ M_1(\partial) \downarrow & & \downarrow M_2(\partial) \\ \mathcal{U}^q & = & \mathcal{U}^q \end{array}.$$

It remains to see that $U_1(\partial)$ determines an isomorphism of $Ker\, R_1(\partial)$ onto $Ker\, R_2(\partial)$. For this, choose V_1, V_2, G and H so that

$$\begin{bmatrix} V_2 & R_1 \\ H & U_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_2 & -R_2 \\ -G & V_1 \end{bmatrix}$$

are inverse to each other. (This is possible by the primeness conditions.) We then have

$$R_1V_1 = V_2R_2, U_1V_1 = I + HR_2 \quad \text{and} \quad V_1U_1 = I + GR_1.$$

The first equality above implies that $V_1(\partial)$ induces a differential operator

$$Ker\, R_2(\partial) \to Ker\, R_1(\partial);$$

the two others imply that $V_1(\partial)U_1(\partial)$ and $U_1(\partial)V_1(\partial)$ induce the identity operators on $Ker\, R_1(\partial)$ and $Ker\, R_2(\partial)$, respectively.

"Only if" By the hypothesis, there exist polynomial matrices U_1 and V_1 such that

$$\forall y \in Ker\, R_1(\partial), \quad U_1(\partial)y \in Ker\, R_2(\partial),$$
$$\forall y \in Ker\, R_2(\partial), \quad V_1(\partial)y \in Ker\, R_1(\partial),$$
$$\forall y \in Ker\, R_1(\partial), \quad M_1(\partial)y = M_2(\partial)U_1(\partial)y,$$
$$\forall y \in Ker\, R_1(\partial), \quad V_1(\partial)U_1(\partial)y = y,$$
$$\forall y \in Ker\, R_2(\partial), \quad U_1(\partial)V_1(\partial)y = y.$$

These conditions can be rewritten respectively as

$$Ker\, R_1(\partial) \subseteq Ker(R_2(\partial)U_1(\partial)),$$
$$Ker\, R_2(\partial) \subseteq Ker(R_1(\partial)V_1(\partial)),$$
$$Ker\, R_1(\partial) \subseteq Ker(M_2(\partial)U_1(\partial) - M_1(\partial)),$$
$$Ker\, R_1(\partial) \subseteq Ker(V_1(\partial)U_1(\partial) - I),$$
$$Ker\, R_2(\partial) \subseteq Ker(U_1(\partial)V_1(\partial) - I).$$

Applying Lemma 1 to these inclusions respectively, we get

$$R_2U_1 = U_2R_1, \tag{3}$$
$$R_1V_1 = V_2R_2,$$
$$M_2U_1 - M_1 = KR_1, \tag{4}$$
$$V_1U_1 - I = GR_1, \tag{5}$$
$$U_1V_1 - I = HR_2$$

for some polynomial matrices U_2, V_2, K, G and H.

Multiplying (from the left) the equality $U_1V_1 - I = HR_2$ by R_2, we get

$$R_2U_1V_1 - R_2 = R_2HR_2.$$

Because $R_2U_1 = U_2R_1$ and because $R_1V_1 = V_2R_2$, it follows that

$$U_2V_2R_2 - R_2 = R_2HR_2.$$

From this, since R_2 is of full row rank, it follows that

$$U_2V_2 - I = R_2H. \tag{6}$$

By (3)–(6), (U_1, U_2, K) is a homotopy.

Obviously, the ARMA-models have the same input number, and the proof is complete. □

4. Duality

Given a left ARMA-model

$$\begin{cases} R(\partial)y = 0 \\ w = M(\partial)y \end{cases},$$

we define its transpose as the right ARMA-model

$$R^{tr}(\partial)z = M^{tr}(\partial)w.$$

The transpose of a right ARMA-model

$$M(\partial)z = R(\partial)w$$

is defined to be

$$\begin{cases} M^{tr}(\partial)y = 0 \\ w = R^{tr}(\partial)y \end{cases}.$$

Example 5. The transpose of an AR-model $0 = R(\partial)w$ is the MA-model $w = R^{tr}(\partial)y$. The transpose of an MA-model $w = M(\partial)z$ is the AR-model $0 = M^{tr}(\partial)w$.

The transpose of an ARMA-model $\mathfrak{A}$ will be denoted by $\mathfrak{A}^{tr}$. For every ARMA-model $\mathfrak{A}$, we clearly have

$$(\mathfrak{A}^{tr})^{tr} = \mathfrak{A}. \tag{7}$$

Before proceeding, we need the following lemma.

Lemma 4. *If two ARMA-models (of the same type) are homotopy equivalent, then so are their transposes.*

Proof. This is very easy. Indeed, suppose that we are given two, say, left ARMA-models

$$\begin{cases} R_1(\partial)y_1 = 0 \\ w = M_1(\partial)y_1 \end{cases} \quad \text{and} \quad \begin{cases} R_2(\partial)y_2 = 0 \\ w = M_2(\partial)y_2 \end{cases},$$

and suppose that they are homotopy equivalent. This means that they have equal input numbers and that there exists a triple of polynomial matrices (U_1, U_2, K) such that

(1) $\begin{bmatrix} U_2 & 0 \\ K & I \end{bmatrix}\begin{bmatrix} R_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} R_2 \\ M_2 \end{bmatrix} U_1;$

(2) $\begin{bmatrix} U_1 \\ R_1 \end{bmatrix}$ is right prime and $\begin{bmatrix} U_2 & R_2 \end{bmatrix}$ is left prime.

Consider the transposes of these models

$$R_1^{tr}(\partial)z_1 = M_1^{tr}(\partial)w \quad \text{and} \quad R_2^{tr}(\partial)z_2 = M_2^{tr}(\partial)w.$$

These are right ARMA-models. Clearly they have equal output numbers, and clearly from the conditions above we have

(1) $\begin{bmatrix} R_1^{tr} & M_1^{tr} \end{bmatrix}\begin{bmatrix} U_2^{tr} & K^{tr} \\ 0 & I \end{bmatrix} = U_1^{tr}\begin{bmatrix} R_2^{tr} & M_2^{tr} \end{bmatrix};$

(2) $\begin{bmatrix} U_1^{tr} & R_1^{tr} \end{bmatrix}$ is left prime and $\begin{bmatrix} U_2^{tr} \\ R_2^{tr} \end{bmatrix}$ is right prime.

The lemma follows. □

Let Σ be a linear system, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two of its ARMA-representations. Then, by Theorem 2, they are homotopy equivalent. By the previous lemma, the ARMA-models $\mathfrak{A}_1^{tr}$ and $\mathfrak{A}_2^{tr}$ also are homotopy equivalent. Consequently, by Theorem 2, they determine the same linear system. We therefore are in a position to make the following definition.

Definition. The dual Σ^* of a linear system Σ is defined by setting

$$\Sigma^* = Sigma(\mathfrak{A}^{tr}),$$

where $\mathfrak{A}$ is a (left or right) ARMA-representation of Σ.

The following theorem is the main result of the note. It states that the above duality is really a duality and that controllability and observability are dual concepts.

Theorem 3. *Let Σ be a linear system. Then $(\Sigma^*)^* = \Sigma$. If Σ is controllable (observable), then Σ^* is observable (controllable).*

Proof. This is immediate from (7) and the above definition. □

5. Conclusion

The starting point of this note has been the following remark made by Willems in his seminal paper Willems (1991): "... *in our* *framework controllability and observability are prima facie not dual concepts. Controllability is an intrinsic property of the behavior of a dynamical system, while the observability remains representation dependent*".

The reason why duality does not hold is that Willems' notion of linear systems is not enough flexible.

Fix a positive integer q, and consider pairs $(\mathcal{B}, \mu)$, where $\mathcal{B}$ is the solution set of a linear constant coefficient differential equation and μ is an injective differential operator from $\mathcal{B}$ to $\mathcal{U}^q$. Say that $(\mathcal{B}_1, \mu_1)$ is equivalent to $(\mathcal{B}_2, \mu_2)$ if there exists an isomorphism $\alpha : \mathcal{B}_1 \simeq \mathcal{B}_2$ such that $\mu_1 = \mu_2 \circ \alpha$. The binary relation is an equivalence relation. A linear differential system in the sense of Willems may equivalently be defined as an equivalence class. Indeed, a pair of the form $(\mathcal{B}, id)$, where $\mathcal{B} \subseteq \mathcal{U}^q$ is the solution set of a linear constant coefficient differential equation (in q variables), is a canonical representative or "normal form" among all pairs in an equivalence class.

In our opinion, there is no particular reason to require "μ" to be injective necessarily. Allowing it be quasi-injective, we get a more flexible notion that leads naturally to a duality.

References

Fuhrmann, P. A. (2001). On behavior homomorphisms and system equivalence. *Systems & Control Letters*, *44*, 127–134.

Fuhrmann, P. A. (2002). A study of behaviors. *Linear Algebra and its Applications*, *351/352*, 303–380.

Kuijper, M. (1994). *First order representations of linear systems*. Boston: Birkhäuser.

Lomadze, V. (2011). On homotopy and similarity in linear systems theory. *Acta Applicandae Mathematicae*, *116*, 87–105.

Pillai, H., Wood, J., & Rogers, E. (2002). On homomorphisms of *n*-D behaviors. *IEEE Transactions on Circuits and Systems*, *49*, 732–742.

Polderman, J. W., & Willems, J. C. (1998). *Introduction to mathematical systems theory*. New York: Springer.

Schumacher, J. M. (1998). Transformations of linear systems under external equivalence. *Linear Algebra and its Applications*, *102*, 1–33.

Willems, J. C. (1991). Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, *36*, 259–294.