Systems & Control Letters 94 (2016) 25–30

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Proper representations of (multivariate) linear differential systems

Vakhtang Lomadze

A. Razmadze Mathematical Institute, Mathematics Department of I. Javakhishvili State University, Georgia

ARTICLE INFO

Article history:
Received 10 December 2015
Received in revised form
4 May 2016
Accepted 5 May 2016
Available online 30 May 2016

Keywords:
LTI differential system
Graded and filtered modules
Regularity at infinity
WPD property
Proper representation

ABSTRACT

A proper representation of a linear differential system is a representation with no singularity at infinity. It is shown that such a representation always exists. It turns out that for proper representations having minimal number of rows is equivalent to having minimal total row degree. One is led therefore to a natural definition of the notion of minimality. What is remarkable is that a minimal proper representation is uniquely determined up to premultiplication by a unimodular polynomial matrix of special form. This uniqueness result allows, in particular, to introduce important integer invariants.

1. Introduction

Throughout, $\mathbb{F}$ is the field of real or complex numbers, n and q are fixed positive integers, $s = (s_1, \dots, s_n)$ is a sequence of indeterminates, and s_0 is an extra ("homogenizing") indeterminate. We let $S = \mathbb{F}[s]$ and $T = \mathbb{F}[s_0, s]$, and denote by $\mathcal{U}$ the space of C^∞-functions (or distributions) defined on some domain of $\mathbb{R}^n$.

Proper polynomial matrices are polynomial matrices (with entries in S) that behave well at infinity. They play a significant role in the classical one-dimensional linear systems theory, and we claim that their role in higher dimensions must be analogous. (The infinity is the complement of the affine space $\mathbb{A}^n$ to the projective space $\mathbb{P}^n$, that is, the hyperplane in $\mathbb{P}^n$ defined by the equation $s_0 = 0$.)

Assume that we have a linear time-invariant (LTI) differential system $\mathcal{B} \subseteq \mathcal{U}^q$, and assume that it is represented by a polynomial matrix $R \in S^{p\times q}$, so that

$$\mathcal{B} = KerR(\partial).$$

As is well-known, the submodule $R^{tr}S^p \subseteq S^q$ is independent of the choice of R and is an intrinsic invariant of $\mathcal{B}$; moreover, by Oberst's duality, this is a full invariant. There is a procedure, called *homogenization* (and denoted here by the superscript "h"), that produces homogeneous things from non-homogeneous ones. Homogenizing the submodule $R^{tr}S^p \subseteq S^q$, we get a homogeneous submodule $(R^{tr}S^p)^h \subseteq T^q$. Like $R^{tr}S^p$, this module also is an intrinsic full invariant. Alternatively, one can homogenize first R and then take the homogeneous submodule $(R^h)^{tr}T^p \subseteq T^q$. The latter, however, depends on R and is *not* an invariant of $\mathcal{B}$. One has

$$(R^h)^{tr}T^p \subseteq (R^{tr}S^p)^h.$$

The equality holds if and only if s_0 is not a zero divisor on the quotient module $T^q/(R^h)^{tr}T^p$. In our opinion, representations having this property are of primary importance, and we call them proper.

We think that it is not proper to represent an LTI differential system via an improper polynomial matrix since it does not provide an adequate description at infinity.

Remark. As explained in the concluding section, properness should be interpreted as the property of "controllability at infinity".

In this paper, we prove that proper representations always exist. Next, we show that for proper representations there is a good notion of minimality. Namely, we show that if R is a proper representation of an LTI differential system $\mathcal{B}$, then the following two conditions are equivalent:

(a) R has the minimum possible number of rows (among all proper representations of $\mathcal{B}$);
(b) R has the minimum possible total row degree (among all proper representations of $\mathcal{B}$).

Proper representations satisfying these conditions are called minimal. The uniqueness result that we prove states that minimal proper representations are uniquely determined up to, the so-called, Brunovsky equivalence.

E-mail address: vakhtang.lomadze@tsu.ge.

http://dx.doi.org/10.1016/j.sysconle.2016.05.003

Remark. As is well-known, in dimension 1, an LTI differential system has a full row rank proper representation, which certainly is a minimal proper representation. This is unfortunately not the case in higher dimensions; even more, an LTI differential system may have a full row rank representation, but not a full row rank proper representation (see Example 10 in Section 6).

For every $d \in \mathbb{Z}$, we shall write $S_{\leq d}$ to denote the space of polynomials (in S) of degree $\leq d$ and T_d for the space of homogeneous polynomials (in T) of degree d. It is worth noting that $S_{\leq d} = \{0\}$ and $T_d = \{0\}$ for negative d. (In Section 6, we shall need homogeneous polynomials in S as well, and S_d will stand for the space of all homogeneous polynomials that have degree d.) For a positive integer p, we write $[1, p]$ for the set $\{1, \dots, p\}$.

This article can be viewed as an attempt to generalize Section X in Willems [1] to higher dimensions. We remark also that much of material presented here is adapted from [2] (which, in turn, is based on [3]).

2. Preliminaries on graded and filtered modules

Powerful tools for the study of LTI differential systems are S-modules. But S-modules disregard the infinity, and therefore are useless when one wants to carry out the study at infinity. Graded T-modules have the advantage that they allow to study LTI differential systems (simultaneously) both on the finite domain and at the infinity.

A graded module over T is a module M together with a gradation, i.e., a sequence $M_0, M_1, M_2, \dots$ of $\mathbb{F}$-linear subspaces of M such that

$$M = \bigoplus_{d \geq 0} M_d \quad \text{and} \quad s_k M_d \subseteq M_{d+1} \quad \forall k, d.$$

(For $d < 0$, one puts $M_d = \{0\}$.) The elements of M_d are called the homogeneous elements of M of degree d. A submodule $N \subseteq M$ is called a graded submodule of M if $N = \bigoplus (N \cap M_d)$.

For a graded T-module M and a nonnegative integer k, one denotes by $M(-k)$ the graded T-module whose homogeneous components are defined by

$$M(-k)_d = M_{d-k}.$$

Example 1. Let p be a positive integer. Then, a function $a : [1, p] \to \mathbb{Z}_+$ determines on T^p a gradation consisting of the spaces

$$T^p(-a)_d = \{f \in T^p \mid \deg(f_i) = d - a(i)\} \quad (d \geq 0).$$

The module T^p equipped with this gradation is denoted by $T^p(-a)$. Notice that

$$T^p(-a) = T(-a(1)) \oplus \cdots \oplus T(-a(p)).$$

A homomorphism of graded modules $M \to N$ is a module homomorphism $u : M \to N$ such that $u(M_d) \subseteq N_d$ for all $d \geq 0$.

Example 2. Let a and b be nonnegative integers. Homomorphisms from $T(-a)$ to $T(-b)$ are exactly multiplications by homogeneous polynomials of degree $a - b$. That is,

$$Hom(T(-a), T(-b)) = T_{a-b}.$$

A polynomial matrix with entries in T is called column-homogeneous if all the entries in each column are homogeneous and have the same degree.

Example 3. A column-homogeneous polynomial matrix H of size $q \times p$ and with column degree function a determines a homomorphism of graded modules

$$H : T^p(-a) \to T^q.$$

The homogenization in degree d is the bijective linear map $\theta_d : S_{\leq d} \to T_d$ defined by the formula

$$\theta_d(f) = s_0^d f(s/s_0).$$

(Here and below s/s_0 stands for $(s_1/s_0, \dots, s_n/s_0)$.)

Example 4. Let $n = 2$ and $f = 2s_1^3 s_2 + 1$. Then

$$\theta_4(f) = 2s_1^3 s_2 + s_0^4 \quad \text{and} \quad \theta_5(f) = 2s_0 s_1^3 s_2 + s_0^5.$$

If $A \subseteq S^q$ is a submodule, the homogenization A^h of A is defined to be

$$A^h = \bigoplus_{d \geq 0} A_d^h,$$

where $A_d^h = \theta_d(A_{\leq d})$. This is the smallest graded submodule of T^q that contains A.

The dehomogenization is the operator $T \to S$ defined by

$$u(s_0, s) \mapsto u(1, s).$$

It is worth noting that if $d \geq 0$, then

$$\begin{aligned} &\forall u \in T_d, \quad \theta_d(u(1, s)) = u \quad \text{and} \\ &\forall f \in S_{\leq d}, \quad (\theta_d f)(1, s) = f. \end{aligned} \tag{1}$$

If $B \subseteq T^q$ is a graded submodule, the dehomogenization B^{dh} of B is its image under the dehomogenization operator, i.e.,

$$B^{dh} = \{u(1, s) \mid u \in B\}.$$

This is a submodule of S^q.

We pass now to filtered S-modules, which are more natural tools than graded T-modules. (However, graded modules are superior from the purely technical point of view.) The point is that the modules associated with LTI differential systems have the structure of a filtered S-module.

Let M be a module over S. A filtration on M is an ascending chain

$$M_{\leq 0} \subseteq M_{\leq 1} \subseteq M_{\leq 2} \subseteq \cdots$$

of linear subspaces of M such that

$$M = \bigcup_{d \geq 0} M_{\leq d} \quad \text{and} \quad s_k M_{\leq d} \subseteq M_{\leq d+1} \quad \forall k, d.$$

A module with a filtration is called a filtered module. A submodule N of a filtered module M is a filtered module with the filtration

$$N_{\leq d} = N \cap M_{\leq d}, \quad d \geq 0.$$

(For $d < 0$, put $M_{\leq d} = \{0\}$.) If M is a filtered module and k a nonnegative integer, we denote by $M[-k]$ the filtered module with the filtration defined by

$$M[-k]_{\leq d} = M_{\leq d-k}.$$

Example 5. Let p be a positive integer. Then, a function $a : [1, p] \to \mathbb{Z}_+$ determines on S^p a filtration consisting of the spaces

$$S^p[-a]_{\leq d} = \{f \in S^p \mid \deg(f_i) \leq d - a(i)\} \quad (d \geq 0).$$

The module S^p equipped with this filtration is denoted by $S^p[-a]$. Notice that

$$S^p[-a] = S[-a(1)] \oplus \cdots \oplus S[-a(p)].$$

A homomorphism of filtered modules $M \to N$ is a module homomorphism $u : M \to N$ such that

$$\forall d \geq 0, \quad u(M_{\leq d}) \subseteq N_{\leq d}.$$

Notice that $Ker(\varphi)$ is a graded submodule of M and $Im(\varphi)$ is a graded submodule of N.

Example 6. Let a and b be nonnegative integers. Homomorphisms from $S[-a]$ to $S[-b]$ are exactly multiplications by polynomials of degree $\leq a-b$. That is,

$$Hom(S[-a], S[-b]) = S_{\leq a-b}.$$

Example 7. A polynomial matrix G of size $q \times p$ and with column degree function a determines a homomorphism of filtered modules

$$G : S^p[-a] \to S^q.$$

One has the obvious notion of isomorphisms of filtered modules.

Lemma 1. *Let $a_1 : [1, p_1] \to \mathbb{Z}_+$ and $a_2 : [1, p_2] \to \mathbb{Z}_+$ be two functions. If*

$$S^{p_1}[-a_1] \simeq S^{p_2}[-a_2],$$

then $p_1 = p_2$ and $a_1 = a_2$ (up to permutation).

Proof. See Lemma 1 in [3]. □

For each $p \geq 1$, one denotes by $GL(p, S)$ the group of unimodular matrices of size p. The following lemma is immediate from Example 6.

Lemma 2. *Let $a : [1, p] \to \mathbb{Z}_+$ be given. Then,*

$$Aut(S^p[-a]) = \{(u_{ij}) \in GL(p, S) \mid \deg(u_{ij}) \leq a(j) - a(i) \ \ \forall i, j\}.$$

3. Regularity at infinity and the WPD property

Given a graded submodule $B \subseteq T^q$, we say that B is *regular* (or non-singular) at infinity if s_0 is not a zero divisor on the module T^q/B, or equivalently, if the linear maps

$$T^q_d/B_d \stackrel{s_0}{\to} T^q_{d+1}/B_{d+1}, \quad d \geq 0$$

are injective, that is,

$$\forall d \geq 0 \quad (f \in T^q_d \text{ and } s_0 f \in B_{d+1} \ \Rightarrow \ f \in B_d).$$

Lemma 3. *The mapping*

$$A \mapsto A^h$$

establishes a one-to-one correspondence between submodules of S^q and graded submodules of T^q that are regular at infinity.

Proof. Let A be a submodule of S^q. We claim that A^h is regular at infinity. Indeed, let $d \geq 0$ and suppose that $u \in T^q_d$ is such that $s_0 u \in A^h_{d+1}$. It is clear that $u(1, s)$ has degree $\leq d$ and belongs to A. Thus, we have $u(1, s) \in A_{\leq d}$. By (1), u is the d-homogenization of $u(1, s)$, and therefore belongs to A^h_d. The claim is proved.

Now, let B be a graded submodule of T^q that is regular at infinity, and put $A = B^{dh}$. We claim that $B = A^h$. Indeed, let again $d \geq 0$. In view of (1), it is clear that $B_d \subseteq A^h_d$. To show the inverse inclusion, take any element in A^h_d and write it as $\theta_d(f)$ with $f \in A_{\leq d}$. There is a sufficiently large $k \geq 0$ and there is $u \in B_{d+k}$ such that $f = u(1, s)$. Using once again (1), we get

$$\begin{aligned} s_0^k \theta_d(f) &= s_0^k s_0^d f(s/s_0) = s_0^{d+k} f(s/s_0) \\ &= \theta_{d+k}(f) = \theta_{d+k}(u(1, s)) = u \in B_{d+k}. \end{aligned}$$

From this, since s_0^k is not a zero divisor on T^q/B, we get that $\theta_d(f) \in B_d$. The claim is proved.

The lemma follows. □

Let G be a polynomial matrix of size $q \times p$ and with column degree function a. Define the homogenization G^h of G by setting

$$G^h = G(s/s_0) s_0^a,$$

where s_0^a stands for the diagonal matrix with $s_0^{a(1)}, \ldots, s_0^{a(p)}$ on the diagonal.

Associated with G there are two graded submodules in T^q, namely,

$$G^h T^p \quad \text{and} \quad (GS^p)^h.$$

Remark. G^h determines a homomorphism $T^p(-a) \to T^q$, and so $G^h T^p = G^h T^p(-a)$ is indeed a graded submodule of T^q.

Using the relations $G(S^p[-a]_{\leq d}) \subseteq (GS^p)_{\leq d}$, one can easily see that

$$G^h T^p \subseteq (GS^p)^h.$$

The following simple example tells us that the equality does not hold always.

Example 8. Let $n = 1$, and consider the matrix $G = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$. This is a unimodular matrix, and hence $GS^2 = S^2$. Consequently, $(GS^2)^h = T^2$. However, the homogenization of G is $G^h = \begin{bmatrix} 1 & s \\ 0 & s_0 \end{bmatrix}$, and we have

$$G^h T^2 = T \oplus s_0 T \neq T^2.$$

Definition. Say that G is regular at infinity if the graded submodule

$$G^h T^p \subseteq T^q$$

is regular at infinity.

Definition. Say that G has the weak predictable degree (WPD) property if the linear map

$$GS^p[-a]_{\leq d} = (GS^p)_{\leq d} \quad \forall d \geq 0.$$

Theorem 1. *The following three conditions are equivalent:*

(a) *G is regular at infinity;*
(b) *$G^h T^p = (GS^p)^h$;*
(c) *G has the* WPD *property.*

Proof. (a) ⇔ (b) follows from Lemma 3 (and the equality $(G^h T^p(-a))^{dh} = GS^p$).

(b) ⇔ (c) follows from the commutative diagrams

$$\begin{array}{ccc} S^p[-a]_{\leq d} & \stackrel{G}{\to} & S^q_{\leq d} \\ \downarrow & & \downarrow \\ T^p(-a)_d & \stackrel{G^h}{\to} & T^q_d \end{array}.$$

(The left vertical arrow here is $\mathrm{diag}(\theta_{d-a_1}, \ldots, \theta_{d-a_p})$ and the right one is θ^q_d.)

The proof is complete. □

4. The integer invariants

Given a filtered module M, for every $d \geq 1$, we define

$$\Gamma_d(M) = \frac{M_{\leq d}}{M_{\leq d-1} + s_1 M_{\leq d-1} + \cdots + s_n M_{\leq d-1}}.$$

This is a linear space over $\mathbb{F}$.

Example 9.

$$\Gamma_d(S[-k]) = \begin{cases} \mathbb{F} & \text{when } d = k; \\ \{0\} & \text{when } d \neq k. \end{cases}$$

Lemma 4. *If $A \subseteq S^q$ is a submodule, then all the linear spaces $\Gamma_d(A)$ have finite dimension. Moreover, they all are trivial except for a finite number.*

Proof. See Theorem 1 in [2]. □

Recall that if $\mathcal{B} \subseteq \mathcal{U}^q$ is an LTI differential system, then its associated module A is defined by setting

$$A = \{f \in \mathbb{F}[s]^q \,|\, f^{tr}(\partial)w = 0\, \forall w \in \mathcal{B}\}.$$

(If R is a representation of $\mathcal{B}$, then $A = ImR^{tr}$.)

The previous lemma permits us to introduce very important integer invariants.

Definition. Let $\mathcal{B}$ be an LTID system, and let A be its associated module. For every $d \geq 0$, set

$$\gamma_{\mathcal{B}}(d) = \dim(\Gamma_d(A)).$$

Following Willems [1], call $\gamma_{\mathcal{B}}(d),\ d \geq 0$ the structural indexes of $\mathcal{B}$. Define $\pi(\mathcal{B})$ and $\delta(\mathcal{B})$ by setting

$$\pi(\mathcal{B}) = \sum \gamma_{\mathcal{B}}(d) \quad \text{and} \quad \delta(\mathcal{B}) = \sum d\gamma_{\mathcal{B}}(d).$$

Call the integers d, for which $\gamma_{\mathcal{B}}(d) \neq 0$, observability indices. Every observability index d is counted with multiplicity; the multiplicity is $\gamma_{\mathcal{B}}(d)$.

5. Proper representations

Given a function $a : [1, p] \to \mathbb{Z}_+$, let $\Gamma(a)$ denote the group of unimodular polynomial matrices $(u_{ij}) \in GL(p, S)$ such that

$$\forall i, j \quad \deg(u_{ij}) \leq a(i) - a(j).$$

Remark. In dimension 1, this group is well-known for systems community (see, for example, Fuhrmann and Willems [4]).

Remark. Assume that a is increasing. Let

$$d_1 < \cdots < d_r$$

be its different values and, for each $k \in [1, r]$, let $\gamma(k)$ be the number of times that a takes the value d_k. Then $\Gamma(a)$ consists of matrices

$$\begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{21} & U_{22} & 0 & \cdots & 0 \\ U_{31} & U_{32} & U_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_{r1} & U_{r2} & U_{r3} & \cdots & U_{rr} \end{bmatrix}$$

with $U_{ij} \in (S_{\leq d_i - d_j})^{\gamma(i)\times\gamma(j)}$ and $U_{11}, U_{22}, \ldots, U_{rr}$ nonsingular. (Compare with unimodular matrices in Section X, Willems [1].)

Two polynomial matrices R_1 and R_2 with the same row number p are said to be Brunovsky equivalent if there exist $a : [1, p] \to \mathbb{Z}_+$, $U \in \Gamma(a)$, and permutation matrices P_1, P_2 such that

$$P_2R_2 = UP_1R_1.$$

Let $u : M \to N$ be a homomorphism of filtered modules such that the linear maps

$$M_{\leq d} \to N_{\leq d}, \quad d \geq 0$$

are surjective. Then, clearly, the linear maps

$$\Gamma_d(u) : \Gamma_d(M) \to \Gamma_d(N), \quad d \geq 0$$

also are surjective. Say that u is a quasi-isomorphism when all these maps are bijective.

A polynomial matrix R is called proper if its transpose is regular at infinity.

Theorem 2. *Let $\mathcal{B}$ be an* LTI *differential system, and let A be its associated module. Then, there exists a proper representation R such that*

$$R^{tr} : S^p[-a] \to A,$$

where p is the row number of R and a the row degree function, is a quasi-isomorphism. Moreover, such a representation is uniquely determined up to Brunovsky equivalence.

Proof. Follows from Theorem 2 in [2]. □

Definition. Any R satisfying the condition of Theorem 2 is called a minimal proper representation of $\mathcal{B}$.

If R is a polynomial matrix, we let $\pi(R)$ denote the number of rows in R and $\delta(R)$ the total row degree of R (i.e., the sum of all its row degrees).

Theorem 3. *Let $\mathcal{B}$ be an* LTI *differential system, and let R be its proper representation. Then,*

$$\pi(\mathcal{B}) \leq \pi(R) \quad \text{and} \quad \delta(\mathcal{B}) \leq \delta(R).$$

Moreover, the following three conditions are equivalent:

(a) *R is minimal;*
(b) $\pi(R) = \pi(\mathcal{B})$;
(c) $\delta(R) = \delta(\mathcal{B})$.

Proof. The proof is the same as that of Theorem 3 in [2]. Since it is very easy, we reproduce it.

Let a denote the row degree function of R. For each $d \geq 0$, let $\gamma_R(d)$ denote the number of values of a equal to d. Then

$$\pi(R) = \sum \gamma_R(d) \quad \text{and} \quad \delta(R) = \sum d\gamma_R(d).$$

In view of Example 9, we have: $\gamma_R(d) = \dim \Gamma_d(S^p[-a])$. Because the linear map $\Gamma_d(S^p[-a]) \to \Gamma_d(A)$ is surjective, $\gamma_R(d) \geq \gamma_{\mathcal{B}}(d)$. Consequently,

$$\pi(R) = \sum \gamma_R(d) \geq \sum \gamma_{\mathcal{B}}(d) = \pi(\mathcal{B}) \quad \text{and}$$
$$\delta(R) = \sum d\gamma_R(d) \geq \sum d\gamma_{\mathcal{B}}(d) = \delta(\mathcal{B}).$$

Certainly, (b) and (c) hold if and only if $\gamma_R(d) = \gamma_{\mathcal{B}}(d)$ for every d. But this, in turn, is equivalent to bijectivity of all the linear maps $\Gamma_d(S^q[-a]) \to \Gamma_d(A)$.

The proof is complete. □

As a consequence, we have the following two characterizations of minimality.

Corollary 1. (a) *A minimal proper representation is a one that has minimal number of rows (among all proper representations).*
(b) *A minimal proper representation is a one that has minimal total row degree (among all proper representations).*

Needless to say that the row degrees of a minimal proper representation coincide with the observability indices.

6. Illustrative examples

Before we consider examples, we want to give the following remark. Assume we have an LTI differential system $\mathcal{B} \subseteq \mathcal{U}^q$, and assume further that R is its proper representation. Let $d_1, \ldots, d_r$ be the different row degrees of R, and let $\gamma_1, \ldots, \gamma_r$ be their "multiplicities". The transpose matrix $G = R^{tr}$ determines a homomorphism of filtered modules

$$G : S[-d_1]^{\gamma_1} \oplus \cdots \oplus S[-d_r]^{\gamma_r} \to S^q.$$

Put $A = ImG$. For every $d \geq 0$, the linear map

$$\Gamma_d(\oplus S[-d_i]^{\gamma_i}) \xrightarrow{G} \Gamma_d(A)$$

is surjective (because R is proper). This map automatically is bijective when d differs from all of $d_1, \ldots, d_r$ since $\Gamma_d(S[-d_i]) = \{0\}$. When $d = d_i$, we have: $\Gamma_d(\oplus S[-d_i]^{\gamma_i}) = \mathbb{F}^{\gamma_i}$. So, if one wants to prove minimality of R, it suffices to check injectivity of the linear maps

$$\mathbb{F}^{\gamma_i} \xrightarrow{G} \Gamma_{d_i}(A), \quad i = 1, \ldots, r.$$

Example 10. Let $n = 2$, and let $\mathcal{B} \subseteq \mathcal{U}^2$ be the solution set of

$$\begin{cases} w_1 + \partial_1^2 w_2 = 0 \\ \partial_2 w_2 = 0 \end{cases}.$$

A representation of $\mathcal{B}$ is

$$\begin{bmatrix} 1 & s_1^2 \\ 0 & s_2 \end{bmatrix}.$$

This is a full row rank representation. However, it is not proper; its transpose

$$\begin{bmatrix} 1 & 0 \\ s_1^2 & s_2 \end{bmatrix} : S[-2] \oplus S[-1] \to S^2$$

does not have the WPD property. (Indeed, for example,

$$\begin{pmatrix} s_2 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ s_1^2 & s_2 \end{bmatrix} \begin{pmatrix} s_2 \\ -s_1^2 \end{pmatrix} \in A_{\leq 1},$$

but it does not belong to $\begin{bmatrix} 1 & 0 \\ s_1^2 & s_2 \end{bmatrix} (\{0\} \oplus \mathbb{F})$.)

The equation above is equivalent to

$$\begin{cases} w_1 + \partial_1^2 w_2 = 0 \\ \partial_2 w_1 = 0 \\ \partial_2 w_2 = 0 \end{cases},$$

and consequently the polynomial matrix

$$R = \begin{bmatrix} 1 & s_1^2 \\ s_2 & 0 \\ 0 & s_2 \end{bmatrix}$$

also is a representation. Its transpose

$$G = \begin{bmatrix} 1 & s_2 & 0 \\ s_1^2 & 0 & s_2 \end{bmatrix} : S[-2] \oplus S[-1]^2 \to S^2$$

has the WPD property. Indeed, assume that $\begin{pmatrix} x \\ s_1^2 x + s_2 y \end{pmatrix}$ is an element in $A_{\leq d}$. Then

$$x \in S_{\leq d} \quad \text{and} \quad y \in S_{\leq d+1}$$

necessarily. Define polynomials $f \in S_{d-1} + S_d$ and $g \in S_d + S_{d+1}$ so that

$$x - f \in S_{\leq d-2} \quad \text{and} \quad y - g \in S_{\leq d-1}.$$

Since $s_1^2 x + s_2 y \in S_{\leq d}$, we must have $s_1^2 f + s_2 g = 0$. It follows that $f = s_2 u$ and $g = -s_1^2 u$ for some $u \in S_{\leq d-1}$. We then have:

$$\begin{pmatrix} x - f \\ u \\ y - g \end{pmatrix} \in S_{\leq d-2} \oplus S_{\leq d-1} \oplus S_{\leq d-1} \quad \text{and}$$

$$G \begin{pmatrix} x - f \\ u \\ y - g \end{pmatrix} = \begin{pmatrix} x \\ s_1^2 x + s_2 y \end{pmatrix}.$$

Thus, R is a proper representation.

To show that R is minimal, we only need to check that the linear maps

$$\mathbb{F}^2 \to \Gamma_1(A) \quad \text{and} \quad \mathbb{F} \to \Gamma_2(A)$$

are injective.

Notice that $A_{\leq 1} = s_2 \mathbb{F}^2$. Because $A_{\leq 0} = \{0\}$, $\Gamma_1(A) = s_2 \mathbb{F}^2$ and the first linear map is

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto s_2 \begin{pmatrix} a \\ b \end{pmatrix}.$$

This, of course, is injective. The second linear map above is given by

$$c \mapsto \begin{pmatrix} c \\ cs_1^2 \end{pmatrix} \operatorname{mod}(A_{\leq 1} + s_1 A_{\leq 1} + s_2 A_{\leq 1}).$$

Obviously, $\begin{pmatrix} 1 \\ s_1^2 \end{pmatrix} \notin A_{\leq 1} + s_1 A_{\leq 1} + s_2 A_{\leq 1}$, and this implies that the map is injective.

We have $\pi(\mathcal{B}) = 3$ and $\delta(\mathcal{B}) = 4$; the observability indices are 2, 1, 1.

The next example is more interesting. (It was suggested to consider by one of the referees.)

Example 11. Let $n = 3$, and let

$$\mathcal{B} = \{w \in \mathcal{U}^3 | \, curl(w) = 0\}.$$

A natural representation of $\mathcal{B}$ is

$$R = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}.$$

The transpose

$$G = \begin{bmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{bmatrix}$$

determines a *homogeneous* homomorphism

$$S(-1)^3 \to S^3.$$

(Like T, S also is a graded ring. So, one can consider graded modules over S. The definition of the "shifted" graded modules $S(-d)$ is the same as that of $T(-d)$ given in Section 2.) Therefore, the image $A = ImG$ is a graded submodule of S^3, i.e.,

$$A = \bigoplus (A \cap S_d^3).$$

The matrix G has a property that is much stronger than the WPD property. Namely, there holds

$$GS_{d-1}^3 = A_d \quad \forall d \geq 0,$$

where $A_d = A \cap S_d^3$. Since $A_{\leq d} = A_d + \cdots + A_1 + A_0$, it is clear from this that

$$\forall d \geq 0, \quad GS_{\leq d-1}^3 = A_{\leq d}.$$

Thus, R is a proper representation.

Further, $A_0 = \{0\}$ and so $\Gamma_1(A) = A_{\leq 1} = A_1$. The linear map $\mathbb{F}^3 \xrightarrow{G} A_1$, given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} bs_3 - cs_2 \\ cs_1 - as_3 \\ as_2 - bs_1 \end{pmatrix},$$

clearly is injective. It follows that R is minimal.

We have $\pi(\mathcal{B}) = 3$ and $\delta(\mathcal{B}) = 3$; the observability indices are 1, 1, 1.

7. Concluding remarks

If $f \in S$ is a nonconstant polynomial, the set of zeros of f in $\mathbb{A}^n$ is called the affine hypersurface defined by f, and is denoted by $V(f)$. Likewise, if $f \in T$ is a nonconstant form, the set of zeros of f in $\mathbb{P}^n$ is called the projective hypersurface defined by f, and is denoted by $V(f)$.

There is a one-to-one correspondence between affine hypersurfaces and nonconstant polynomials with no multiple factors (and determined up to multiplication by a nonzero constant). Likewise, there is a one-to-one correspondence between projective hypersurfaces and nonconstant forms with no multiple factors (and determined up to multiplication by a nonzero constant).

The infinite hyperplane is the hypersurface defined by the form s_0.

If $H = V(f)$ is an affine hypersurface, then $\overline{H} = V(f^h)$ is called the projective closure of H.

Of special interest are irreducible hypersurfaces, i.e., hypersurfaces defined by irreducible polynomials. The assignment $H \mapsto \overline{H}$ yields a bijective correspondence between irreducible affine hypersurfaces and irreducible projective hypersurfaces that are distinct from the infinite hyperplane.

A frequency is an irreducible projective hypersurface. The infinite frequency, denoted by ∞, is the infinite hyperplane. All other frequencies are called finite and can be identified with irreducible affine hypersurfaces.

If B is a graded submodule of T^q and if ϕ is a frequency, say that B is *regular* at ϕ if the defining form of ϕ is not a zero divisor on T^q/B.

A submodule $A \subseteq S^q$ can be homogenized, and by homogenizing one gets a graded submodule $A^h \subseteq T^q$. A simple observation is that A^h is regular at a finite frequency ϕ if and only if the dehomogenization of the defining polynomial of ϕ is not a zero divisor on S^q/A. Further, one can easily show that A^h is regular at ∞.

Let now $\mathcal{B} \subseteq \mathcal{U}^q$ be an LTI differential system, and let $A \subseteq S^q$ be the ("non-homogeneous") associated module. Define the associated homogeneous module to be the module A^h. This is an intrinsic invariant that contains the whole information about the behavior of $\mathcal{B}$ at all frequencies. Recall that $\mathcal{B}$ is called controllable if S^q/A is torsion-free. Saying that S^q/A is torsion-free is clearly equivalent to saying that no irreducible polynomial is a zero divisor on S^q/A. One naturally comes to a local definition of controllability. Say that $\mathcal{B}$ is controllable at a frequency ϕ if A^h is regular at ϕ. According to this definition, $\mathcal{B}$ is *a priori* controllable at ∞. (In our opinion, this is as it should be!). We obviously have that $\mathcal{B}$ is controllable if and only if it is controllable at all frequencies.

All representations of an LTI differential system provide an adequate description at finite frequencies but not necessarily at the infinite one. Proper representations are those representations that take into account the structure at infinity as well. The main result of this paper is exactly the definition of proper representations.

We have proved that proper representations exist and then explored the minimality problem.

References

[1] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Automat. Control 36 (1991) 259–294.

[2] V. Lomadze, The weak predictable degree property and minimality in multidimensional convolutional coding, Adv. Math. Commun. (submitted for publication).

[3] V. Lomadze, "Reduced polynomial matrices" in several variables, SIAM J. Control Optim. 51 (2013) 3258–3273.

[4] P. Fuhrmann, J.C. Willems, Factorization indices at infinity for rational matrix functions, Int. Equ. Oper. Theory 2 (1979) 287–301.