Linear Algebra and its Applications 351–352 (2002) 519–532
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Rosenbrock models and their homotopy equivalence

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Received 10 March 2001; accepted 4 September 2001

Submitted by J. Rosenthal

Abstract

As is known, the notion of homotopy equivalence is a fundamental notion of mathematics and was introduced in order to formalize a relation that is weaker than isomorphism. In this note we define a homotopy equivalence of Rosenbrock systems and show that it coincides with the classical equivalences of Rosenbrock and Fuhrmann. Next, we show that the homotopy equivalence does preserve the important properties of a system (including the properties at infinity when these are properly understood). Finally, we define in a simple manner the states and motions of a system and claim that they are homotopy invariants. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Rosenbrock model; Homotopy; Kalman model; Pole modules; Zero modules; State space; Motions

1. Introduction

The general idea of homotopy is to “deform” slightly a complicated object into another object, which is simpler and shares important properties of the original one. The idea comes from algebraic topology (see, for example, [10]) and homological algebra (see, for example, [9]). The reader recognizes of course that the idea of Rosenbrock concerning “polynomial system matrices” is very similar. In [15] Rosenbrock determined a sort of “deformation” applying which a general system of the type

$$T(s)z = U(s)u, \qquad y = V(s)z + W(s)u \tag{1}$$

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PII: S0024-3795(01)00506-7

can be brought to a special system of the type

$$(sI - A)x = Bu, \qquad y = Cx + D(s)u. \tag{2}$$

As one knows, several important properties are left invariant in the process.

In this paper we construct a homotopy theory of Rosenbrock models in full analogy to the classical homotopy theories. We then demonstrate that the homotopy equivalence coincides with the equivalences defined by Rosenbrock [15,17] and Fuhrmann [3]. Next, we show that it leaves unchanged fundamental structural properties (including the properties at infinite frequency). We also address the question of how to define the states and motions of a Rosenbrock model.

It should be emphasized that our approach to the notion of structure at infinity is different from those that exist in the literature (see, for example, [2,16,19]). In our approach systems of the form (1) and (2) have no infinite decoupling zeros. In other words, our standpoint is that most of the systems studied in the literature are in fact controllable and observable at infinity.

To explain the naturalness of this point of view we need to recall the Willems idea of considering systems in which external variables are not classified into inputs and outputs (see [20]). Let k be a ground field, and let q be a fixed integer. Call a Willems model (with signal number q) a quintuple (X, Y, E, F, G), where X and Y are finite-dimensional linear spaces and $E, F : X \to Y, G : k^q \to Y$ are linear maps such that E is injective and $[E\ G]$ is surjective. We can express this model as

$$sEx = Fx + Gw,$$

where w stands for the signal variable. Let $p = \dim Y - \dim X$ and $m = q - p$; these are respectively the output number and the input number. A partition of $\{1, \ldots, q\}$ in two subsets with cardinalities m and p gives rise to a decomposition $k^q \simeq k^m \oplus k^p$, which in turn determines a representation $G = [G_1\ G_2]$. A partition is called an I/O structure if $\det[sE - F \vdots - G_2] \neq 0$. A Willems model together with an I/O structure can be written as

$$sEx = Fx + G_1u + G_2v. \tag{3}$$

The transfer function of this I/O model is defined to be the composition

$$k(s)^m \to Y(s) \to X(s) \oplus k(s)^p \to k(s)^p,$$

where the first arrow is G_1, the second one is $[sE - F \vdots - G_2]^{-1}$ and the third is the canonical projection. In general, the transfer function of course is not proper. A simple remarkable fact is that there always exists an I/O structure such that the resulting model is an ordinary classical model. Indeed, the canonical map $k^q \to Y/E(X)$ is surjective, and clearly we can find a partition such that

$$k^p \simeq Y/E(X).$$

This allows us to identify Y with $X \oplus k^p$. Doing this we obtain

$$E = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad F = \begin{bmatrix} A \\ C \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} B & 0 \\ D & -I \end{bmatrix},$$

and consequently our model becomes

$$(sI - A)x = Bu, \qquad v = Cx + Du.$$

One knows that controllability and observability are independent from the I/O structure. (This principle is to be found in [1,20].) Since a classical linear system is controllable and observable at infinity, we therefore conclude that so is (3). Now, it can be shown that there is a *canonical* one-to-one correspondence between models of the type (2) and models of the type (3). This strongly suggests that one think of models of the type (2) as being controllable and observable at infinity. General models of the type (1) should also be regarded as such, if we want the homotopy equivalence to leave invariant the infinite structure.

As one knows, the structure at infinity in the traditional understanding is lost under the equivalence of Rosenbrock and Fuhrmann. Hayton et al. [4] were led therefore to propose the notion of *full system equivalence*. This kind of equivalence has been thoroughly studied in [6,7,12,13]. We remind the reader that using the full system equivalence, one can bring (1) to the system

$$sEx = Ax + Bu, \qquad v = Cx + Du. \tag{4}$$

In our opinion, the classical notion of system equivalence is very nice and it is rather the notion of infinite structure that should be reexamined. The reader may ask: which approach is more correct? This question reduces essentially to the following. Which is the more correct generalization of a classical linear system, (3) or (4)? (The author personally thinks that (3) is, notwithstanding the fact that (4) is the usual starting point for a singular theory of linear systems.)

Concluding the introduction we point out that the concept of homotopy equivalence has already been introduced into linear system theory (see [11,14]). However, the present note has a different slant (and was written independently). It is our belief that it will help the reader to understand well the true mathematical nature of one of the classical topics of linear systems theory.

Throughout, k is an arbitrary field, s an indeterminate, m an input number and p an output number. We let O denote the ring of proper rational functions. Given a finitely generated torsion module M over $k[s]$ or O, we shall write H^0M to denote the underlying k-linear space of M.

2. Preliminaries

A Kalman model is a quintuple (X, A, B, C, D), where X is a finite-dimensional linear space, $A : X \to X$, $B : k^m \to X$ and $C : X \to k^p$ are linear maps, and D is a polynomial $p \times m$ matrix. (Usually one expresses it as (2).) There is an equivalent definition, which will be used in the sequel. A Kalman model is a quadruple $(Q; \varphi, \psi, D)$, where Q is a finite $k[s]$-module, $\varphi : k[s]^m \to Q$ and $\psi : Q \to k(s)^p/k[s]^p$ are homomorphisms, and D is as above. That the two definitions are equivalent is the main result of [5, Chapter 10]. One introduces in an evident way the notion of transformations for Kalman models. Obviously Kalman models and their transformations form a category. Isomorphic Kalman models are said to be similar.

By an operational calculus we understand a pair $(\mathscr{H}, L)$, where $\mathscr{H}$ is a non-torsion module over O and $L : O \to \mathscr{H}$ an injective homomorphism over O. Typical examples can be found in [8]. Intuitively $\mathscr{H}$ is a "function" space over $\mathbb{R}_+$ and L is the (inverse) Laplace transform. Functions that belong to $L(O)$ are interpreted as exponential functions and functions that belong to $L(k)$ as constant functions. Multiplication by s^{-1} is regarded as integration. We require that

$$\bar{h} \in \mathscr{H} \text{ and } s^{-1}\bar{h} \in L(O) \quad \Rightarrow \quad \bar{h} \in L(O). \tag{5}$$

This axiom says that a function is exponential whenever its primitive is exponential. Let $\mathscr{M}$ denote the fraction space of $\mathscr{H}$. This is a linear space over $k(s)$, and we call its elements Mikusinski functions. There is a canonical embedding of $\mathscr{H}$ into $\mathscr{M}$ given by $\bar{h} \mapsto \bar{h}/1$, and we shall identify $\mathscr{H}$ with its image under this embedding. The homomorphism L can be continued to a $k(s)$-linear map $k(s) \to \mathscr{M}$, which will be denoted by L again. One can easily introduce "vector functions". Given a finite-dimensional k-linear space X, we set $\mathscr{H}(X) = \mathscr{H} \otimes X$ and $\mathscr{M}(X) = \mathscr{M} \otimes X$. (The tensor products are taken over k.) Again, we shall use the letter L to denote the canonical $k(s)$-linear map $X(s) \to \mathscr{M}(X)$.

We recall now the well-known Bezout lemma.

Lemma 1. *Let Z and Z' be finite-dimensional linear spaces. Assume we have a commutative square*

$$\begin{array}{ccc} Z[s] & \xrightarrow{T} & Z[s] \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle M} \\ Z'[s] & \xrightarrow{T'} & Z'[s] \end{array},$$

where K and T are right coprime and M and T' are left coprime. Then, there exist H, K', H' and M' such that

$$\begin{bmatrix} M & T' \\ -H & -K' \end{bmatrix} \begin{bmatrix} M' & T \\ -H' & -K \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

that is,

$$MM' = T'H' + I, \qquad HM' = K'H', \quad \textit{and} \quad K'K = HT + I.$$

Proof. We have an exact sequence

$$0 \to Z[s] \xrightarrow{\begin{bmatrix} T \\ -K \end{bmatrix}} Z[s] \oplus Z'[s] \xrightarrow{[M \quad T']} Z'[s] \to 0.$$

This sequence splits, and the lemma follows. □

Corollary 1. *In the situation of the lemma we have*

$$\begin{bmatrix} M' & T \\ -H' & -K \end{bmatrix}\begin{bmatrix} M & T' \\ -H & -K' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} M' & I \\ -T'H' & -M \end{bmatrix}\begin{bmatrix} M & I \\ -TH & -M' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Proof. Obvious. □

3. Models and transformations

A Rosenbrock model is a quintuple $(Z; T, U, V, W)$, where Z is a finite-dimensional linear space, $T : Z[s] \to Z[s]$ is an injective homomorphism and $U : k[s]^m \to Z[s]$, $V : Z[s] \to k[s]^p$, $W : k[s]^m \to k[s]^p$ are arbitrary homomorphisms. The space Z is the space of latent variables. (The terminology is due to Willems [20].) The transfer function is defined as the rational matrix $VT^{-1}U + W$. A model is called regular if its transfer function is proper.

Example. For each $r \geqslant 0$, $\Omega^r = (k^r; I_r, 0, 0, 0)$ is a Rosenbrock model. The zero model 0 is defined in an evident way. We put $\Omega^0 = 0$.

A transformation of a Rosenbrock model $(Z; T, U, V, W)$ into a Rosenbrock model $(Z'; T', U', V', W')$ is a quadruple (K, L, M, N) consisting of homomorphisms $K : Z[s] \to Z'[s]$, $L : k[s]^m \to Z'[s]$, $M : Z[s] \to Z'[s]$, and $N : Z[s] \to k[s]^p$ such that

$$\begin{bmatrix} M & 0 \\ N & I \end{bmatrix}\begin{bmatrix} T & U \\ -V & W \end{bmatrix} = \begin{bmatrix} T' & U' \\ -V' & W' \end{bmatrix}\begin{bmatrix} K & -L \\ 0 & I \end{bmatrix},$$

that is,

$$MT = T'K, \quad MU = -T'L + U', \quad NT - V = -V'K,$$
$$NU + W = V'L + W'.$$

(Compare with (4.13) in [3] and (22) in [4].)

If $\Phi_1 = (K_1, L_1, M_1, N_1)$ and $\Phi_2 = (K_2, L_2, M_2, N_2)$ are two transformations such that the range of the first one is equal to the domain of the second, then their composition is defined to be

$$\Phi_2 \circ \Phi_1 = (K_2K_1, K_2L_1 + L_2, M_2M_1, N_2M_1 + N_1).$$

The identity transformation of a Rosenbrock model $\Sigma = (Z; T, U, V, W)$ is defined as $I_\Sigma = (I, 0, I, 0)$.

One can check easily that Rosenbrock models together with transformations form a category. One therefore has, in particular, the notion of isomorphism. Isomorphisms of Rosenbrock models are called strict equivalences. Thus, a transformation $\Phi : \Sigma \to \Sigma'$ is a strict equivalence if there exists a transformation $\Phi' : \Sigma' \to \Sigma$ such that

$$\Phi' \circ \Phi = I_\Sigma \quad \text{and} \quad \Phi \circ \Phi' = I_{\Sigma'}.$$

The following states that our notion of strict equivalence coincides with that of Rosenbrock (see [15, Chapter 2]).

Proposition 1. *A transformation (K, L, M, N) is a strict equivalence if and only if K and M are unimodular.*

Proof. Straightforward and easy. □

Proposition 2. *Any transformation leaves invariant the transfer function.*

Proof. Let $(K, L, M, N) : (Z; T, U, V, W) \to (Z'; T', U', V', W')$ be a transformation. Then

$$T'^{-1}U' - L = T'^{-1}MU \quad \text{and} \quad T'^{-1}M = KT^{-1}.$$

Using these relations, we see that

$$\begin{aligned} -V'T'^{-1}U' + V'L &= -V'T'^{-1}MU = -V'KT^{-1}U \\ &= (NT - V)T^{-1}U = NU - VT^{-1}U. \end{aligned}$$

This together with the relation $NU + W = V'L + W'$ proves the proposition. □

If $\Sigma = (Z; T, U, V, W)$ is a Rosenbrock model, then we define the Kalman representation $\mathrm{KR}(\Sigma)$ as the quadruple consisting of the module $Z[s]/TZ[s]$, the homomorphisms

$$\begin{aligned} &u \mapsto (Uu) \bmod TZ[s] \quad (u \in k[s]^m), \\ &z \bmod TZ[s] \mapsto (VT^{-1}z) \bmod k[s]^p \quad (z \in Z[s]), \end{aligned}$$

and the polynomial part of the transfer function of Σ.

Let $\Sigma = (Z; T, U, V, W)$ and $\Sigma' = (Z'; T', U', V', W')$ be Rosenbrock models, and let $(Q; \varphi, \psi, D)$ and $(Q'; \varphi', \psi', D')$ be their Kalman representations, respectively. If $\Phi = (K, L, M, N)$ is a transformation of Σ into Σ', define $\theta : Q \to Q'$ by the formula

$$\theta(z \bmod TZ[s]) = (Mz) \bmod T'Z'[s].$$

One can check without difficulty that θ is a transformation of $\mathrm{KR}(\Sigma)$ into $\mathrm{KR}(\Sigma')$. We call this the Kalman representation of Φ and denote it by $\mathrm{KR}(\Phi)$.

It is easy to check that KR is a (covariant) functor from the category of Rosenbrock models to that of Kalman models.

4. Homotopy equivalence

Let $\Sigma = (Z; T, U, V, W)$ and $\Sigma' = (Z'; T', U', V', W')$ be Rosenbrock models. If $\Phi_1 = (K_1, L_1, M_1, N_1)$ and $\Phi_2 = (K_2, L_2, M_2, N_2)$ are transformations of the first one into the second, then we say that Φ_1 is homotopic to Φ_2 if there exists a homomorphism $H : Z[s] \to Z'[s]$ satisfying the following two equivalent conditions:

$$T'H = M_1 - M_2 \quad \text{and} \quad HT = K_1 - K_2.$$

Such a homomorphism is called a homotopy. We write $\Phi_1 \approx \Phi_2$ for "Φ_1 is homotopic to Φ_2".

Lemma 2. *Let $\Sigma = (Z; T, U, V, W)$ be a Rosenbrock model. Then, all transformations that are homotopic to I_Σ have the form*

$$(HT + I, -HU, TH + I, -VH),$$

where $H \in \mathrm{Hom}(Z[s], Z[s])$.

Proof. Left to the reader. □

Proposition 3. *Homotopy is an equivalence relation on the set of all transformations of one Rosenbrock model into another.*

Proof. *Reflexivity*: If Φ is a transformation, then clearly 0 is a homotopy of Φ onto itself.

Symmetry: If H is a homotopy of Φ_1 into Φ_2, then clearly $-H$ is a homotopy of Φ_2 into Φ_1.

Transitivity: If H is a homotopy of Φ_1 into Φ_2 and H' a homotopy of Φ_2 into Φ_3, then $H + H'$ is a homotopy of Φ_1 into Φ_3. □

Two transformations are homotopy equivalent if there is a homotopy between them. Speaking intuitively, two transformations are homotopy equivalent if one of them can be obtained from the other "perturbing" a little.

A transformation $\Phi : \Sigma \to \Sigma'$ is a homotopy equivalence if there exists a transformation $\Phi' : \Sigma' \to \Sigma$ such that

$$\Phi' \circ \Phi \approx I_\Sigma \quad \text{and} \quad \Phi \circ \Phi' \approx I_{\Sigma'}.$$

(Compare with the notion of strict equivalence.) Two Rosenbrock models Σ and Σ' are homotopy equivalent if there exists a homotopy equivalence $\Phi : \Sigma \to \Sigma'$. We write $\Sigma \approx \Sigma'$ to denote that Σ and Σ' are homotopy equivalent. Certainly strictly equivalent models are homotopy equivalent, i.e., the homotopy equivalence is a weaker relation than the strict equivalence.

Proposition 4. *The relation between Rosenbrock models of being homotopy equivalent is an equivalence relation.*

Proof. Follows from the previous proposition. □

Lemma 3. *A Rosenbrock model Σ, with "latent" dimension $r \geqslant 1$, is homotopy equivalent to* 0 *if and only if it is strictly equivalent to Ω^r.*

Proof. *If*: Clearly Ω^r is homotopy equivalent to 0; the homotopy equivalence is $(0, 0, 0, 0)$.

Only if: Let $\Sigma = (T, U, V, W)$ be a Rosenbrock model homotopy equivalent to 0, and suppose that (K, L, M, N) establishes this equivalence. We then have

$$\begin{bmatrix} M & 0 \\ N & I \end{bmatrix} \begin{bmatrix} T & U \\ -V & W \end{bmatrix} = 0.$$

This in particular gives $NU + W = 0$. Further, we have

$$(0, 0, 0, 0) \circ (K, L, M, N) = (0, 0, 0, N)$$

and consequently, by Lemma 2,

$$(HT + 1, -HU, TH + 1, -VH) = (0, 0, 0, N),$$

where H is a polynomial matrix. We get that T is unimodular, $U = 0$ and $W = 0$. Finally,

$$\begin{bmatrix} -H & 0 \\ -VH & I \end{bmatrix} \begin{bmatrix} T & 0 \\ -V & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

The lemma is proved. □

Proposition 5. *Two transformations are homotopy equivalent if and only if their Kalman representations are equal.*

Proof. Let $(Z; T, U, V, W)$ and $(Z'; T', U', V', W')$ be Rosenbrock models, and let (K_1, L_1, M_1, N_1) and (K_2, L_2, M_2, N_2) be two transformations of the first one into the second.

If: Assume that $M_2 - M_1 = T'H$. Then

$$M_2 z \bmod T'Z'[s] = (M_1 z + T'Hz) \bmod T'Z'[s] = M_1 z \bmod T'Z'[s].$$

Only if: Assume that $M_2 z \bmod T'Z'[s] = M_1 z \bmod T'Z'[s]$ for each $z \in Z[s]$. Then

$$H = T'^{-1}(M_2 - M_1)$$

is a homotopy. □

Corollary 2. *Two Rosenbrock models are homotopy equivalent if and only if their Kalman representations are similar.*

Proof. Obvious. □

Theorem 1. *The homotopy category of Rosenbrock models is canonically equivalent to the category of Kalman models.*

Proof. Follows from the previous proposition and corollary. □

5. Classical equivalences

We say that two Rosenbrock models $(Z; T, U, V, W)$ and $(Z'; T', U', V', W')$ are equivalent in the sense of Fuhrmann if there exists a transformation (K, L, M, N) of the first one into the other for which K and T are right coprime and M and T' are left coprime (see [3]).

Theorem 2. *Two Rosenbrock models are equivalent in the sense of Fuhrmann if and only if they are homotopy equivalent.*

Proof. The "If" part of the theorem is obvious. To prove the "Only if" part assume that $(K, L, M, N) : (Z; T, U, V, W) \to (Z'; T', U', V', W')$ is a transformation, where M and T' are left coprime, and K and T are right coprime. Define H, H', K' and M' as in the Bezout lemma. Next, define L' and N' by

$$K'L + L' = -HU \quad \text{and} \quad NM' + N' = -V'H',$$

respectively.

One can check easily that (K', L', M', N') is a transformation. (It suffices to use the Bezout lemma and its corollary.) Further, we have

$$\begin{aligned} N'M + N &= (-V'H' - NM')M + N = -V'H'M + N(I - M'M) \\ &= -V'H'M - NTH = -V'H'M + (V'K - V)H \\ &= V'(KH - H'M) - VH = -VH. \end{aligned}$$

Likewise, we have

$$\begin{aligned} KL' + L &= K(-HU - K'L) + L = (I - KK')L - KHU \\ &= -H'T'L - KHU = H'(MU - U') - KHU \\ &= -H'U' + (H'M - KH)U = -H'U'. \end{aligned}$$

It follows that

$$(K'K, K'L + L', M'M, N'M + N) = (HT + I, -HU, TH + I, -VH)$$

and

$$\begin{aligned}&(KK', KL' + L, MM', NM' + N')\\ &\quad = (H'T' + I, -H'U', T'H' + I, -V'H').\end{aligned}$$

The theorem is proved. □

If $(Z_1; T_1, U_1, V_1, W_1)$ and $(Z_2; T_2, U_2, V_2, W_2)$ are Rosenbrock models, their parallel connection is defined to be

$$\left(Z_1 \oplus Z_2; \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \begin{bmatrix} V_1 & V_2 \end{bmatrix}, W_1 + W_2\right).$$

We say that two Rosenbrock models Σ and Σ' are equivalent in the sense of Rosenbrock if

$$\Omega^l \oplus \Sigma \simeq \Omega^{l'} \oplus \Sigma'$$

for some nonnegative integers l and l' (see [15,17]).

Theorem 3. *Two Rosenbrock models are equivalent in the sense of Rosenbrock if and only if they are homotopy equivalent.*

Proof. Let $\Sigma = (T, U, V, W)$ and $\Sigma' = (T', U', V', W')$ be Rosenbrock models.

If: Let (K, L, M, N) be a homotopy equivalence. Choose H, H', K' and M' as in the Bezout lemma. We have

$$\begin{aligned}&\begin{bmatrix} M' & I & 0 \\ -T'H' & -M & 0 \\ V'H' & -N & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & T & U \\ 0 & -V & W \end{bmatrix}\\ &\quad = \begin{bmatrix} I & 0 & 0 \\ 0 & T' & U' \\ 0 & -V' & W' \end{bmatrix} \begin{bmatrix} M' & T & U \\ -H' & -K & L \\ 0 & 0 & -I \end{bmatrix}.\end{aligned}$$

By Corollary 1, the extreme matrices are unimodular.

Only if: Using Lemma 3, we have

$$\Sigma \approx \Omega^l \oplus \Sigma \simeq \Omega^{l'} \oplus \Sigma' \approx \Sigma'.$$

The theorem is proved. □

6. Poles and zeros

Poles and zeros of linear systems were defined first by Rosenbrock [15], and a great number of works have been devoted to the study of them. For a nice historical survey we refer the reader to [18]. Below we define finite poles and zeros following

Bourlès and Fliess [1]. As emphasized, a Rosenbrock model has no decoupling zeros at infinity. This tells us in particular that its infinite structure can be defined in terms of the transfer function. The theory of poles and zeros of transfer functions has been established well by Wyman et al. [21], and we follow them when we define infinite poles and infinite (invariant) zeros.

Let $\Sigma = (Z; T, U, V, W)$ be a Rosenbrock model, and let G be its transfer function. We define the finite and infinite pole modules to be

$$\mathscr{P}_{\mathrm{f}} = \frac{Z[s]}{TZ[s]} \quad \text{and} \quad \mathscr{P}_{\infty} = \frac{O^p + GO^m}{O^p}.$$

We define the finite (invariant) and infinite (invariant) zero modules to be

$$\mathscr{Z}_{\mathrm{f}} = \mathrm{TP}\left(Z[s] \oplus k[s]^p \Big/ \begin{bmatrix} T & U \\ -V & W \end{bmatrix} (Z[s] \oplus k[s]^m) \right)$$

and

$$\mathscr{Z}_{\infty} = \mathrm{TP}\left(\frac{O^p + GO^m}{GO^m} \right).$$

We define the input-decoupling and output-decoupling zero modules to be

$$\mathscr{Z}^{\mathrm{i.d.}} = \frac{Z[s]}{TZ[s] + Uk[s]^m}$$

and

$$\mathscr{Z}^{\mathrm{o.d.}} = \mathrm{TP}\left(Z[s] \oplus k[s]^p \Big/ \begin{bmatrix} T \\ -V \end{bmatrix} Z[s] \right).$$

(Above: “TP” stands for “torsion part”.) All these modules surely are finitely generated and torsion.

It should be noted that the pole modules are especially important. The dimension of $H^0\mathscr{P}_{\mathrm{f}}$ is called the number of finite poles and the dimension of $H^0\mathscr{P}_{\infty}$ the number of infinite poles. The first clearly is equal to $\deg(\det T)$. Choose a rational matrix U such that $O^p + GO^m = U^{-1}O^p$. It is clear that U exists and is defined uniquely up to biproper multiplier. Since $O^p \subseteq U^{-1}O^p$, we have $UO^p \subseteq O^p$ and consequently U is proper. The number of infinite poles is equal to $\mathrm{ord}_{\infty}(\det U)$. Obviously the model is regular if and only if it has no infinite poles.

Remark that the model is controllable if and only if T and U are left coprime, and observable if and only if T and V are right coprime.

It is easily seen that the constructions of poles and zeros are functorial. That is, a transformation $\Sigma \to \Sigma'$ gives rise to canonical homomorphisms

$$\mathscr{P}_{\mathrm{f}} \to \mathscr{P}'_{\mathrm{f}} \quad \text{and} \quad \mathscr{P}_{\infty} \to \mathscr{P}'_{\infty},$$

$$\mathscr{Z}_{\mathrm{f}} \to \mathscr{Z}'_{\mathrm{f}} \quad \text{and} \quad \mathscr{Z}_{\infty} \to \mathscr{Z}'_{\infty},$$

$$\mathscr{Z}^{\mathrm{i.d.}} \to \mathscr{Z}'^{\mathrm{i.d.}} \quad \text{and} \quad \mathscr{Z}^{\mathrm{o.d.}} \to \mathscr{Z}'^{\mathrm{o.d.}}.$$

Proposition 6. *Two homotopy equivalent transformations induce the same homomorphisms of pole and zero modules.*

Proof. Straightforward and easy. □

Remark. In view of Proposition 2, the case of infinite poles and zeros is trivial.

Theorem 4. *Homotopy equivalence leaves invariant pole and zero modules.*

Proof. Follows immediately from the previous proposition. □

7. States and motions

Let $\Sigma = (Z; T, U, V, W)$ be a Rosenbrock model, and let G be its transfer function. We define the state space X by the formula

$$X = \frac{\{(z, v) \in Z[s] \oplus k[s]^p \mid VT^{-1}z + v \in s^{-1}(O^p + GO^m)\}}{\begin{bmatrix} T \\ -V \end{bmatrix} Z[s]}.$$

There is a canonical linear map $\phi : X \to k(s)^p$, which in fact has values in $s^{-1}(O^p + GO^m)$. If x is a state and if (z, v) is its representative, then $\phi(x) = VT^{-1}z + v$.

Let $\mathscr{P}_{\mathrm{f}}$ and $\mathscr{P}_\infty$ be pole modules of Σ. For each n, we let $\mathscr{P}_\infty(n) = s^n(O^p + GO^m)/s^n O^p$. (Certainly all $\mathscr{P}_\infty(n)$ are isomorphic to each other.)

Theorem 5. *There is a canonical isomorphism*

$$X \simeq H^0\mathscr{P}_{\mathrm{f}} \oplus H^0\mathscr{P}_\infty(-1).$$

Proof. If x is a state, assign to it the pair

$$(z \bmod TZ[s], (VT^{-1}z + v) \bmod s^{-1}O^p),$$

where (z, v) is a representative of x.

Suppose a pair (z, v) represents a state going to 0. Then $z \in TZ[s]$ and $VT^{-1}z + v \in s^{-1}O^p$. It follows that

$$z = Tz_0 \quad \text{and} \quad Vz_0 + v \in s^{-1}O^p$$

for some $z_0 \in Z[s]$. Because $Vz_0 + v \in k[s]^p$, we get $Vz_0 + v = 0$. Hence $(z, v) = (Tz_0, -Vz_0)$, and so the state is equal to 0.

To show the surjectivity take $z \in Z[s]$ and $v \in s^{-1}(O^p + GO^m)$. Let v_1 denote the polynomial part of $v - VT^{-1}z$. It is easily seen that (z, v_1) represents a state and its image is equal to $(z \bmod TZ[s], v \bmod s^{-1}O^p)$.

Thus, our linear map is bijective. □

The McMillan degree of the model is defined to be the number of all its poles.

Corollary 3. *The dimension of the state space is equal to the McMillan degree.*

We now pass to the notion of motions. To define them assume we have an operational calculus $(\mathscr{H}, L)$.

Let $x_0 \in X$ and $\bar{u} \in \mathscr{M}^m$. Choose a representative (z_0, v_0) of the state x_0 and consider the equations

$$T\bar{z} = U\bar{u} + sL(z_0), \qquad \bar{v} = V\bar{z} + W\bar{u} + sL(v_0).$$

Here $\bar{z} \in \mathscr{M}(Z)$ and $\bar{v} \in \mathscr{M}^p$. From the first equation we have

$$\bar{z} = T^{-1}U\bar{u} + sL(T^{-1}z_0).$$

Substituting this in the second equation, we obtain

$$\bar{v} = VT^{-1}U\bar{u} + W\bar{u} + sL(VT^{-1}z_0) + sL(v_0) = G\bar{u} + sL(\phi(x_0)).$$

This is the motion associated with the initial state x_0 and the input function $\bar{u}$. We call

$$X \times \mathscr{M}^m \to \mathscr{M}^p, \qquad (x_0, \bar{u}) \mapsto G\bar{u} + sL(\phi(x_0))$$

the behavior of Σ. The map $x_0 \mapsto sL(\phi(x_0))$ is called the free response map.

Proposition 7. *The following conditions are equivalent*: (a) Σ *is regular*; (b) $G\mathscr{H}^m \subseteq \mathscr{H}^p$; (c) *all free motions belong to* $\mathscr{H}^p$.

Proof. (a) $\Leftrightarrow$ (b). If G is proper, clearly we have (b). Conversely, if (b) holds, then $GL(O^m) \subseteq \mathscr{H}^p$. Using the axiom (5), we see that $GL(O^m) \subseteq L(O^p)$, whence $GO^m \subseteq O^p$.

(a) $\Leftrightarrow$ (c). Note that (c) is equivalent to saying that $\phi(X) \subseteq s^{-1}O^p$. This in turn is equivalent to saying that the canonical linear map $X \to H^0\mathscr{P}_\infty(-1)$ is zero. In view of Theorem 5, this is possible if and only if $\mathscr{P}_\infty(-1) = 0$. $\square$

Closing, we remark that the state spaces and behaviors are respected by transformations and that they are homotopy invariants. The reader easily verifies this.

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