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On duality for partial differential (and difference) equations ☆

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Abstract

In this paper we provide a simple version of Oberst's duality between finitely generated polynomial modules and the solution sets of partial differential (and difference) equations.

The space $C^\infty(\Omega)$, where Ω is a connected open subset of $\mathbb{R}^r$, can be viewed in an obvious way as a module over $\mathbb{R}[s] = \mathbb{R}[s_1, \ldots, s_r]$. If R is a matrix with entries in $\mathbb{R}[s]$, then the solution set of the associated partial differential equation $R(D)\xi = 0$ is equal (as noticed by Malgrange [4]) to $\mathrm{Hom}_{\mathbb{R}[s]}(M, C^\infty(\Omega))$ with $M = \mathrm{Coker}\, R^{\mathrm{tr}}$. One is led therefore to consider the functor $\mathrm{Hom}_{\mathbb{R}[s]}(-, C^\infty(\Omega))$ defined on finitely generated polynomial modules. It is a theorem of Ehrenpreis, Malgrange and Palamodov that this functor is exact, i.e., the module $C^\infty(\Omega)$ is injective. (For this deep result we refer the reader to Hörmander [2], for example.) As is well known (see Matlis [5]), every injective module over a commutative noetherian ring is a direct sum of indecomposable injective modules, which are classified via the prime ideals. In [7] Oberst has proved that $C^\infty(\Omega)$ is a "large" injective module (i.e., $C^\infty(\Omega)$ contains at least one indecomposable injective module of each type). From this (difficult) theorem, using the general duality theory developed in Oberst [6] and Roos [10], Oberst has obtained that the functor above establishes a duality between the category of finitely generated polynomial modules and the category of the solution sets of partial differential equations. Oberst has shown that

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$k^{\mathbb{Z}_+^r}$, where k is an arbitrary discrete field, also is a large injective module, and hence this function space also gives rise to a duality. (However, this case is considerably easier.)

The goal of this paper is to present a simple version of Oberst's duality.

Let k be an arbitrary field equipped with an absolute value $|\ |$ and complete with respect to this absolute value. (One may assume that k is either $\mathbb{R}$ or $\mathbb{C}$, or an arbitrary discrete field.) Let r be an arbitrary positive integer, and let $s_1, \dots, s_r$ be indeterminates. Put $t_1 = s_1^{-1}, \dots, t_r = s_r^{-1}$, and put $s = (s_1, \dots, s_r)$ and $t = (t_1, \dots, t_r)$. By a multi-index we mean any element of $\mathbb{Z}_+^r$. If $f = (f_1, \dots, f_r)$ is any sequence of rational functions, then for each multi-index $i = (i_1, \dots, i_r) \in \mathbb{Z}_+^r$, we write $f^i = f_1^{i_1} \cdots f_r^{i_r}$. The degree of a polynomial $f = \sum a_i s^i$, denoted by $\deg f$, is the supremum of those multi-indices i for which $a_i \neq 0$. (We consider the componentwise order on the set of multi-indices.) Following [7, 6.60], we call a rational function f/g proper, if $\deg f \leqslant \deg g$ and the coefficient at $s^{\deg g}$ in g is distinct from 0. Let O denote the ring of proper rational functions. (It is interesting to note that O is the local ring of the "most infinite" point in $(\mathbb{P}^1)^r$.) Clearly, O is contained in the ring of formal series $k[[t]]$. On the latter one has left shift operators $\sigma_1, \dots, \sigma_r$ making it a $k[s]$-module. Throughout, O will be regarded as a $k[s]$-submodule of $k[[t]]$.

Call a (finite) point any sequence $p = (p_1, \dots, p_r)$, where $p_1 \in k[s_1], \dots, p_r \in k[s_r]$ are monic irreducible univariate polynomials. Let I_p denote the ideal in $k[s]$ generated by the entries of p. By Hilbert's Nullstellensatz (see Proposition 2(iii) in [1, Chapter V]), points are in a canonical one-to-one correspondence with maximal ideals of $k[s]$ (i.e., with closed points of the affine space $\mathbb{A}^r$).

Let $p = (p_1, \dots, p_r)$ be a point, and let $d_1, \dots, d_r$ be the degrees of the polynomials $p_1, \dots, p_r$. For each nonnegative integer n, define

$$E_p(n) = \bigoplus_{i,j} k \cdot \frac{s^i}{p^j},$$

where

$$(1, \dots, 1) \leqslant i \leqslant (d_1, \dots, d_r) \quad \text{and} \quad (1, \dots, 1) \leqslant j \leqslant (n, \dots, n).$$

This is a $k[s]$-submodule of O. To see this it suffices to note that $E_p(n)$ can be defined also as the image of the canonical embedding

$$\bigotimes_{l=1}^{r} \frac{s_l}{p_l^n} k[s_l]/s_l k[s_l] \to O.$$

Set $E_p = \bigcup E_p(n)$.

The module

$$E = \bigoplus_p E_p,$$

which is a submodule in O, will be of special significance for this paper. This module has been investigated comprehensively in [8]. In particular, it has been shown there that this is the minimal injective cogenerator over $k[s]$. For the sake of completeness, we include here the following

Lemma 1. *The module E is a cogenerator.*

Proof. By [3, Theorem 19.8], we have to show that for each maximal ideal $\mathfrak{m}$, E contains a copy of the injective hull of $k[s]/\mathfrak{m}$.

Let $\mathfrak{m}$ be any maximal ideal of $k[s]$, and let $p = (p_1, \dots, p_r)$ be the associated point. Consider the canonical homomorphism

$$\bigotimes_{l=1}^{r} k[s_l]/p_l^n k[s_l] \to k[s]/I_p^n,$$

which clearly is injective. Because both of the modules have the same dimension over k it must be an isomorphism. It follows that there is a canonical isomorphism

$$\bigotimes_{l=1}^{r} \operatorname{Hom}\bigl(k[s_l]/p_l^n k[s_l], k\bigr) \simeq \operatorname{Hom}\bigl(k[s]/I_p^n, k\bigr).$$

Further, for each $1 \leqslant l \leqslant r$, there are canonical isomorphisms

$$\frac{s_l}{p_l^n} k[s_l]/s_l k[s_l] \simeq \frac{1}{p_l^n} k[s_l]/k[s_l] \simeq k[s_l]/p_l^n k[s_l] \simeq \operatorname{Hom}_k\bigl(k[s_l]/p_l^n k[s_l], k\bigr),$$

and we see that $E_p(n)$ is canonically isomorphic to $\operatorname{Hom}_k(k[s]/I_p^n, k)$. Thus, we have a canonical isomorphism

$$E_p \simeq \varinjlim_n \operatorname{Hom}_k\bigl(k[s]/I_p^n, k\bigr).$$

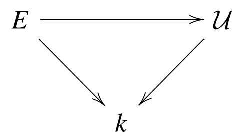
The right-hand side represents the functor

$$M \mapsto \varinjlim_n \operatorname{Hom}_k\bigl(M/I_p^n M, k\bigr),$$

which is exact by the Artin–Rees lemma (see [1, Chapter III, §3, Proposition 1]). Hence, the module E_p is injective. Finally, since $E_p(1)$ is isomorphic to $k[s]/I_p$, which obviously contains $k[s]/\mathfrak{m}$, we conclude that E_p contains the injective hull of $k[s]/\mathfrak{m}$. (It can be shown easily that, in fact, E_p coincides with this injective hull.) □

Remark. For $f \in k[s]$ and $g \in O$, define $\langle g, f\rangle$ to be the constant term of $fg \in k((t))$. We then get a k-bilinear form. Every $g \in O$ determines therefore a canonical linear functional on $k[s]$. From the proof above we see that E_p consists of those $g \in O$ for which the linear functional $\langle g, -\rangle$ vanishes on some power of I_p.

Assume now we are given a triple $(\mathcal{U}, L, ev)$, where $\mathcal{U}$ is a $k[s]$-module, L an injective $k[s]$-homomorphism of E into $\mathcal{U}$ and ev a k-linear functional on $\mathcal{U}$ such that the diagram



is commutative. (By $E \to k$ we mean the canonical k-linear functional on E determined by $\sum b_i t^i \mapsto b_0$.)

We shall think of elements of $\mathcal{U}$ as functions that we can differentiate as many times as we please. Multiplications by $s_1, \ldots, s_r$ in $\mathcal{U}$ will be interpreted as partial differentiations and will be denoted by $D_1, \ldots, D_r$, respectively. If $h = (h_1, \ldots, h_r)$ is a multi-index, we let D^h denote the partial differentiation operator $D^h = D_1^{h_1} \cdots D_r^{h_r}$. We shall think of L as the (inverse) Laplace transform and of functions of the form Lg as exponential functions. The map ev can be viewed as the "evaluation map at 0". Given a function $\xi \in \mathcal{U}$, we shall write $\xi(0)$ for $ev(\xi)$.

Here are examples that are of interest.

Example 1. Let $k = \mathbb{R}$ or $\mathbb{C}$, and let Ω be a connected open subset in $\mathbb{R}^r$. (Without loss of generality we may assume that Ω contains the origin.) Set $\mathcal{U} = C^\infty(\Omega)$. For $g = \sum b_i t^i \in E$, define $Lg \in \mathcal{U}$ by the formula

$$(Lg)(x) = \sum_{i \geqslant 0} b_i \frac{x^i}{i!}, \quad x \in \Omega.$$

(Here $i = (i_1, \ldots, i_r)$, $x = (x_1, \ldots, x_r)$, $x^i = x_1^{i_1} \cdots x_r^{i_r}$, $i! = i_1! \cdots i_r!$.) Finally, for every $\xi \in \mathcal{U}$, put

$$ev(\xi) = \xi(0).$$

Example 2. Let $\mathcal{U} = k^{\mathbb{Z}_+^r}$. For $g = \sum b_i t^i \in E$, define $Lg \in \mathcal{U}$ by the formula

$$(Lg)(i) = b_i, \quad i \in \mathbb{Z}_+^r.$$

For every $\xi \in \mathcal{U}$, put

$$ev(\xi) = \xi(0).$$

Remark. In view of Example 2, all what follows is applicable to difference equations as well, and this justifies the title of the paper.

For each multi-index n and positive number δ, define

$$\mathcal{U}_{n,\delta} = \bigl\{\xi \in \mathcal{U} \bigm| \bigl|D^i \xi(0)\bigr| < \delta \text{ for all } i \leqslant n\bigr\}.$$

We declare the sets $\mathcal{U}_{n,\delta}$ as a basis of neighbourhoods of 0 for a topology on $\mathcal{U}$. Remark that this topology, in general, is not separate. (This is the case in Example 1.) Note that

$$\{\bar{0}\} = \{\xi \in \mathcal{U} \mid D^i\xi(0) = 0 \text{ for all } i \geqslant 0\}.$$

Obviously, $D^h\mathcal{U}_{n+h,\delta} \subseteq \mathcal{U}_{n,\delta}$. It immediately follows from this that $D^h : \mathcal{U} \to \mathcal{U}$ is continuous, and hence all differential operators of the form $R(D) : \mathcal{U}^q \to \mathcal{U}^p$, where R is a polynomial $p \times q$ matrix, are continuous.

For each multi-index n, set

$$\mathcal{U}(n) = \{\xi \in \mathcal{U} \mid D^i\xi(0) = 0 \text{ for all } i \leqslant n\}.$$

The Taylor polynomial of degree n of a function $\xi \in \mathcal{U}$ is defined to be

$$T_n(\xi) = \sum_{i \leqslant n} D^i\xi(0)t^i.$$

For a fixed n, $\xi \mapsto T_n(\xi)$ is a k-linear map into $k[t]_{\leqslant n}$, the space of polynomials in t of degree $\leqslant n$. Because E contains $E_s = k[t]$, this map must be surjective. The kernel obviously is $\mathcal{U}(n)$, and hence T_n induces a canonical k-linear isomorphism

$$\mathcal{U}/\mathcal{U}(n) \simeq k[t]_{\leqslant n}.$$

Given a topological k-linear space $\mathcal{X}$, let us write $\mathcal{X}^*$ for $\mathrm{Hom}_k^{\mathrm{cont}}(\mathcal{X}, k)$. Likewise, if ϕ is a continuous linear map of topological k-linear spaces, write ϕ^* to denote $\mathrm{Hom}_k^{\mathrm{cont}}(\phi, k)$.

Lemma 2. *We have*

$$\mathcal{U}^* = k[s] \quad \text{and} \quad (D^h)^* = s^h.$$

Proof. One can see that a linear functional on $\mathcal{U}$ is continuous if and only if it vanishes on some $\mathcal{U}(n)$. Linear functionals vanishing on $\mathcal{U}(n)$ can be identified with linear functionals on $\mathcal{U}/\mathcal{U}(n)$ and hence with linear functionals on $k[t]_{\leqslant n}$. Linear functionals on this latter can be identified with polynomials in $k[s]_{\leqslant n}$, the space of polynomials in s of degree $\leqslant n$. The union of all $k[s]_{\leqslant n}$ is $k[s]$, and the first equality is shown.

Further, for each n, we have a canonical linear map

$$D^h : \mathcal{U}/\mathcal{U}(n+h) \to \mathcal{U}/\mathcal{U}(n),$$

which can be identified with

$$\sigma^h : k[t]_{\leqslant n+h} \to k[t]_{\leqslant n}.$$

(Here $\sigma^h = \sigma_1^{h_1} \cdots \sigma_r^{h_r}$.) The dual of this latter can be identified with

$$s^h : k[s]_{\leqslant n} \to k[s]_{\leqslant n+h}.$$

The direct limit of these linear maps is $s^h : k[s] \to k[s]$, and the proof of the second equality is complete. □

Corollary 3. *For each nonnegative integer l and for each polynomial matrix R we have*

$$\left(\mathcal{U}^l\right)^* = k[s]^l \quad \text{and} \quad R(D)^* = R^{\mathrm{tr}}.$$

Lemma 4. *Let $\mathcal{X}$ be a linear subspace in $\mathcal{U}^q$ (equipped with the induced topology). Then*

$$\left(\mathcal{U}^q\right)^* \to \mathcal{X}^*$$

is surjective.

Proof. We have injective linear maps of finite dimensional linear spaces

$$\mathcal{X}/\left(\mathcal{X} \cap \mathcal{U}^q(n)\right) \to \mathcal{U}^q/\mathcal{U}^q(n).$$

Consequently, we have surjective linear maps

$$\left(\mathcal{U}^q/\mathcal{U}^q(n)\right)^* \to \left(\mathcal{X}/\left(\mathcal{X} \cap \mathcal{U}^q(n)\right)\right)^*.$$

Taking the direct limit, we complete the proof. □

We remark that there are sufficiently many continuous linear functionals on $\mathcal{X} \subseteq \mathcal{U}^q$ (in other words, for each $\xi \in \mathcal{X}$ that does not belong to $\{\bar{0}\}$, there exists a continuous linear functional $f : \mathcal{X} \to k$ such that $f(\xi) \neq 0$).

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological k-linear spaces, and let ϕ_1 and ϕ_2 be two continuous linear maps from the first one to the other. Let us say that ϕ_1 and ϕ_2 are essentially equal (and denote this by $\phi_1 \equiv \phi_2$) if

$$\forall x \in \mathcal{X}, \quad \phi_1(x) \equiv \phi_2(x) \mod \left\{\bar{0}\right\}.$$

Assuming that $\mathcal{Y}$ has sufficiently many continuous linear functionals, we have the following obvious

Lemma 5. $\phi_1 \equiv \phi_2$ *if and only if* $\phi_1^* = \phi_2^*$.

The ring $k[s]$ can be viewed as a topological ring. (The topology on $k[s]$ is the inductive limit of the canonical topologies on $k[s]_{\leqslant n}$, $n \in \mathbb{Z}_+^r$.) One therefore has the notion of a topological $k[s]$-module. It is worth noting that giving a topological $k[s]$-module is equivalent to giving a topological k-linear space together with r pairwise commuting continuous k-linear endomorphisms.

Let R be a polynomial matrix, say, of size $p \times q$. Then, the solution set of the differential equation

$$R(D)\xi = 0, \quad \xi \in \mathcal{U}^q$$

(together with the topology induced from $\mathcal{U}^q$) is a topological $k[s]$-module. We define a linear (dynamical) system as a topological $k[s]$-module that is isomorphic to the solution set of a differential equation. (*Warning*: This definition is not exactly the same as that given in [7,9], since it takes into account a specific topology.) Define a differential operator from one linear system to another as an equivalence class of essentially equal continuous $k[s]$-homomorphisms. The set of all differential operators from $\mathcal{S}$ to $\mathcal{T}$ denote by $\mathrm{Diff}(\mathcal{S},\mathcal{T})$. The following example, where differential operators from $\mathcal{U}^q$ to $\mathcal{U}^p$ are described, can serve as a justification of this definition.

Example 3. Let ϕ be any continuous $k[s]$-homomorphism from $\mathcal{U}^q$ to $\mathcal{U}^p$. By Corollary 3, there exists a polynomial matrix R such that $\phi^* = R^{\mathrm{tr}}$. We clearly have $R(D)^* = \phi^*$ and, using Lemma 5, we see that $\phi \equiv R(D)$. Let now R_1 and R_2 be polynomial matrices of size $p \times q$ such that $R_1(D) \equiv R_2(D)$. Then, $R_1(D)^* = R_2(D)^*$. Applying again Corollary 3, we see that $R_1^{\mathrm{tr}} = R_2^{\mathrm{tr}}$. Hence, $R_1(D) = R_2(D)$. We conclude that

$$\mathrm{Diff}\big(\mathcal{U}^q,\mathcal{U}^p\big) = \big\{R(D) \mid R \in k[s]^{p\times q}\big\}.$$

One can define in an obvious way compositions of differential operators, and one can check easily that linear systems (together with differential operators) form a category. Let us denote this category by $\underline{\mathrm{Syst}}$.

Denote by $\underline{\mathrm{Mod}}$ the category of finitely generated $k[s]$-modules. We are going to show that $\underline{\mathrm{Mod}}$ and $\underline{\mathrm{Syst}}$ are dual to each other.

Let $M \in \underline{\mathrm{Mod}}$. Choose generators $m_1, \ldots, m_q$ of M. It is easy to see that the sets

$$\big\{\varphi \in \mathrm{Hom}(M,\mathcal{U}) \mid \varphi(m_1), \ldots, \varphi(m_q) \in \mathcal{U}_{n,\delta}\big\},$$

where n is a multi-index and δ a positive number, determine a topology on $\mathrm{Hom}(M,\mathcal{U})$ for which they constitute a fundamental system of neighbourhoods of 0. One can check easily that this topology does not depend on the choice of a set of generators and the module $\mathrm{Hom}(M,\mathcal{U})$ together with this topology is a topological $k[s]$-module. We call it the behavior of M and denote by $\mathrm{Bh}(M)$.

The following simple nice fact was observed by Malgrange (see [4]).

Lemma 6. *Let R be a polynomial matrix. Then* $\mathrm{Ker}\, R(D)$ *is canonically isomorphic to* $\mathrm{Bh}(M)$*, where M denotes the cokernel of R^{tr}.*

Proof. Let $p \times q$ be the size of R. We then have an exact sequence

$$k[s]^p \xrightarrow{R^{\mathrm{tr}}} k[s]^q \longrightarrow M \longrightarrow 0.$$

From this we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M,\mathcal{U}) \longrightarrow \mathcal{U}^q \xrightarrow{R(D)} \mathcal{U}^p,$$

which yields a canonical isomorphism

$$\mathrm{Hom}(M, \mathcal{U}) \simeq \mathrm{Ker}\, R(D).$$

Taking the generators of M determined by the epimorphism $k[s]^q \to M$, one can easily see that the isomorphism is valid in the topological sense as well. The proof is complete. □

The lemma above implies immediately that if M is a finitely generated $k[s]$-module, then $\mathrm{Bh}(M)$ is a linear system. Given a $k[s]$-homomorphism f of finitely generated modules, define $\mathrm{Bh}(f)$ to be $\mathrm{Hom}_{k[s]}(f, \mathcal{U})$. Clearly this is a continuous homomorphism. Thus Bh is a functor from $\underline{\mathrm{Mod}}$ to the category of topological $k[s]$-modules.

It is clear that if $\mathcal{X}$ is a topological $k[s]$-module, then $\mathcal{X}^*$ is a $k[s]$-module. Likewise, if ϕ is a continuous $k[s]$-homomorphism, then ϕ^* is a $k[s]$-homomorphism. The following says that the composition $\mathrm{I} = {}^* \circ \mathrm{Bh}$ is canonically isomorphic to the identity functor (of $\underline{\mathrm{Mod}}$).

Theorem 7. (a) *If M is a finitely generated $k[s]$-module, then the canonical homomorphism*

$$M \to \mathrm{I}(M)$$

is an isomorphism.

(b) *If $f : M \to N$ is a $k[s]$-homomorphism of finitely generated modules, then the canonical diagram*

$$\begin{array}{ccc} M & \simeq & \mathrm{I}(M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{I}(f)} \\ N & \simeq & \mathrm{I}(N) \end{array}$$

is commutative.

Proof. (a) By Corollary 3, the assertion is true when M is of the form $k[s]^l$. In the general case, M admits a finite presentation

$$k[s]^p \to k[s]^q \to M \to 0.$$

Applying the functor I to this exact sequence (and using again Corollary 3), we get a sequence

$$k[s]^p \to k[s]^q \to \mathrm{I}(M) \to 0.$$

This is a complex, of course. By Lemma 4, this complex is exact at $\mathrm{I}(M)$, and hence our homomorphism is surjective.

To show the injectivity, take any $0 \neq x \in M$. Because E is a cogenerator we can find a homomorphism $g : M \to E$ such that $g(x) \neq 0$. At least one of the coefficients in the expansion of $g(x)$, say, the coefficient at t^i, is distinct from 0. Because $s^i g$ is also a homomorphism of M into E and $(s^i g(x))(0) \neq 0$, we see that the image of x under our homomorphism is not zero.

(b) The homomorphism f can be included into a commutative diagram

$$\begin{array}{ccccccc}
k[s]^p & \longrightarrow & k[s]^q & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
k[s]^l & \longrightarrow & k[s]^m & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

with exact rows. From this we get a commutative diagram

$$\begin{array}{ccccccc}
k[s]^p & \longrightarrow & k[s]^q & \longrightarrow & \mathrm{I}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
k[s]^l & \longrightarrow & k[s]^m & \longrightarrow & \mathrm{I}(N) & \longrightarrow & 0
\end{array}$$

also having exact rows. (The exactness is seen from the proof of (a).) This completes the proof. □

The following important result was obtained by Oberst (see [7, Theorem 2.61]).

Corollary 8. *Let R_1 and R_2 be polynomial matrices with sizes $m_1 \times q$ and $m_2 \times q$, respectively, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the corresponding linear systems. Then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if there is a polynomial matrix X such that $R_2 = X R_1$.*

Proof. The "if" part is obvious.

"Only if". Let $M_1 = \mathcal{S}_1^*$ and $M_2 = \mathcal{S}_2^*$. From $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{U}^q$, using the functoriality of * and Corollary 3, we see that the diagram

$$\begin{array}{ccccccc}
k[s]^{m_2} & \xrightarrow{R_2^{\mathrm{tr}}} & k[s]^q & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \\
k[s]^{m_1} & \xrightarrow{R_1^{\mathrm{tr}}} & k[s]^q & \longrightarrow & M_1 & \longrightarrow & 0
\end{array}$$

is commutative. By the theorem above, the rows in this diagram are exact. This implies the assertion. □

The following can be viewed as a generalization of Example 3.

Theorem 9. *Let M and N be finitely generated $k[s]$-modules.*

(a) *Every continuous* $k[s]$*-homomorphism* $\phi : \mathrm{Bh}(N) \to \mathrm{Bh}(M)$ *is essentially equal to* $\mathrm{Bh}(f)$ *for some* $k[s]$*-homomorphism* $f : M \to N$*.*
(b) *If* $f_1, f_2 : M \to N$ *are two* $k[s]$*-homomorphisms, then* $\mathrm{Bh}(f_1)$ *and* $\mathrm{Bh}(f_2)$ *are essentially equal if and only if they are equal.*

Proof. The arguments are as in Example 3. (We need only apply Theorem 7 instead of Corollary 3.) □

The theorem above says that differential operators from $\mathrm{Bh}(N)$ to $\mathrm{Bh}(M)$ can be identified with continuous $k[s]$-homomorphisms of the form $\mathrm{Bh}(f)$.

We call an observable of a linear system $\mathcal{S}$ any continuous linear functional on it. (The term is borrowed from [9].) Denote by $\mathrm{Ob}(\mathcal{S})$ the module of observables of $\mathcal{S}$ (i.e., the module $\mathcal{S}^*$). Given a differential operator $\Phi : \mathcal{S} \to \mathcal{T}$, define $\mathrm{Ob}(\Phi)$ to be ϕ^*, where ϕ is a representative of Φ.

Theorem 10. *The functors* Bh *and* Ob *establish a duality between the categories* $\underline{\mathrm{Mod}}$ *and* $\underline{\mathrm{Syst}}$*.*

Proof. By the Malgrange lemma, every object of $\underline{\mathrm{Syst}}$ is isomorphic to an object of the form $\mathrm{Bh}(M)$ with $M \in \underline{\mathrm{Mod}}$. By the previous theorem, if $M, N \in \underline{\mathrm{Mod}}$, then the canonical map

$$\mathrm{Hom}(M, N) \to \mathrm{Diff}\big(\mathrm{Bh}(N), \mathrm{Bh}(M)\big)$$

is bijective. The proof is complete. □

An important consequence of this theorem is that the category $\underline{\mathrm{Syst}}$ is abelian. This means, in particular, that one can speak about the kernels, images and cokernels of differential operators of linear systems.

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