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How to define the dual of a higher-dimensional linear system

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The classical duality of linear systems is generalized to higher dimensions. Consequently, a new insight is given to the concepts of controllability and observability, and autonomy.

0. Introduction

One defines in an obvious way the dual of a classical (input/state/output) linear system

$$\begin{cases} x' = Ax + Bu, \\ y = Cx + Du. \end{cases}$$

In higher dimensions, the class of linear systems admitting input/state/output representations is very limited, and the question of how to define the dual of a (general) linear system is not obvious at all.

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One approach to the problem has already been presented by Pommaret [8]. Here we pursue a different point of view.

Let $s_1, \ldots, s_r$ be indeterminates and $\partial_1, \ldots, \partial_r$ partial differentiation (or shift) operators, and let q be a positive integer. Put $s = (s_1, \ldots, s_r)$ and $\partial = (\partial_1, \ldots, \partial_r)$. We remind (see [11]) that an AR-model with signal number q is an equation of the form

$$R(\partial)w = 0, \tag{1}$$

with $R(s) \in \mathbb{R}[s]^{g\times q}$, and an MA-model with signal number q is an equation of the form

$$w = M(\partial)l, \tag{2}$$

with $M(s) \in \mathbb{R}[s]^{q\times g}$. There is an evident duality between AR-models and MA-models: The dual of (1) is

$$w = R^t(\partial)l$$

and the dual of (2) is

$$M^t(\partial)w = 0.$$

(Here and in the text the superscript "t" stands for the transpose.)

The so-called ARMA-models (see [11]) comprise AR-models as well as MA-models. An ARMA-model with signal number q is defined to be

$$M(\partial)l = R(\partial)w,$$

where R and M are as above. The duality between AR- and MA-models, in our opinion, is very natural, and our purpose in this paper is to extend it to all ARMA-models.

We are based on our own paper [3], where, among other things, the duals for special type ARMA-models (1st order in the latent variable and 0th order in the manifest variable) are defined. We construct the duals here in the same manner as in the mentioned paper. But the double dual in the general setting of the present paper is no longer isomorphic to the original system. Instead, it is homotopy equivalent to it. Thus, in order to obtain a real duality we need a category in which ARMA-models are isomorphic whenever they are homotopy equivalent. In other words, we need a homotopy category of ARMA-models.

Homotopy is well-known for systems community; this is none other than the classical strict system equivalence of Fuhrmann and Rosenbrock (see [4]).

Throughout, D is a ring of polynomials in r indeterminates (with coefficients in any field), q a fixed positive integer, and $\mathcal{U}$ is a module over D. The number q will serve as the signal number, and the module $\mathcal{U}$ will be regarded as a function space. So, $\mathcal{U}^q$ will be our "universum". (See [11] for this concept.) Given a homomorphism $P : X \to Y$, we shall denote by $P^{\odot}$ the homomorphism $P \otimes \mathcal{U} : X \otimes \mathcal{U} \to Y \otimes \mathcal{U}$. We shall use the abbreviation "f.g." for "finitely generated".

The module $\mathcal{U}$ will be assumed to be either an injective cogenerator or a faithfully flat module. The importance of the injective cogenerator property is well-known since Oberst's fundamental paper [5]. It turned out that the property of faithful flatness also is of importance for systems theory, and Shankar [10] was the first who realized this importance. (We remind that $\mathcal{U}$ is an injective cogenerator if it is injective and satisfies the condition: $Hom(X, \mathcal{U}) = 0 \Leftrightarrow X = 0$. Likewise, for $\mathcal{U}$ to be faithfully flat means to be flat and satisfy the condition: $X \otimes \mathcal{U} = 0 \Leftrightarrow X = 0$.) Examples of an injective cogenerator are the space of C^∞-functions and the space of distributions (see Ref. [5]). Examples of a faithfully flat module are the space of compactly supported C^∞-functions and the space of compactly supported distributions (see [10]).

The content of the paper is as follows.

Section 1. ARMA-models
Section 2. Homotopy
Section 3. The dual of an ARMA-model
Section 4. The structural modules
Section 5. Elimination theorems
Section 6. Long exact behavioral sequence
Section 7. Controllability and observability, and autonomy

Closing Introduction, we remark that every linear map between modules can be viewed as a chain complex concentrated in the degrees 1 and 0: A linear map $X_1 \xrightarrow{u} X_0$ can be identified with the chain complex

$$\cdots \to 0 \to X_1 \xrightarrow{u} X_0 \to 0 \to \cdots$$

We can speak therefore about the homologies of a linear map, chain maps between linear maps, a homotopy equivalence of chain maps, and a homotopy equivalence of linear maps: The homologies of $u : X_1 \to X_0$ are $H_1(u) = Ker(u)$ and $H_0(u) = Coker(u)$; a chain map between $X = \{X_1 \xrightarrow{u} X_0\}$ and $Y = \{Y_1 \xrightarrow{v} Y_0\}$ is a pair $\varphi = (f_1, f_0)$ consisting of linear maps $f_1 : X_1 \to Y_1$ and $f_0 : X_0 \to Y_0$ such that $f_0 u = v f_1$; two chain maps (f_1, f_0) and (g_1, g_0) are homotopy equivalent if there is a linear map $h : X_0 \to Y_1$ such that $g_1 - f_1 = hu$ and $g_0 - f_0 = vh$; two linear maps $X = \{X_1 \xrightarrow{u} X_0\}$ and $Y = \{Y_1 \xrightarrow{v} Y_0\}$ are homotopy equivalent, if there exist chain maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are homotopy equivalent to the "identity" chain maps 1_X and 1_Y, respectively. It is an easy exercise to show that homotopy equivalent linear maps have isomorphic homologies. (An excellent book on homological algebra is Gelfand and Manin [2].)

1. ARMA-models

We shall mean by an AR-model a pair (Z, R), where Z is a f.g. free D-module and R is a D-linear map from D^q to Z. Similarly, we shall mean by an MA-model a pair (Z, M), where again Z is a f.g. free D-module and M is a linear map from Z to D^q. A map from one AR-model (Z_1, R_1) to another (Z_2, R_2) is a linear map $f : Z_1 \to Z_2$ such that $R_2 = fR_1$. A map from one MA-model (Z_1, M_1) to another (Z_2, M_2) is a linear map $f : Z_1 \to Z_2$ such that $M_1 = M_2 f$. The category of AR-models will be denoted by **AR**, and the category of MA-models by **MA**.

An ARMA-model is a diagram

$$Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q,$$

where Z_1, Z_0 are f.g. free D-modules and M, R are D-linear maps.

Example 1. Let (Z, R) be an AR-model. Then

$$0 \to Z \xleftarrow{R} D^q$$

is an ARMA-model.

Example 2. Let (Z, M) be an MA-model. Then

$$Z \xrightarrow{M} D^q \xleftarrow{id} D^q$$

is an ARMA-model.

Example 3. Let X be a f.g. free module. Then

$$X \to 0 \leftarrow D^q$$

is an ARMA-model. Such an ARMA-model will be called trivial.

Example 4. Let X be a f.g. free module. Then

$$0 \to X \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q$$

is an ARMA-model. Such an ARMA-model will be called incorrect.

As is known, one needs not just objects, but also maps (morphisms) between them. A map between ARMA models

$$\{Y_1 \stackrel{L}{\rightarrow} Y_0 \stackrel{Q}{\leftarrow} D^q\} \;\rightarrow\; \{Z_1 \stackrel{M}{\rightarrow} Z_0 \stackrel{R}{\leftarrow} D^q\}$$

is a triple (f_1, f_0, k) consisting of linear maps $f_1 : Y_1 \to Z_1$, $f_0 : Y_0 \to Z_0$ and $k : D^q \to Z_1$ such that

$$f_0L = Mf_1 \quad \text{and} \quad f_0Q = R + Mk;$$

that is,

$$\left(\begin{bmatrix} f_1 & -k \\ 0 & I \end{bmatrix}, f_0\right)$$

is a chain map between the linear maps

$$Y_1 \oplus D^q \stackrel{[L \quad -Q]}{\longrightarrow} Y_0 \quad \text{and} \quad Z_1 \oplus D^q \stackrel{[M \quad -R]}{\longrightarrow} Z_0.$$

Example 5. Let f be a map between AR-models (Y, Q) and (Z, R). Then clearly $(0, f, 0)$ is a chain map from $0 \to Y \stackrel{Q}{\leftarrow} D^q$ to $0 \to Z \stackrel{R}{\leftarrow} D^q$. Every map from $0 \to Y \stackrel{Q}{\leftarrow} D^q$ to $0 \to Z \stackrel{R}{\leftarrow} D^q$ is obtained this way.

Example 6. Let f be a map between MA-models (Y, L) and (Z, M). Then clearly $(f, I, 0)$ is a map from $Y \stackrel{L}{\rightarrow} D^q \stackrel{id}{\leftarrow} D^q$ to $Z \stackrel{M}{\rightarrow} D^q \stackrel{id}{\leftarrow} D^q$. However, not every map from $Y \stackrel{L}{\rightarrow} D^q \stackrel{id}{\leftarrow} D^q$ to $Z \stackrel{M}{\rightarrow} D^q \stackrel{id}{\leftarrow} D^q$ is obtained this way.

We remark that if $\mathfrak{A} = \{X \to Z \leftarrow D^q\}$ is an ARMA-model, then $1_{\mathfrak{A}} = (I, I, 0)$ is a map of $\mathfrak{A}$ into itself, called the identity map. The composition of two maps

$$\{X_1 \to X_0 \leftarrow D^q\} \stackrel{(f_1, f_0, k)}{\longrightarrow} \{Y_1 \to Y_0 \leftarrow D^q\} \stackrel{(g_1, g_0, l)}{\longrightarrow} \{Z_1 \to Z_0 \leftarrow D^q\}$$

is defined by the formula

$$(g_1, g_0, l) \circ (f_1, f_0, k) = (g_1f_1, g_0f_0, g_1k + l).$$

We let **ARMA** denote the category of ARMA-models and their maps. The Examples 5 and 6 say that **AR** is a full subcategory of **ARMA**, but **MA** not. (We remind that a subcategory $\mathcal{C}_0$ of a category $\mathcal{C}$ is said to be full, if for every pair (A, B) of objects in $\mathcal{C}_0$, the set $Hom_{\mathcal{C}_0}(A, B)$ coincides with the whole $Hom_{\mathcal{C}}(A, B)$.)

2. Homotopy

Our aim in this section is to introduce a new category, a kind of quotient of **ARMA**, in which it is better to work. The point is that **ARMA**, in fact, is not so good: It contains too many maps! (We have seen this in Example 6.)

Suppose

$$Y_1 \stackrel{L}{\rightarrow} Y_0 \stackrel{Q}{\leftarrow} D^q \quad \text{and} \quad Z_1 \stackrel{M}{\rightarrow} Z_0 \stackrel{R}{\leftarrow} D^q$$

are ARMA-models, and suppose (f_1, f_0, k) and (g_1, g_0, l) are two maps from the first one to the second. We say that these maps are homotopic if there is $h : Y_0 \to Z_1$ such that

$$g_1 = f_1 + hL, \quad g_0 = f_0 + Mh \quad \text{and} \quad l = k + hQ.$$

(Intuitively: (g_1, g_0, l) is obtained from (f_1, f_0, k) by a slight change). We shall write $\approx$ to denote homotopy equivalence.

Proposition 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ARMA-models. Homotopy is an equivalence relation in the set $Mor_{\mathbf{ARMA}}(\mathfrak{A}, \mathfrak{B})$.*

Proof. Left to the reader. □

The following says that composing homotopic maps yields homotopic compositions.

Proposition 2. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be ARMA-models, and let $\varphi_1, \varphi_2 : \mathfrak{A} \to \mathfrak{B}$ and $\psi_1, \psi_2 : \mathfrak{B} \to \mathfrak{C}$ be chain maps. If $\varphi_1 \approx \varphi_2$ and $\psi_1 \approx \psi_2$, then $\psi_1 \circ \varphi_1 \approx \psi_2 \circ \varphi_2$.*

Proof. Left to the reader. □

We want to treat homotopic maps between ARMA-models as being equal; in other words, we want homotopy equivalence classes to be morphisms between ARMA-models.

Define the category $K(\textbf{ARMA})$ as follows: The objects are ARMA-models and the morphisms between ARMA-models are maps modulo the homotopy equivalence relation; that is,

$$Mor_{K(\textbf{ARMA})}(\mathfrak{A}, \mathfrak{B}) = Mor_{\textbf{ARMA}}(\mathfrak{A}, \mathfrak{B}) / \approx .$$

That this is a category indeed follows from the previous proposition. (For the notion of the homotopy category $K(\textbf{C})$ of a category $\textbf{C}$, the reader may consult [2, Chapter III.4].)

Isomorphisms in $K(\textbf{ARMA})$ will be called also homotopy equivalences in **ARMA**. Thus, a mapping $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a homotopy equivalence if there exists $\psi : \mathfrak{B} \to \mathfrak{A}$ such that $\varphi \circ \psi$ is homotopic to $id_{\mathfrak{B}}$ and $\psi \circ \varphi$ is homotopic to $id_{\mathfrak{A}}$. We shall use the same symbol "≈" to denote the homotopy equivalence between ARMA-models.

We define linear systems to be homotopy equivalence classes of ARMA-models, that is, isomorphism classes in $K(\textbf{ARMA})$.

Example 7. Let (Y, Q) and (Z, R) be two AR-models. If f and g are two maps of (Y, Q) into (Z, R), then the maps $(0, f, 0)$ and $(0, g, 0)$ from $0 \to Y \overset{Q}{\leftarrow} D^q$ to $0 \to Z \overset{R}{\leftarrow} D^q$ are homotopy equivalent if and only if $f = g$.

Example 8. Let (Y, L) and (Z, M) be two MA-models. If (f, g, k) is a map

$$\{Y \to D^q = D^q\} \to \{Z \to D^q = D^q\},$$

then, it is easily seen that $f - kL$ is a map of (Y, L) into (Z, M) and (f, g, k) is homotopy equivalent to $(f - kL, I, 0)$. Next, if f and g are two maps of (Y, L) into (Z, M), then $(f, I, 0)$ and $(g, I, 0)$ are homotopy equivalent if and only if $f = g$.

Proposition 3. *Both* **AR** *and* **MA** *are full subcategories of $K(\textbf{ARMA})$.*

Proof. Follows from the previous two examples. □

Remark 1. It is worthwhile to note that the category of f.g. free modules also can be viewed (in two different ways) as full subcategories of $K(\textbf{ARMA})$. (See Examples 3 and 4.)

We remark that if two ARMA-models

$$X_1 \overset{L}{\to} X_0 \overset{Q}{\leftarrow} D^q \quad \text{and} \quad Y_1 \overset{M}{\to} Y_0 \overset{R}{\leftarrow} D^q$$

are homotopy equivalent, then necessarily

$$rk(X_1) + rk(Y_0) = rk(X_0) + rk(Y_1).$$

(See [4].) The following theorem is a generalization of Fuhrmann's classical result in [1]. It is very useful when one wants to check whether a given map from one ARMA-model to another is a homotopy equivalence.

Theorem 1. *Let*

$$X_1 \xrightarrow{L} X_0 \xleftarrow{Q} D^q \quad \text{and} \quad Y_1 \xrightarrow{M} Y_0 \xleftarrow{R} D^q$$

be two ARMA-models satisfying the rank condition above. Then a map $\varphi = (f_1, f_0, k)$ *between them is a homotopy equivalence if and only if it satisfies the following two conditions ("Fuhrmann's conditions"):*
(a) f_0 *and* M *are right coprime*;
(b) f_1 *and* L *are left coprime*.

Proof. See [4]. □

We close the section by the following relevant

Lemma 1. *Let*

$$Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q$$

be an ARMA-model. Then the ARMA-model

$$Z_1 \oplus D^q \xrightarrow{\begin{bmatrix} M & -R \\ 0 & -I \end{bmatrix}} Z_0 \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q$$

is canonically isomorphic, in $K(\mathbf{ARMA})$, *to the given one*.

Proof. It is easily seen that

$$([I \quad 0], [I \quad -R], 0)$$

is a map from the ARMA-model (in which we are interested) to the given one. This map satisfies Furmann's conditions; hence, it is a homotopy equivalence. □

3. The dual of an ARMA-model

Let $\mathfrak{A} = \{Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q\}$ be an ARMA-model. We define the dual of $\mathfrak{A}$ as the ARMA-model

$$\mathfrak{A}^* = \left\{ Z_0^* \xrightarrow{\begin{bmatrix} M^t \\ -R^t \end{bmatrix}} Z_1^* \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q \right\}.$$

The following two examples say that the dual of an AR-model is an MA-model and the dual of MA-model is an AR-model.

Example 9. Let (Z, R) be an AR-model. Then the dual of $0 \to Z \xleftarrow{R} D^q$ is

$$Z^* \xrightarrow{-R^t} D^q \xleftarrow{id} D^q.$$

This certainly is isomorphic to

$$Z^* \xrightarrow{R^t} D^q \xleftarrow{id} D^q,$$

which corresponds to the MA-model (Z^*, R^t).

Example 10. Let (Z, M) be an MA-model. Then the dual of $Z \xrightarrow{M} D^q \xleftarrow{id} D^q$ is

$$D^q \xrightarrow{\begin{bmatrix} M^t \\ -I \end{bmatrix}} Z^* \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q.$$

By Lemma 1, this one is homotopy equivalent to

$$0 \to Z^* \xleftarrow{M^t} D^q,$$

which corresponds to the AR-model (Z^*, M^t).

The following two examples say that the dual of a trivial model is incorrect and the dual of an incorrect model is trivial.

Example 11. The dual of $X \to 0 \leftarrow D^q$ is

$$0 \to X^* \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q.$$

Example 12. The dual of $0 \to X \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q$ is

$$X^* \oplus D^q \xrightarrow{[0 \quad -I]} D^q \leftarrow D^q,$$

which, by Lemma 1, is homotopy equivalent to $X^* \to 0 \leftarrow D^q$.

The following example says, in particular, that our definition of the dual agrees with that in the classical case.

Example 13. Let m and p be integers such that $m + p = q$. Consider a Rosenbrock model

$$(Z; T, U, V, W),$$

where Z is a free module over D, and $T : Z \to Z$ is a linear map with $det(T) \neq 0$ and $U : D^m \to Z$, $V : Z \to D^p$, $W : D^m \to D^p$ are arbitrary linear maps. As an ARMA-model, this is

$$Z \xrightarrow{\begin{bmatrix} T \\ -V \end{bmatrix}} Z \oplus D^p \xleftarrow{\begin{bmatrix} U & 0 \\ W & -I \end{bmatrix}} D^m \oplus D^p.$$

The dual of this ARMA-model is

$$Z^* \oplus D^p \xrightarrow{\begin{bmatrix} T^t & -V^t \\ -U^t & -W^t \\ 0 & I \end{bmatrix}} Z^* \oplus D^m \oplus D^p \xleftarrow{\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}} D^m \oplus D^p.$$

In $K(\mathbf{ARMA})$, the latter is canonically isomorphic to

$$Z^* \xrightarrow{\begin{bmatrix} -T^t \\ U^t \end{bmatrix}} Z^* \oplus D^m \xleftarrow{\begin{bmatrix} 0 & -V^t \\ -I & -W^t \end{bmatrix}} D^m \oplus D^p.$$

The isomorphism is given by

$$\left([I \quad 0], \begin{bmatrix} -I & 0 & -V^t \\ 0 & -I & -W^t \end{bmatrix}, [0 \quad 0] \right).$$

(One can check easily that Fuhrmann's conditions are satisfied.)

As a Rosenbrock model, this is

$$(Z; -T^t, -V^t, -U^t, -W^t).$$

Remark 2. It is harmless to define the dual of $(Z; T, U, V, W)$ as $(Z^*; -T^t, -V^t, -U^t, -W^t)$.

Let $\varphi = (f_1, f_0, k)$ be a map from $\mathfrak{A} = \{Y_1 \to Y_0 \leftarrow D^q\}$ to $\mathfrak{B} = \{Z_1 \to Z_0 \leftarrow D^q\}$. Define the dual of φ by the formula

$$\varphi^* = \left(f_0^t, \begin{bmatrix} f_1^t & 0 \\ -k^t & I \end{bmatrix}, 0 \right).$$

One can easily see that this indeed is a map from $\mathfrak{B}^*$ to $\mathfrak{A}^*$.

It is clear that if $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{C}$ are maps between ARMA-models, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Proposition 4. *If $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ are homotopy equivalent maps, then so are the maps $\varphi^*, \psi^* : \mathfrak{B}^* \to \mathfrak{A}^*$.*

Proof. Obvious. □

The following is our main theorem.

Theorem 2. *For every ARMA-model $\mathfrak{A}$, in $K(\mathbf{ARMA})$, there exists a canonical isomorphism*

$$\mathfrak{A}^{**} \simeq \mathfrak{A}.$$

Proof. This is exactly Lemma 1. □

One defines in an obvious way the dual Σ^* of a linear system Σ, and we have

Corollary 1. $\Sigma^{**} = \Sigma$.

We shall need the following

Lemma 2. *Let $Z_1 \stackrel{M}{\to} Z_0 \stackrel{R}{\leftarrow} D^q$ be an ARMA-model. Then there are canonical isomorphisms*

$$H_0 \begin{bmatrix} M & -R \\ 0 & -I \end{bmatrix} \simeq H_0 M \quad \textit{and} \quad H_0 \begin{bmatrix} M^t & 0 \\ -R^t & -I \end{bmatrix} \simeq H_0 M^t.$$

Proof. The homomorphisms

$$Z_1 \oplus D^q \stackrel{\begin{bmatrix} M & -R \\ 0 & I \end{bmatrix}}{\longrightarrow} Z_0 \oplus D^q \quad \text{and} \quad Z_1 \stackrel{M}{\to} Z_0$$

are homotopy equivalent. (The homotopy equivalence is $([I \quad 0], [I \quad -R]$.)

Likewise, the homomorphisms

$$Z_0^* \oplus D^q \stackrel{\begin{bmatrix} -M^t & 0 \\ R^t & -I \end{bmatrix}}{\longrightarrow} Z_1^* \oplus D^q \quad \text{and} \quad Z_0^* \stackrel{M^t}{\to} Z_1^*$$

are homotopy equivalent as well. (The homotopy equivalence is $([I \quad 0], [I \quad 0]$.)

The proof is complete. □

4. Structural modules

In this section we are interested in four modules that are very much related to the structure of ARMA-models.

Let

$$\mathfrak{A} = \left\{ Z_1 \stackrel{M}{\to} Z_0 \stackrel{R}{\leftarrow} D^q \right\}$$

be an ARMA-model. Associated with $\mathfrak{A}$ there are, as we already could notice, two important homomorphisms

$$\begin{bmatrix} M & -R \end{bmatrix} : Z_1 \oplus D^q \to Z_0 \quad \text{and} \quad M : Z_1 \to Z_0.$$

We set

$$ID(\mathfrak{A}) = H_0 \begin{bmatrix} M & -R \end{bmatrix}^t, \quad OD(\mathfrak{A}) = H_0(M), \quad Co(\mathfrak{A}) = H_0 \begin{bmatrix} M & -R \end{bmatrix} \quad \text{and}$$
$$Ob(\mathfrak{A}) = H_0(M^t).$$

Using Rosenbrock's terminology (see [9]), we call $ID(\mathfrak{A})$ and $OD(\mathfrak{A})$ the input and output decoupling modules. We call $Co(\mathfrak{A})$ and $Ob(\mathfrak{A})$ the controllability and observability modules.

The following example may justify the names we have given.

Example 14. Let $\mathfrak{A}$ be a Rosenbrock model $(Z; T, U, V, W)$. Then

$$ID(\mathfrak{A}) = H_0 \begin{bmatrix} T^t \\ -U^t \end{bmatrix}, \quad OD(\mathfrak{A}) = H_0 \begin{bmatrix} T \\ -V \end{bmatrix}, \quad Co(\mathfrak{A}) \simeq H_0 \begin{bmatrix} T & -U \end{bmatrix},$$
$$Ob(\mathfrak{A}) \simeq H_0 \begin{bmatrix} T^t & -V^t \end{bmatrix}.$$

The following examples are of special interest.

Example 15. Let $\mathfrak{A}$ be an AR-model $0 \to Z \xrightarrow{R} D^q$. Then

$$ID(\mathfrak{A}) = H_0(R^t), \quad OD(\mathfrak{A}) = Z, \quad Co(\mathfrak{A}) = H_0(R), \quad Ob(\mathfrak{A}) = 0.$$

Example 16. Let $\mathfrak{A}$ be an MA-model $Z \xrightarrow{M} D^q = D^q$. Then

$$ID(\mathfrak{A}) = Z^*, \quad OD(\mathfrak{A}) = H_0(M), \quad Co(\mathfrak{A}) = 0, \quad Ob(\mathfrak{A}) = H_0(M^t).$$

Example 17. Let $\mathfrak{A}$ be a trivial ARMA-model $X \to 0 \leftarrow D^q$. Then

$$ID(\mathfrak{A}) = X^* \oplus D^q, \quad OD(\mathfrak{A}) = 0, \quad Co(\mathfrak{A}) = 0, \quad Ob(\mathfrak{A}) = X^*.$$

Example 18. Let $\mathfrak{A}$ be an incorrect ARMA-model $0 \to X \oplus D^q \leftarrow D^q$. Then

$$ID(\mathfrak{A}) = 0, \quad OD(\mathfrak{A}) = X \oplus D^q, \quad Co(\mathfrak{A}) = X, \quad Ob(\mathfrak{A}) = 0.$$

The four modules are related to each other, as the following says.

Proposition 5. *Given an ARMA-model $\mathfrak{A}$, one has the following canonical exact sequences*

$$D^q \to ID(\mathfrak{A}) \to Ob(\mathfrak{A}) \to 0 \quad \textit{and} \quad D^q \to OD(\mathfrak{A}) \to Co(\mathfrak{A}) \to 0.$$

Proof. The first exact sequence follows from the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & \to & Z_0^* & = & Z_0^* & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & D^q & \to & Z_1^* \oplus D^q & \to & Z_1^* & \to & 0 \end{array}$$

by the snake lemma. Similarly, the second one follows from the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Z_1 & \to & Z_1 \oplus D^q & \to & D^q & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z_0 & = & Z_0 & \to & 0 & & . \end{array}$$ □

If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a map of ARMA-models, then there are canonical homomorphisms

$$ID(\varphi) : ID(\mathfrak{A}) \to ID(\mathfrak{B}) \quad \text{and} \quad OD(\varphi) : OD(\mathfrak{A}) \to OD(\mathfrak{B}).$$

Likewise, we have canonical homomorphisms

$$Co(\varphi) : Co(\mathfrak{A}) \to Co(\mathfrak{B}) \quad \text{and} \quad Ob(\varphi) : Ob(\mathfrak{A}) \to Ob(\mathfrak{B}).$$

Lemma 3. *Suppose $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ are homotopy equivalent maps. Then*

$$ID(\varphi) = ID(\psi), \quad OD(\varphi) = OD(\psi), \quad Co(\varphi) = Co(\psi) \quad \text{and} \quad Ob(\varphi) = Ob(\psi).$$

Proof. Let

$$\mathfrak{A} = \left\{ Y_1 \xrightarrow{L} Y_0 \xleftarrow{Q} D^q \right\} \quad \text{and} \quad \mathfrak{B} = \left\{ Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q \right\},$$

and let $\varphi = (f_1, f_0, k)$ and $\psi = (g_1, g_0, l)$. Suppose $h : Y_0 \to Z_1$ is a homotopy between φ and ψ.

Then, $\begin{bmatrix} h \\ 0 \end{bmatrix}$ determines a homotopy between the chain maps

$$\left(\begin{bmatrix} f_1 & -k \\ 0 & I \end{bmatrix}, f_0 \right), \left(\begin{bmatrix} g_1 & -l \\ 0 & I \end{bmatrix}, g_0 \right) : \left\{ Y_1 \oplus D^q \overset{[L \quad -Q]}{\longrightarrow} Y_0 \right\} \to \left\{ Z_1 \oplus D^q \overset{[M \quad -R]}{\longrightarrow} Z_0 \right\}$$

and $\begin{bmatrix} h^t & 0 \end{bmatrix}$ a homotopy between their transposes.

Obviously, h determines a homotopy between the chain maps

$$(f_1, f_0), \ (g_1, g_0) : \ \{Y_1 \to Y_0\} \ \to \ \{Z_1 \to Z_0\}$$

and h^t a homotopy between their transposes.

This completes the proof, because homotopy equivalent chain maps yield equal maps of the homologies. □

The following is in the spirit of Rosenbrock's book [9].

Theorem 3. *Homotopy equivalence preserves the structural modules.*

Proof. Follows from the previous lemma. □

Theorem 4. *There are canonical isomorphisms*

$$ID(\mathfrak{A}^*) \simeq OD(\mathfrak{A}), \quad OD(\mathfrak{A}^*) \simeq ID(\mathfrak{A}), \quad Co(\mathfrak{A}^*) \simeq Ob(\mathfrak{A}) \quad \textit{and} \quad Ob(\mathfrak{A}^*) \simeq Co(\mathfrak{A}).$$

Moreover, the diagrams

$$\begin{array}{ccccccc} D^q & \to & ID(A^*) & \to & Ob(A^*) & \to 0 \\ || & & \downarrow & & \downarrow & \\ D^q & \to & OD(A) & \to & Co(A) & \to 0 \end{array} \quad \textit{and}$$

$$\begin{array}{ccccccc} D^q & \to & OD(A^*) & \to & Co(A^*) & \to 0 \\ || & & \downarrow & & \downarrow & \\ D^q & \to & ID(A) & \to & Ob(A) & \to 0 \end{array}$$

are commutative.

Proof. That $ID(\mathfrak{A}^*) \simeq OD(\mathfrak{A})$ and $Co(\mathfrak{A}^*) \simeq Ob(\mathfrak{A})$ follows from Lemma 2. By the very definition, $OD(\mathfrak{A}^*) = ID(\mathfrak{A})$ and $Ob(\mathfrak{A}^*) = Co(\mathfrak{A})$.

The proof of the second statement is left to the reader. □

5. Elimination theorems

Let an ARMA-model $\mathfrak{A} = \left\{ Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q \right\}$ be given.

In this section we are interested in the question: When $\mathfrak{A}$ can be brought to the AR-, MA-, trivial or incorrect form?

Theorem 5. $\mathfrak{A}$ *is homotopically equivalent to an AR-model if and only if its observability module is zero.*

Proof. “If”: The linear map M^* is surjective, and therefore its kernel must be a free module. Letting Y denote the dual of this kernel, we have a canonical short exact sequence

$$0 \to X \to Z \to Y \to 0.$$

Denote by U the canonical linear map $Z \to Y$.

The canonical map

$$(0, U, 0) : \mathfrak{A} \to \left\{0 \to Y \xleftarrow{Q} D^q\right\}$$

determines an isomorphism in $K(\mathbf{ARMA})$. (It satisfies Fuhrmann's conditions: U is right invertible and M is left invertible.)

“Only if”: Follows from Example 15.

The proof is complete. □

Theorem 6. *$\mathfrak{A}$ is homotopically equivalent to an MA-model if and only if its controllability module is zero.*

Proof. “If”: Letting Y denote the kernel of $[M \;\; -R]$, we have a canonical exact sequence

$$0 \to Y \to X \oplus D^q \to Z \to 0.$$

Let U and L denote the canonical linear maps $Y \to X$ and $Y \to D^q$, respectively.

The canonical map

$$(U, R, 0) : \{Y \xrightarrow{L} D^q \;=\; D^q\} \to \mathfrak{A}$$

determines an isomorphism in $K(\mathbf{ARMA})$. (It satisfies Fuhrmann's conditions: $[R \quad M]$ is right invertible and $\begin{bmatrix} U \\ L \end{bmatrix}$ is left invertible.)

“Only if”: Follows from Example 16.

The proof is complete. □

Theorem 7. *$\mathfrak{A}$ is homotopically equivalent to a trivial model if and only if its output-decoupling module is zero.*

Proof. “If”: The hypothesis implies that there is an exact sequence

$$0 \to Y \xrightarrow{j} Z \to X \to 0,$$

where Y is a free module. It is easily seen that $(j, 0, 0)$ is a homotopy equivalence of $Y \to 0 \leftarrow D^q$ with $\mathfrak{A}$.

“Only if”: Follows from Example 17. □

Theorem 8. *$\mathfrak{A}$ is homotopically equivalent to an incorrect model if and only if its input-decoupling module is zero.*

Proof. “If”: There is an exact sequence

$$0 \to X \oplus D^q \to Z \to Y \to 0,$$

where Y is a f.g. free module. The sequence splits, and hence there exists a linear map $j : Y \to Z$ such that $[M \;\; j \;\; R]$ is an isomorphism of $X \oplus Y \oplus D^q$ onto Z. One can easily check that $(0, [j \;\; R], 0)$ is a homotopy equivalence of $0 \to Y \oplus D^q \leftarrow D^q$ with $\mathfrak{A}$.

“Only if”: Follows from Example 18. □

6. Long exact behavioral sequence

We begin with the following simple lemma (“Malgrange lemma”).

Lemma 4. *Let $P : X \to Y$ be a homomorphism of free D-modules of finite rank. Then*

$$H_1(P^{\odot}) = Hom(H_0(P^t), \mathcal{U}) \quad \text{and} \quad H_0(P^{\odot}) = H_0(P) \otimes \mathcal{U}.$$

In other words, there is a canonical exact sequence

$$0 \to Hom(H_0(P^t), \mathcal{U}) \to X \otimes \mathcal{U} \stackrel{P^{\odot}}{\to} Y \otimes \mathcal{U} \to H_0(P) \otimes \mathcal{U} \to 0.$$

Proof. Applying the functor $Hom(-, \mathcal{U})$ to the exact sequence

$$Y^* \stackrel{P^t}{\to} X^* \to H_0(P^t) \to 0,$$

we get an exact sequence

$$0 \to Hom(H_0(P^t), \mathcal{U}) \to X \otimes \mathcal{U} \stackrel{P^{\odot}}{\to} Y \otimes \mathcal{U}.$$

Next, applying the functor $- \otimes \mathcal{U}$ to the exact sequence

$$X \stackrel{P}{\to} Y \to H_0(P) \to 0,$$

we get an exact sequence

$$X \otimes \mathcal{U} \stackrel{P^{\odot}}{\to} Y \otimes \mathcal{U} \to H_0(P) \otimes \mathcal{U} \to 0.$$

The proof is complete. □

Theorem 9. *Let* $\mathfrak{A} = \left\{ Z_1 \stackrel{M}{\to} Z_0 \stackrel{R}{\leftarrow} D^q \right\}$ *be an ARMA-model. There is an exact sequence*

$$0 \to Hom(Ob(\mathfrak{A}), \mathcal{U}) \to Hom(ID(\mathfrak{A}), \mathcal{U}) \to \mathcal{U}^q \to OD(\mathfrak{A}) \otimes \mathcal{U} \to Co(\mathfrak{A}) \otimes \mathcal{U} \to 0.$$

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Z_1 & \to & Z_1 \oplus D^q & \to & D^q & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z_0 & = & Z_0 & \to & 0 & & \end{array} .$$

Tensoring this by $\mathcal{U}$ and then applying the snake lemma, we obtain an exact sequence

$$0 \to H_1(M^{\odot}) \to H_1[M^{\odot} \quad -R^{\odot}] \to \mathcal{U}^q \to H_0(M^{\odot}) \to H_0[M^{\odot} \quad -R^{\odot}] \to 0.$$

From this, in view of the previous lemma, the statement follows. □

We call the sequence in the theorem the long exact behavioral sequence of $\mathfrak{A}$ and denote it by $Bh(\mathfrak{A})$.

We set

$$\mathcal{B}_{id}(\mathfrak{A}) = Hom(ID(\mathfrak{A}), \mathcal{U}) \quad \text{and} \quad \mathcal{B}_{od}(\mathfrak{A}) = OD(\mathfrak{A}) \otimes \mathcal{U}.$$

With this notation, the middle part of the above sequence can be rewritten as

$$\mathcal{B}_{id}(\mathfrak{A}) \to \mathcal{U}^q \to \mathcal{B}_{od}(\mathfrak{A}).$$

The image of the first homomorphism and the kernel of the second one coincide, and this common set is called the *manifest behavior* of $\mathfrak{A}$. We shall denote it by $\mathcal{B}_{mf}(\mathfrak{A})$. Notice that $\mathcal{B}_{id}(\mathfrak{A})$ is equal to the solution set of the equation

$$Mz = Rw \quad (z \in Z_1 \otimes \mathcal{U},\ w \in \mathcal{U}^q),$$

which in Willems [11] is called the *full behavior* of $\mathfrak{A}$. This space certainly is very important, and we want to emphasize that the space $\mathcal{B}_{od}(\mathfrak{A})$ is equally important. (It is suggestive to think of $\mathcal{B}_{id}(\mathfrak{A})$ and $\mathcal{B}_{od}(\mathfrak{A})$ as two "black boxes" associated with $\mathfrak{A}$.)

Example 19. The long exact behavioral sequence of AR-model (X, R) is

$$0 \to 0 \to Ker(R) \to \mathcal{U}^q \stackrel{R}{\to} X \otimes \mathcal{U} \to H_0(R) \otimes \mathcal{U} \to 0.$$

Example 20. The long exact behavioral sequence of MA-model (X, M) is

$$0 \to Ker(M) \to X \otimes \mathcal{U} \xrightarrow{M} \mathcal{U}^q \to H_0(M) \otimes \mathcal{U} \to 0 \to 0.$$

Example 21. The long exact behavioral sequence of $X \to 0 \leftarrow D^q$ is

$$0 \to X \otimes \mathcal{U} \to X \otimes \mathcal{U} \oplus \mathcal{U}^q \to \mathcal{U}^q \to 0 \to 0 \to 0.$$

(Notice that the manifest behavior is equal to $\mathcal{U}^q$.)

Example 22. The long exact behavioral sequence of $0 \to X \oplus D^q \xleftarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} D^q$ is

$$0 \to 0 \to 0 \to \mathcal{U}^q \to X \otimes \mathcal{U} \oplus \mathcal{U}^q \to X \otimes \mathcal{U} \to 0.$$

(Notice that the manifest behavior is equal to $\{0\}$.)

If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a map of ARMA-models, then there is an obvious chain map

$$Bh(\varphi) : Bh(\mathfrak{A}) \to Bh(\mathfrak{A}).$$

It is clear that homotopy equivalent maps of ARMA-models induce the same chain map of the associated long exact behavioral sequences. It follows from this that homotopy equivalent ARMA-models have isomorphic long exact behavioral sequences.

In the sequel we shall need $\mathcal{B}_{id}(\varphi)$ and $\mathcal{B}_{od}(\varphi)$, which are defined in an obvious way. (They are, by the way, two components of $Bh(\varphi)$.)

Given two ARMA-models $\mathfrak{A}$ and $\mathfrak{B}$, we say that $\mathfrak{A}$ is more powerful than $\mathfrak{B}$ (and write $\mathfrak{A} \succeq \mathfrak{B}$) if there is a map $\varphi : \mathfrak{A} \to \mathfrak{B}$.

Proposition 6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ARMA-models. Then*

$$\mathfrak{A} \succeq \mathfrak{B} \quad \Rightarrow \quad \mathcal{B}_{mf}(\mathfrak{A}) \subseteq \mathcal{B}_{mf}(\mathfrak{B}).$$

Proof. Let $\mathfrak{A} = \left\{Y_1 \xrightarrow{L} Y_0 \xleftarrow{Q} D^q\right\}$ and $\mathfrak{B} = \left\{Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q\right\}$, and let $\varphi = (f_0, f_1, k)$ be a map from $\mathfrak{A}$ to $\mathfrak{B}$.

If w is a manifest trajectory of $\mathfrak{A}$, then $Qw = Ly$ for some $y \in \mathcal{U} \otimes Y_1$. And we have

$$Rw = f_0 Q - Mkw = f_0 Ly - Mkw = Mf_1 y - Mkw = M(f_1 y - kw).$$

Hence, w is a manifest trajectory of $\mathfrak{B}$ as well. $\square$

The following two theorems are elimination theorems. The first one is well-known (see, for example, [10,11]), and it says that if the function module is injective, then the manifest behavior of any ARMA-model has a "kernel representation". The second is analogous, and it says that if the function module is flat, then the manifest behavior of any ARMA-model has an "image representation".

Theorem 10. *Suppose $\mathcal{U}$ is injective. Then, for every ARMA-model $\mathfrak{A}$, there exists an AR-model $\mathfrak{B}$ such that*

$$\mathfrak{A} \succeq \mathfrak{B} \quad \text{and} \quad \mathcal{B}_{mf}(\mathfrak{A}) = \mathcal{B}_{mf}(\mathfrak{B}).$$

Proof. Let $\mathfrak{A}$ be $Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q$. There is an exact sequence

$$X^* \xrightarrow{f^*} Z_0^* \to Z_1^*,$$

where X is a f.g. free module and $f : Z_0 \to X$ is a homomorphism. We claim that the AR-model $\mathfrak{B} = (X, fR)$ satisfies the theorem. Indeed, $(0, f, 0)$ is a map from $\mathfrak{A}$ to $\mathfrak{B}$. Further, applying the functor $Hom(-, \mathcal{U})$ to the exact sequence above, we obtain an exact sequence

$$Z_1 \otimes \mathcal{U} \to Z_0 \otimes \mathcal{U} \to X \otimes \mathcal{U}.$$

It follows from this that the canonical map

$$\mathcal{B}_{od}(\mathfrak{A}) \to X \otimes \mathcal{U}$$

is injective. From the commutative diagram

$$\begin{array}{ccc} \mathcal{U}^q & = & \mathcal{U}^q \\ \downarrow & & \downarrow \\ \mathcal{B}_{od}(\mathfrak{A}) & \to & X \otimes \mathcal{U} \end{array}$$

we see that $\mathfrak{B}$ has the same manifest behavior as $\mathfrak{A}$. □

Theorem 11. *Suppose $\mathcal{U}$ is flat. Then, for every ARMA-model $\mathfrak{A}$, there exists an MA-model $\mathfrak{B}$ such that*

$$\mathfrak{B} \succeq \mathfrak{A} \quad \textit{and} \quad \mathcal{B}_{mf}(\mathfrak{B}) = \mathcal{B}_{mf}(\mathfrak{A}).$$

Proof. Let $\mathfrak{A}$ be $Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q$. There is an exact sequence

$$X \xrightarrow{\begin{bmatrix} f \\ L \end{bmatrix}} Z_1 \oplus D^q \to Z_0,$$

where X is a f.g. free module and $f : X \to Z_1$, $L : X \to D^q$ are homomorphisms. We claim that the MA-model $\mathfrak{B} = (X, L)$ satisfies the theorem. Indeed, $(f, R, 0)$ is map from $\mathfrak{B}$ to $\mathfrak{A}$. Next, applying the functor $- \otimes \mathcal{U}$ to the sequence above, we obtain an exact sequence

$$X \otimes \mathcal{U} \to Z_1 \otimes \mathcal{U} \oplus \mathcal{U}^q \to Z_0 \otimes \mathcal{U}.$$

It follows from this that the canonical map

$$X \otimes \mathcal{U} \to \mathcal{B}_{id}(\mathfrak{A})$$

is surjective. From the commutative diagram

$$\begin{array}{ccc} X \otimes \mathcal{U} & \to & \mathcal{B}_{id}(\mathfrak{A}) \\ \downarrow & & \downarrow \\ \mathcal{U}^q & = & \mathcal{U}^q \end{array}$$

we see that $\mathfrak{B}$ has the same manifest behavior as $\mathfrak{A}$. □

We close the section by the following:

Theorem 12. *The long exact behavioral sequence of the dual of $\mathfrak{A} = \left\{ Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q \right\}$ is canonically isomorphic to*

$$0 \to Hom(Co(\mathfrak{A}), \mathcal{U}) \to Hom(OD(\mathfrak{A}), \mathcal{U}) \to \mathcal{U}^q \to ID(\mathfrak{A}) \otimes \mathcal{U} \to Ob(\mathfrak{A}) \otimes \mathcal{U} \to 0.$$

Proof. Follows immediately from Theorem 4. □

The reader can notice that the relation between $Bh(\mathfrak{A})$ and $Bh(\mathfrak{A}^*)$ is symmetric.

7. Controllability and observability, and autonomy

The concepts of controllability, observability and autonomy are fundamental, as one knows well. For higher-dimensional linear systems, several attempts were made by various authors to develop a unified approach to the notions of controllability and autonomy. The results culminate in Pommaret and Quadrat [6,7], where very good definitions for these notions have been proposed, using *Ext*. We recommend Zerz [12] for a nice expository account. In the book [8], Pommaret gave the definition of observability as well.

For convenience of the reader, we briefly review the approach of Pommaret and Quadrat.

Let E be a f.g. D-module, and suppose $P : X \to Y$ is a homomorphism of f.g. free D-modules such that $E = H_0(P)$. A crucial fact is that the "dual" module $F = H_0(P^t)$ is uniquely determined up to projective equivalence (see [7]). The module E, regarded as a linear system without signals, is said to be controllable if $Ext^1(F, D) = 0$. If E is controllable, then the maximum integer d such that

$$Ext^1(F, D) = \cdots = Ext^d(F, D) = 0.$$

is called the controllability degree of E. The definition is based on the observation that

$$E \text{ is torsion free } \Leftrightarrow d \geqslant 1; \quad E \text{ is reflexive } \Leftrightarrow d \geqslant 2; \quad E \text{ is free } \Leftrightarrow d = r.$$

So, the controllability degree classifies the "higher" analogues of torsion freeness.

Further, E is said to be autonomous if $Ext^0(E, D) = 0$, and one says that its autonomy degree is k, if k is the maximum integer such that

$$Ext^0(E, D) = \cdots = Ext^k(E, D) = 0.$$

There is a nice formula

$$k = r - \dim(E) - 1.$$

(The dimension of E is the Krull dimension of the ring $D/Ann(E)$). In particular, we have

$$Ext^0(E, D) = 0 \Leftrightarrow E \quad \text{is a torsion module.}$$

Finally, Pommaret [8] defines E to be observable if $Ext^1(E, D) = 0$.

We apply the above machinary to our situation in the following way.

Let

$$\mathfrak{A} = \{Z_1 \overset{M}{\to} Z_0 \overset{R}{\leftarrow} D^q\}$$

be an ARMA-model. Associated with $\mathfrak{A}$ there are, as we know, four (structural) modules

$$ID(\mathfrak{A}), \quad OD(\mathfrak{A}), \quad Co(\mathfrak{A}) \quad \text{and} \quad Ob(\mathfrak{A}).$$

Notice that $Co(\mathfrak{A})$ is the dual of $ID(\mathfrak{A})$ (in the sense of Pommaret-Quadrat) and $Ob(\mathfrak{A})$ is the dual of $OD(\mathfrak{A})$. We say:

(Co) $\mathfrak{A}$ is controllable if it satisfies the following equivalent conditions

$$Ext^1(Co(\mathfrak{A}), D) = 0 \Leftrightarrow ID(\mathfrak{A}) \text{ is torsion free;}$$

(Ob) $\mathfrak{A}$ is observable if it satisfies the following equivalent conditions

$$Ext^1(Ob(\mathfrak{A}), D) = 0 \Leftrightarrow OD(\mathfrak{A}) \text{ is torsion free}$$

(IA) $\mathfrak{A}$ is input-autonomous if it satisfies the following equivalent conditions

$$Ext^0(ID(\mathfrak{A}), D) = 0 \Leftrightarrow ID(\mathfrak{A}) \text{ is a torsion module}$$

(OA) $\mathfrak{A}$ is output-autonomous if it satisfies the following equivalent conditions:

$$Ext^0(OD(\mathfrak{A}), D) = 0 \Leftrightarrow OD(\mathfrak{A}) \text{ is a torsion module.}$$

In case $\mathfrak{A}$ is controllable, its controllability degree is defined to be the maximum integer d such that

$$Ext^1(Co(\mathfrak{A}), D) = \cdots = Ext^d(Co(\mathfrak{A}), D) = 0.$$

If the degree is equal to r, we say that $\mathfrak{A}$ is strongly controllable. Similarly, in case $\mathfrak{A}$ is observable, its observability degree is the maximum integer d such that

$$Ext^1(Ob(\mathfrak{A}), D) = \cdots = Ext^d(Ob(\mathfrak{A}), D) = 0.$$

If the degree is equal to r, we say that $\mathfrak{A}$ is strongly observable.

One can introduce in an analogous way various degrees of autonomy as well.

We remark that

$$\mathfrak{A} \text{ is strongly controllable} \Leftrightarrow ID(\mathfrak{A}) \text{ is free} \Leftrightarrow Co(\mathfrak{A}) = 0 \Leftrightarrow \mathfrak{A} \approx \text{ an MA-model;}$$
$$\mathfrak{A} \text{ is strongly observable} \Leftrightarrow OD(\mathfrak{A}) \text{ is free} \Leftrightarrow Ob(\mathfrak{A}) = 0 \Leftrightarrow \mathfrak{A} \approx \text{ an AR-model.}$$

So, the controllability module of an ARMA-model is the obstruction to strong controllability and the observability module is the obstruction to strong observability. We can also say: The controllability module of an ARMA-model is the obstruction to representability in the MA-form and the observability module is the obstruction to representability in the AR-form.

Of course, in the one-dimensional case, "controllability" = "strong controllability" and "observability" = "strong observability". And in this case the definitions above coincide with those given by Willems [11, Section VI].

Under the hypothesis that $\mathcal{U}$ is faithfully flat, saying that $Co(A) = 0$ is equivalent to saying that $\mathcal{U}^q \to \mathcal{B}_{od}(\mathfrak{A})$ is surjective. Hence, strong controllability means that

"the "od"- variable is (completely) controlled via the manifest variable".

Under the hypothesis that $\mathcal{U}$ is an injective cogenerator, saying that $Ob(A) = 0$ is equivalent to saying that $\mathcal{B}_{id}(\mathfrak{A}) \to \mathcal{U}^q$ is injective. Hence, strong observability means that

"the "id"- variable is (completely) observed via the manifest variable".

(This interpretation was given in Willems [11, Section VI]).

The intuitive meaning of controllability and observability is quite clear. To get it we have to replace above the word "completely", say, by the word "essentially".

We shall now make an attempt to interpret more precisely the notions of controllability and observability.

Assume we have an ARMA-model

$$\mathfrak{A} = \left\{ Z_1 \xrightarrow{M} Z_0 \xleftarrow{R} D^q \right\}.$$

We have seen that controllability is a feature of MA-models and observability is a feature of AR-models. It is a natural idea therefore to express controllability using MA-models and observability using AR-models.

The following theorem generalizes the standard fact that the property of controllability is equivalent to the property of having MA-representation. (Recall that the "$\mathcal{B}_{id}$" of MA-model (X, L) is equal to $X \otimes \mathcal{U}$).

Theorem 13. *Suppose $\mathcal{U}$ is an injective cogenerator. Then $\mathfrak{A}$ is controllable if and only if there exists a pair (a "controller") $(\mathfrak{B}, \varphi)$, where $\mathfrak{B}$ is an MA-model and $\varphi : \mathfrak{B} \to \mathfrak{A}$ a map such that*

$$\mathcal{B}_{id}(\varphi) : \mathcal{B}_{id}(\mathfrak{B}) \to \mathcal{B}_{id}(\mathfrak{A})$$

is surjective.

Proof. "If": Let $\mathfrak{B} = \{X \xrightarrow{L} D^q = D^q\}$ be an MA-model and $\varphi = (f, g, k)$ a map that satisfy the condition above. Notice that $(f - kL, R, 0)$ is homotopy equivalent to (f, g, k), and therefore we may assume that $\varphi = (f, R, 0)$. We then have an exact sequence

$$X \otimes \mathcal{U} \to Z_1 \otimes \mathcal{U} \oplus \mathcal{U}^q \to Z_0 \otimes \mathcal{U}.$$

This can be rewritten as

$$Hom(X^*, \mathcal{U}) \to Hom(Z_1^* \oplus D^q, \mathcal{U}) \to Hom(Z_0^*, \mathcal{U}).$$

Since $\mathcal{U}$ is an injective cogenerator, we get that the sequence

$$Z_0^* \to Z_1^* \oplus D^q \to X^*$$

is exact. It follows from this that $Co(\mathfrak{A})$ is torsion free.

"Only if": This is left to the reader. (The proof of this part requires only injectiveness of $\mathcal{U}$.) □

Comment. $\mathfrak{A}$ is controllable if and only if its “id”-trajectories can be controlled via the “id”-trajectories of some more powerful MA-model.

Remark 3. If happens that $\mathcal{B}_{id}(\varphi)$ is bijective, then $\mathfrak{A}$ is strongly controllable. (The proof of this also is left to the reader.)

The following theorem is an obvious analog of the previous one. (Recall that the “$\mathcal{B}_{od}$” of AR-model (X, Q) is equal to $X \otimes \mathcal{U}$.)

Theorem 14. *Suppose $\mathcal{U}$ is faithfully flat. Then $\mathfrak{A}$ is observable if and only if there exists a pair (an “observer”) $(\mathfrak{B}, \varphi)$, where $\mathfrak{B}$ is an AR-model and $\varphi : \mathfrak{A} \to \mathfrak{B}$ a map such that*

$$\mathcal{B}_{od}(\varphi) : \mathcal{B}_{od}(\mathfrak{A}) \to \mathcal{B}_{od}(\mathfrak{B})$$

is injective.

Proof. “If”: Let $\mathfrak{B} = \{0 \to X \stackrel{Q}{\leftarrow} D^q\}$ be an AR-model that satisfies the condition above. We then have an exact sequence

$$Z_1 \otimes \mathcal{U} \to Z_0 \otimes \mathcal{U} \to X \otimes \mathcal{U}.$$

Since $\mathcal{U}$ is faithfully flat, we get that the sequence

$$Z_1 \to Z_0 \to X$$

is exact. It follows from this that $Ob(\mathfrak{A})$ is torsion free.

“Only if”: This is left to the reader. (The proof of this part requires only flatness of $\mathcal{U}$.) □

Comment. $\mathfrak{A}$ is observable if and only if its “od”-trajectories can be observed via the “od”-trajectories of some less powerful AR-model.

Remark 4. If happens that $\mathcal{B}_{od}(\varphi)$ is bijective, then $\mathfrak{A}$ is strongly observable. (The proof of this also is left to the reader.)

We turn now to an interpretation of input-autonomy and output-autonomy.

The following theorem generalizes the standard fact that a linear system is autonomous if and only if its only compact support trajectory is the zero one. (See [12] and references there.)

Theorem 15. *Suppose $\mathcal{U}$ is faithfully flat. Then $\mathfrak{A}$ is input-autonomous if and only if*

$$\mathcal{B}_{id}(\mathfrak{A}) = 0.$$

Proof. The exact sequence

$$Z_0^* \to Z_1^* \oplus D^q \to ID(\mathfrak{A}) \to 0$$

yields the following two exact sequences

$$0 \to Hom(ID(\mathfrak{A}), D) \to Z_1 \oplus D^q \to Z_0 \quad \text{and}$$
$$0 \to Hom(ID(\mathfrak{A}), \mathcal{U}) \to Z_1 \otimes \mathcal{U} \oplus \mathcal{U}^q \to Z_0 \otimes \mathcal{U}.$$

We can see

$$Ext^0(ID(\mathfrak{A}), D) = 0 \Leftrightarrow 0 \to Z_1 \oplus D^q \to Z_0 \text{ is exact } \Leftrightarrow$$
$$0 \to Z_1 \otimes \mathcal{U} \oplus \mathcal{U}^q \to Z_0 \otimes \mathcal{U} \text{ is exact} \Leftrightarrow \mathcal{B}_{id}(\mathfrak{A}) = 0.$$

The proof is complete. □

The following theorem is an obvious analog of the previous one.

Theorem 16. *Suppose $\mathcal{U}$ is an injective cogenerator. Then $\mathfrak{A}$ is output-autonomous if and only if*

$$\mathcal{B}_{od}(\mathfrak{A}) = 0.$$

Proof. The exact sequence

$$Z_1 \to Z_0 \to OD(\mathfrak{A}) \to 0$$

yields the following two exact sequences

$$0 \to Hom(OD(\mathfrak{A}), D) \to Z_0^* \to Z_1^* \quad \text{and} \quad Z_1 \otimes \mathcal{U} \to Z_0 \otimes \mathcal{U} \to OD(\mathfrak{A}) \otimes \mathcal{U} \to 0.$$

We can see

$$Ext^0(OD(\mathfrak{A}), D) = 0 \Leftrightarrow 0 \to Z_0^* \to Z_1^* \text{ is exact } \Leftrightarrow$$
$$Z_1 \otimes \mathcal{U} \to Z_0 \otimes \mathcal{U} \to 0 \text{ is exact } \Leftrightarrow \mathcal{B}_{od}(\mathfrak{A}) = 0.$$

The proof is complete. □

We close by the following theorem, which is an immediate consequence of Theorem 4.

Theorem 17. *Controllability and observability are dual concepts; so are input-autonomy and output-autonomy.*

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