Linear Algebra and its Applications 433 (2010) 1997–2009

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Smooth/impulsive linear systems: Axiomatic description

Vakhtang Lomadze [a,b,*], Hasan Mahmood [a]

[a] Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
[b] A. Razmadze Mathematical Institute, Tbilisi 0193, Georgia

ARTICLE INFO

Article history:
Received 16 November 2009
Accepted 28 July 2010

Submitted by P. Fuhrmann

AMS classification:
93A05
93B10
93B25

Keywords:
Generalized AR-model
Mikusinski functions
sm/imp linear system
Transfer function
McMillan degree

ABSTRACT

The paper concerns with linear systems having arbitrary singularity at infinity. These systems deserve to study since they are limits of familiar classical linear systems. An axiomatic description of such systems is obtained.

1. Introduction

Linear systems with impulsive trajectories have been studied in numerous papers since the work Verghese [20]. The bibliography gives a list only of a very few papers. (These are mostly the papers of which we are aware.)

In [10] the following definition has been offered as the starting point for the "singular" behavioral theory of linear systems. A generalized AR-model is a pair (A, B), where A is a full row rank polynomial matrix and B is a full row rank proper rational matrix such that $A = DB$ for some nonsingular rational matrix D. (The "D" is uniquely determined, and is called the transition matrix. The column numbers

* Corresponding author at: A. Razmadze Mathematical Institute, Tbilisi 0193, Georgia.
E-mail addresses: vakholoma@yahoo.com (V. Lomadze), hasan.mahmood1@yahoo.com (H. Mahmood).

doi:10.1016/j.laa.2010.08.003

of A and B are equal, and this common number is called the signal number.) We say that (A, B) is nonsingular if B is right invertible (as a proper rational matrix). Nonsingular generalized AR-models can be identified with standard AR-models.

Since the definition above may seem strange at first sight, we shall try to give a motivation for it.

Let s be an indeterminate, and put $t = s^{-1}$. Suppose for simplicity that the ground field is $\mathbb{C}$, and consider the case where the signal number is equal to 1. In this special case one can give three equivalent definitions of a classical AR-model (and its McMillan degree):

Definition 1. An AR-model with signal number 1 is a sequence $(a_0, \ldots, a_n)$ of elements in $\mathbb{C}$, where $a_0 \neq 0$. The number n is called the McMillan degree.

Definition 2. An AR-model with signal number 1 is a pair (a, μ), where a is a nonzero complex number and $\mu : \mathbb{C} \to \mathbb{Z}_+$ is a function with a finite support. The McMillan degree is defined as the number $\sum_{x \in \mathbb{C}} \mu(x) = n$.

Definition 3. An AR-model with signal number 1 is a nonzero polynomial $f \in \mathbb{C}[s]$. The McMillan degree is defined simply as the degree of the polynomial.

The definitions above admit respectively the following evident generalizations.

Definition 1′. A generalized AR-model with signal number 1 is a sequence $(a_0, \ldots, a_n)$ of elements in $\mathbb{C}$, where $a_i \neq 0$ for some i. The number n is called the McMillan degree.

Definition 2′. A generalized AR-model with signal number 1 is a pair (a, μ), where a is a nonzero complex number and $\mu : \mathbb{C} \cup \infty \to \mathbb{Z}_+$ is a function with a finite support. The McMillan degree is defined as the number $\sum_{x \in \mathbb{C} \cup \infty} \mu(x) = n$.

Definition 3′. A generalized AR-model with signal number 1 is a pair (f, g) with nonzero polynomials $f \in \mathbb{C}[s]$ and $g \in \mathbb{C}[t]$ such that $f = s^n g$ for some nonnegative integer n. The "n" is uniquely determined, and is called the McMillan degree.

The previous three definitions are equivalent. Indeed, suppose that $(a_0, \ldots, a_n)$ is as in Definition 1′. Letting a to be the leading coefficient of the polynomial $a_0 s^n + \cdots + a_n$ and defining μ by the formula

$$\mu(x) = \begin{cases} \text{multiplicity of } x \text{ in } a_0 s^n + \cdots + a_n, & \text{if } x \in \mathbb{C}, \\ n - deg(a_0 s^n + \cdots + a_n), & \text{if } x = \infty, \end{cases}$$

we obtain a generalized model in the sense of Definition 2′. Further, setting $f = a_0 s^n + \cdots + a_n$ and $g = a_0 + \cdots + a_n t^n$, we obtain a pair as in Definition 3′. Both of the constructions can be inverted.

It should be noted that the infinite frequency is treated with the same emphasis as finite ones; in other words, no constraint is imposed on the behavior at infinity. (This is best seen via Definition 2′.) Note also that, as in the regular case, two generalized AR-models are isomorphic if they differ by a (nonzero) constant multiple.

Definition 3′ is especially interesting for us, as it can be easily generalized to the case of arbitrary signal number. The generalization is as follows: A generalized AR-model with signal number q is a pair (A, B) consisting of full rank matrices $A \in \mathbb{C}[s]^{\bullet \times q}$ and $B \in \mathbb{C}[t]^{\bullet \times q}$ having the same rank and such that

$$A = diag(s^{n_1}, \ldots, s^{n_p})B,$$

where p is the common rank and $n_1, \ldots, n_p$ are nonnegative integers. Because B is right invertible as a rational matrix, these integers are uniquely determined. Using Willems' terminology (see [21]), we call them the lag indices. The number p is called the output number; the McMillan degree is defined as the sum of the lag indices.

The definition that we have recalled at the beginning is equivalent to the definition above and has the advantage to be more flexible. The equivalence follows immediately from the Wiener–Hopf factorization theorem (see Proposition 2). Generalized AR-models in the sense of the previous paragraph are somewhat special and are analogous to row proper AR-models in the regular theory. We call them reduced. (It is interesting to note that saying that (A, B) is a nonsingular reduced generalized AR-model with lag indices $n_1, \ldots, n_p$ is the same as saying that A is a row proper polynomial matrix with row degrees $n_1, \ldots, n_p$.)

The question that arises naturally is: Why generalized AR-models should be studied?

Consider first the following simple example.

Example 1. Let $\varepsilon w^{(n)} + w = 0$ be an AR-model depending on the parameter ε. The model has McMillan degree n. However, when $\varepsilon \to 0$, it converges to $w = 0$, which has the degree 0. We conclude that $w = 0$ can not be the right limit, since this would be in apparent contradiction with the fundamental idea of continuity. To find the right limit, write the given model as the polynomial $\varepsilon s^n + 1$, which, in turn, can be rewritten in the form $(\varepsilon s^n + 1, \varepsilon + t^n)$. Letting now $\varepsilon \to 0$, we see that our model converges to $(1, t^n)$, which is the right limit. The behavior of this limit model is spanned by $\delta, \ldots, \delta^{(n-1)}$, where δ is the delta-function (see Example 2).

The example indicates the presence of “holes" in the class of Willems' AR-models. (These “holes" were discovered by Hazewinkel [5].) The point of generalized AR-models is that they form a “compactification" of Willems' AR-models. This means that the class of generalized models is a minimal class that contains all standard models and is closed with respect to taking limits. The “compactification" theorem is a consequence of Grothendieck's deep result about quotient schemes (see [7]). The fact can be easily explained in the case of signal number 1. In this special case the set of isomorphic classes of classical AR-models of degree n clearly is $\mathbb{A}^n$ (affine space of dimension n) whereas the set of isomorphic classes of generalized AR-models of degree n (according to Definition 1′) is $\mathbb{P}^n$ (projective space of dimension n).

The generalized AR-models in more complicated terms were introduced in [8]. In another form they were proposed and studied by other authors as well (see “eligible pairs" in Geerts and Schumacher [3,4] and “homogeneous behaviors" in Ravi et al. [15]).

The paper is organized as follows.

In Section 1 we introduce generalized functions. (To describe the behaviors of generalized AR-models, we certainly need generalized functions.) Then we give a purely algebraic definition of a vector bundle. Vector bundles are going to play the role of f.g. torsion free polynomial modules. (“f.g." stands for “finitely generated".)

In Section 2 we relax the requirement that a generalized AR-model be of full row rank. (As is known, in the regular theory AR-models are not required to be full rank polynomial matrices necessarily.) Then we define various important invariants (the output number, the initial condition space, the transfer function, the associated vector bundle). Finally, we define the behavior.

In Section 3 we show that there is a good notion of McMillan degree for every linear subspace of the “universum". Then we extend the main result of [9]; we prove that the behaviors of generalized AR-models are characterized by the property of having finite McMillan degree and two evident invariance properties. The paradox is that derivation of this characterization in the general case is easier than in the regular case.

Throughout, $\mathbb{F}$ will be an arbitrary field and s an indeterminate. We let O be the ring of proper rational functions and t the “distinguished" element s^{-1}. We assume that given is a pair $(\mathcal{U}, \hbar)$, where $\mathcal{U}$ is a torsion-free module over O and $\hbar$ a non-zero element of $\mathcal{U}$ satisfying the following axiom

$$\mathcal{U} = t\mathcal{U} \oplus \mathbb{F}\hbar.$$

This abstract setting allows us to treat simultaneously both the continuous-time and the discrete-time cases:

- $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$; $\mathcal{U}$ is $C^{\infty}(\mathbb{T}, \mathbb{F})$, where $\mathbb{T} \subseteq \mathbb{R}$ is a time interval with initial time; $\hbar$ is the function which is identically 1 in $\mathbb{T}$. Let $\int$ be the integral operator mapping a function to its "normalized" primitive, i.e., the primitive that is zero at the initial time. There is exactly one (continuous) action of O on $\mathcal{U}$ for which $tw = \int w$ (see [9]). This action makes $\mathcal{U}$ a torsion-free module over O. The axiom holds by the Newton–Leibniz formula.
- $\mathbb{F}$ is arbitrary; $\mathcal{U}$ is $C(\mathbb{Z}_+, \mathbb{F})$; $\hbar$ is the function defined by

$$\hbar(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases}$$

Identifying $\mathcal{U}$ with $\mathbb{F}[[t]]$, one makes $\mathcal{U}$ a torsion-free O-module in an obvious way. The axiom holds certainly. Indeed, if $g = b_0 + b_1 t + \cdots$, then $g = tg_1 + b_0\hbar$ with $g_1 = b_1 + b_2 t + \cdots$

Since we are primarily interested in the continuous-time case, in what follows we shall interpret elements of $\mathcal{U}$ as smooth functions.

We shall often make of use the functor Q. For any module M over an integral domain, $Q(M)$ denotes the quotient space of M.

2. Some preliminaries

We shall need generalized functions. Our choice is generalized functions, which were introduced in Yosida [22] (and independently in [10]). These functions constitute a very small part of the field of Mikusinski's operators [11].

We define Mikusinski (or generalized) functions simply as elements of $\mathcal{M} = Q(\mathcal{U})$. Because a nonzero proper rational function is congruent to a power of t modulo invertible ones, every generalized function can be written as a ratio u/t^m, where $u \in \mathcal{U}$ and $m \geqslant 0$. We identify $\mathcal{U}$ with its image in $\mathcal{M}$ under the canonical map $u \mapsto u/1$. This identification allows us to represent every Mikusinski function in the form $s^m u$.

The Mikusinski function $\delta = s\hbar$ will be interpreted as the Dirac delta-function. The functions $g\hbar$ with $g \in O$ will be called exponential functions; the functions $f\delta$ with $f \in \mathbb{F}[s]$ will be called (purely) impulsive functions. The space of all impulsive functions will be denoted by Δ. Functions from $\mathbb{F}(s)\hbar$ will be referred to as exponential-impulsive functions.

To explain how Mikusinski functions work, consider the continuous-time case and take $(a_0, \ldots, a_n)$ to be a generalized AR-model in the sense of Definition 1$'$. If $a_0 \neq 0$, then its trajectories, as is well-known, are defined to be the solutions of the differential equation

$$a_0 w^{(n)} + \cdots + a_n w = 0.$$

This equation is equivalent to the integral equation

$$a_0 w + a_1 \int w + \cdots + a_n \int^n w = \text{a polynomial fuction of degree} \leqslant n - 1.$$

(Here $\int^l$ means the l-fold integral.) This, in turn, can be rewritten as

$$(a_0 + a_1 t + \cdots + a_n t^n) w = c_0 t^{n-1}\hbar + \cdots + c_{n-1}\hbar. \tag{1}$$

Notice that the latter makes sense in the class of Mikusinski functions even when the condition $a_0 \neq 0$ does not hold. Indeed, the polynomial $a_0 + a_1 t + \cdots + a_n t^n$ is not zero. Hence we can divide both sides by it. Doing this, we get

$$w = (a_0 + a_1 t + \cdots + a_n t^n)^{-1} (c_0 t^{n-1} + \cdots + c_{n-1})\hbar.$$

We see, in particular, that the solution space of (1) always has dimension n. It is reasonable therefore to declare these solutions as the trajectories of the model.

Here is a concrete example.

Example 2. Suppose that $a_0 = 0, \ldots, a_{n-1} = 0, a_n = 1$; in other words, suppose we are given the model $(1, t^n)$, the limit model from Example 1. Then (1) becomes

$$t^n w = c_0 t^{n-1}\hbar + \cdots + c_{n-1}\hbar.$$

Multiplying both sides by s^n, we find

$$w = c_0 s\hbar + \cdots + c_{n-1}s^n\hbar;$$

whence $w = c_0\delta + \cdots + c_{n-1}\delta^{(n-1)}$. (For $i \geqslant 1$, $\delta^{(i)} = s^i\delta$, the ith "derivative" of δ.)

From the axiom imposed on $(\mathcal{U}, \hbar)$, one can easily derive (see [9]) that

$$\mathcal{M} = \mathcal{U} \oplus \Delta.$$

We therefore have two canonical projection maps

$$\Pi_+ : \mathcal{M} \to \mathcal{U} \quad \text{and} \quad \Pi_- : \mathcal{M} \to \Delta.$$

There are two canonical operators

$$\sigma : \mathcal{U} \to \mathcal{U} \quad \text{and} \quad \tau : \Delta \to \Delta;$$

they are defined respectively as the compositions

$$\mathcal{U} \xrightarrow{s} \mathcal{M} \xrightarrow{\Pi_+} \mathcal{U} \quad \text{and} \quad \Delta \xrightarrow{t} \mathcal{M} \xrightarrow{\Pi_-} \Delta.$$

(Certainly, in the continuous time case σ is the differentiation operator; in the discrete time case this is the (backward) shift operator.)

Suppose given are $A \in \mathbb{F}[s]^{r\times l}$ and $B \in O^{r\times l}$. They determine the operators

$$\mathcal{U}^l \to \mathcal{M}^r \ (u \mapsto Au) \quad \text{and} \quad \Delta^l \to \mathcal{M}^r \ (v \mapsto Bv).$$

Composing these respectively with the projections Π_+ and Π_-, we get the operators

$$\Pi_+ \circ A : \mathcal{U}^l \to \mathcal{U}^r \quad \text{and} \quad \Pi_- \circ B : \Delta^l \to \Delta^r.$$

We also have

$$A(\sigma) : \mathcal{U}^l \to \mathcal{U}^r \quad \text{and} \quad B(\tau) : \Delta^l \to \Delta^r.$$

Not surprisingly,

$$A(\sigma) = \Pi_+ \circ A \quad \text{and} \quad B(\tau) = \Pi_- \circ B.$$

It follows that

$$A(\sigma)u = 0 \Leftrightarrow Au \in \Delta^r \quad \text{and} \quad B(\tau)v = 0 \Leftrightarrow Bv \in \mathcal{U}^r. \tag{2}$$

for every $u \in \mathcal{U}^l$ and $v \in \Delta^l$.

We turn now to some "algebraic geometry".

By a vector bundle we shall mean a triple (M, N, ϕ), where M and N are free modules of finite rank over $\mathbb{F}[s]$ and O, respectively, and ϕ is an isomorphism of the $\mathbb{F}(s)$-linear space $Q(M)$ onto the $\mathbb{F}(s)$-linear space $Q(N)$. A vector bundle should be thought of as a f.g. torsion free "module" over the whole frequency domain.

The basic example of a vector bundle is $\mathcal{O} = (\mathbb{F}[s], O, id)$. Given a nonsingular rational matrix D, the triple $\mathcal{O}(D) = (\mathbb{F}[s]^r, O^r, D)$, where r is the size of D, is a vector bundle.

If (M, N, ϕ) is a vector bundle, then every pair (M_1, N_1) consisting of submodules $M_1 \subseteq M$ and $N_1 \subseteq N$ such that $\phi(Q(M_1)) = Q(N_1)$ will be called a subbundle.

One defines in an obvious way direct sums of vector bundles. Vector bundles of the form $\mathcal{O}^q$ (and their subbundles) will play the same role as modules of the form $\mathbb{F}[s]^q$ (and their submodules) play in Willems' theory [12, 20].

3. Generalized AR-models and their behaviors

A generalized AR-model is a pair (A, B), where A is a polynomial matrix and B a proper rational matrix that have the same size and satisfy the following equivalent conditions:

(a) A and B have the same kernel as rational matrices;
(b) $\exists$ a nonsingular rational matrix D such that $A = DB$.

One should think of A as a representation at the finite domain and of B as a representation at the infinity. The equivalent conditions express compatibility of these representations. The "D" is no longer uniquely determined. Every nonsingular rational matrix D for which $A = DB$ will be called a transition matrix. The number of columns is called the signal number.

Example 4. Assume A is a polynomial matrix with row number p.

(a) Let n be any integer that is greater than or equal to the maximum of the degrees of the entries of A. Then $(A, t^n A)$ is a generalized AR-model.
(b) Let $n_1, \ldots, n_p$ be integers that are greater than or equal to the row degrees of A. Then $(A, \mathrm{diag}(t^{n_1}, \ldots, t^{n_p})A)$ is a generalized AR-model.

Let (A, B) be a generalized AR-model with signal number q. The dimensions of $\mathbb{F}(s)$-linear subspaces $A\mathbb{F}(s)^q \subseteq \mathbb{F}(s)^p$ and $B\mathbb{F}(s)^q \subseteq \mathbb{F}(s)^p$ are equal. We call this common dimension the rank (or, the output number) and denote by $rk(A, B)$. Further, if D is any transition matrix, then

$$Q(A^{\mathrm{tr}}\mathbb{F}[s]^p) = A^{\mathrm{tr}}\mathbb{F}(s)^p = B^{\mathrm{tr}}D^{\mathrm{tr}}\mathbb{F}(s)^p = B^{\mathrm{tr}}\mathbb{F}(s)^p = Q(B^{\mathrm{tr}}O^p).$$

It follows that $(A^{\mathrm{tr}}\mathbb{F}[s]^p, B^{\mathrm{tr}}O^p)$ is a subbundle of $\mathcal{O}^q$. We call this the associated vector bundle and denote by $Ass(A, B)$.

Given two AR-models (A_1, B_1) and (A_2, B_2) with equal signal numbers, we shall say that (A_1, B_1) is more powerful than (A_2, B_2) (and write $(A_1, B_1) \succeq (A_2, B_2)$) if there exist a polynomial matrix U and a proper rational matrix V such that $A_2 = UA_1$ and $B_2 = VB_1$. Clearly we have

$$(A_1, B_1) \succeq (A_2, B_2) \quad \Leftrightarrow \quad Ass(A_2, B_2) \subseteq Ass(A_1, B_1).$$

Two generalized AR-models are said to be equivalent if each of them is more powerful than the other.

Certainly, the rank of a generalized AR-model is greater than or equal to the rank of the associated vector bundle. In case of equality one says that an AR-model is minimal. Notice that a generalized AR-model (A, B) is minimal if and only if both A and B have full row rank.

As already mentioned, a minimal generalized AR-model has one transition matrix only. Minimal generalized AR-models are of special interest thanks to the following

Proposition 1. *Every generalized AR-model is equivalent to a minimal one.*

Proof. This is clear. $\square$

We remark that if (A_1, B_1) and (A_2, B_2) are two AR-models and if (A_1, B_1) is minimal, then they are equivalent if and only if $A_2 = UA_1$ for some left unimodular polynomial matrix U and $B_2 = VB_1$ for some left biproper rational matrix V. Obviously, two minimal AR-models (A_1, B_1) and (A_2, B_2) are equivalent if and only if $A_2 = UA_1$ with unimodular polynomial matrix U and $B_2 = VB_1$ with biproper rational matrix V.

By a reduced generalized AR-model we shall understand a minimal generalized AR-model whose transition matrix is of the form $diag(s^{n_1}, \ldots, s^{n_p})$, where $n_1, \ldots, n_p$ are nonnegative integers. Reduced generalized AR-models should be viewed as generalizations of row proper polynomial matrices.

Example 5. The following pairs

$$\left(\begin{bmatrix} s^3 & 1 \\ 1 & s \end{bmatrix}, \begin{bmatrix} 1 & t^3 \\ t & 1 \end{bmatrix}\right) \quad \text{and} \quad \left(\begin{bmatrix} s^3 & 1 \\ 1 & s \end{bmatrix}, \begin{bmatrix} 1 & t^3 \\ t^2 & t \end{bmatrix}\right)$$

are reduced generalized AR-models. Their transition matrices are

$$\left(\begin{bmatrix} s^3 & 0 \\ 0 & s \end{bmatrix} \text{ and } \begin{bmatrix} s^3 & 0 \\ 0 & s^2 \end{bmatrix}\right)$$

respectively. The first model is nonsingular, the second not. The first one can be identified with

$$\begin{bmatrix} s^3 & 1 \\ 1 & s \end{bmatrix},$$

which is row proper.

Proposition 2. *Every generalized AR-model can be brought into reduced form.*

Proof. Let (A, B) be a generalized AR-model. In view of the previous proposition, we may assume that it is minimal. Let D be the transition matrix, and let

$$D = U\, diag(s^{n_1}, \ldots, s^{n_p})V$$

be a (left) Wiener–Hopf factorization of D. (Here U is a unimodular polynomial matrix and V is a biproper rational matrix.) Then $(U^{-1}A, VB)$ is a reduced generalized AR-model, which is equivalent to the given one. $\square$

By a transfer function with signal number q, we shall mean any $\mathbb{F}(s)$-linear subspace of $\mathbb{F}(s)^q$. Every transfer function has a representation $G\mathbb{F}(s)^m$, where m is the dimension and G is a full column rank rational matrix of size $q \times m$. Given a transfer function $T \subseteq \mathbb{F}(s)^q$, we let $T\mathcal{M}$ denote the set of all finite sums of the form Σfw, where $f \in T$ and $w \in \mathcal{M}$. Obviously, if $T = G\mathbb{F}(s)^m$, then $T\mathcal{M} = G\mathcal{M}^m$. Clearly, $T\mathcal{M}$ is a $\mathbb{F}(s)$-linear subspace of $\mathcal{M}^q$.

Assume we are given a generalized AR-model (A, B) of size $p \times q$.

By definition, the $\mathbb{F}(s)$-linear maps

$$\mathbb{F}(s)^q \xrightarrow{A} \mathbb{F}(s)^p \quad \text{and} \quad \mathbb{F}(s)^q \xrightarrow{B} \mathbb{F}(s)^p$$

have the same kernel. Let T denote this common kernel. This certainly is a transfer function, and call it the transfer function of (A, B).

Let D be any transition matrix of our model. We define the initial condition space X by the formula

$$X = A\mathbb{F}(s)^q \cap \mathbb{F}[s]^p \cap tDO^p.$$

Lemma 1. *The space X is well-defined and finite-dimensional.*

Proof. We have to prove that X does not depend on the choice of D. We have canonical isomorphisms

$$\mathbb{F}(s)^q/T \simeq A\mathbb{F}(s)^q \quad \text{and} \quad \mathbb{F}(s)^q/T \simeq B\mathbb{F}(s)^q.$$

It follows that there is a unique isomorphism $\phi : B\mathbb{F}(s)^q \simeq A\mathbb{F}(s)^q$ making the diagram

$$\begin{array}{ccc} \mathbb{F}(s)^q & \rightarrow & B\mathbb{F}(s)^q \\ || & & \downarrow \phi \\ \mathbb{F}(s)^q & \rightarrow & A\mathbb{F}(s)^q \end{array}$$

commutative. On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{F}(s)^q & \rightarrow & \mathbb{F}(s)^p \\ || & & \downarrow D. \\ \mathbb{F}(s)^q & \rightarrow & \mathbb{F}(s)^p \end{array}$$

It follows that the restriction of $D : \mathbb{F}(s)^p \to \mathbb{F}(s)^p$ on $B\mathbb{F}(s)^q$ coincides with ϕ. Therefore,

$$A\mathbb{F}(s)^q \cap \mathbb{F}[s]^p \cap tDO^p = \mathbb{F}[s]^p \cap D(B\mathbb{F}(s)^q \cap tO^p) = \mathbb{F}[s]^p \cap \phi(B\mathbb{F}(s)^q \cap tO^p).$$

Further, choose k so large that the entries in t^kD be proper. Then $DO^p \subseteq s^kO^p$, and therefore

$$X \subseteq \mathbb{F}[s]^p \cap tDO^p \subseteq \mathbb{F}[s]^p \cap s^{k-1}O^p.$$

This implies that X has finite-dimension.

The proof is complete. □

We define the behavior of (A, B) to be the set

$$Bh(A, B) = Ker\, A(\sigma) \oplus Ker\, B(\tau).$$

Thus, by definition, the behavior consists of two parts; one part is regular and the other is impulsive.

Associated with the model there is an "operational" equation

$$Aw = x\delta \quad (x \in X). \tag{3}$$

(The unknown w is a Mikusinski function.)

Example 6. Let $a_0, \ldots, a_n$ be as in Definition 1′. We can take the "D" to be s^n. Because $a_0s^n + \cdots + a_n \neq 0$, the initial condition space is equal to

$$\mathbb{F}(s) \cap \mathbb{F}[s] \cap ts^nO = \{c_0s^{n-1} + \cdots + c_{n-1} \mid c_i \in \mathbb{F}\}.$$

Therefore, the operational equation has the form

$$(a_0s^n + \cdots + a_n)w = (c_0s^{n-1} + \cdots + c_{n-1})\delta.$$

Multiplying this by t^n, we get the equation

$$(a_0 + \cdots + a_nt^n)w = (c_0 + \cdots + c_{n-1}t^{n-1})\hbar,$$

which is the same as (1).

We need the following

Lemma 2. *If R is a rational matrix of size $p \times q$, then*

$$R\mathcal{M}^q \cap \Delta^p = R\mathbb{F}(s)^q\hbar \cap \Delta^p.$$

Proof. Let r be the rank of R, and choose a full column rank rational matrix P of size $p \times r$ so that $P\mathbb{F}(s)^r = R\mathbb{F}(s)^q$.

We claim that

$$P\mathcal{M}^r \cap \mathbb{F}(s)^p\hbar = P\mathbb{F}(s)^r\hbar.$$

Indeed, suppose $w \in \mathcal{M}^r$ is such that $Pw \in \mathbb{F}(s)^p\hbar$. We need to show that $w \in \mathbb{F}(s)^r\hbar$. For this, choose any left inverse matrix Q of P. We then have $w = Q(Pw) \in \mathbb{F}(s)^r\hbar$, as desired.

The claim is proved, and we have

$$R\mathcal{M}^q \cap \Delta^p = P\mathcal{M}^r \cap \Delta^p = P\mathcal{M}^r \cap \mathbb{F}(s)^p\hbar \cap \Delta^p = P\mathbb{F}(s)^r\hbar \cap \Delta^p = R\mathbb{F}(s)^q\hbar \cap \Delta^p.$$

The proof is complete. □

Proposition 3. *The behavior of (A, B) can be defined by Eq. (3).*

Proof. Let $w = u + v$ with $u \in \mathcal{U}^q$ and $v \in \Delta^q$. We have to show that

$$A(\sigma)u = 0 \quad \text{and} \quad B(\tau)v = 0 \quad \Leftrightarrow \quad Aw \in X\delta.$$

We clearly have

$$Aw \in X\delta \;\Leftrightarrow\; Aw \in A\mathbb{F}(s)^q\hbar \cap \Delta^p \quad \text{and} \quad Aw \in D\mathcal{U}^p$$

Because $Av \in A\mathbb{F}(s)^q\hbar \cap \Delta^p$ and because $A\mathbb{F}(s)^q\hbar \cap \Delta^p = A\mathcal{M}^q \cap \Delta^p$ (by the previous lemma),

$$Aw \in A\mathbb{F}(s)^q\hbar \cap \Delta^p \;\Leftrightarrow\; Au \in A\mathbb{F}(s)^q\hbar \cap \Delta^p \;\Leftrightarrow\; Au \in A\mathcal{M}^q \cap \Delta^p \;\Leftrightarrow\; Au \in \Delta^p.$$

Next, because $Au \in D\mathcal{U}^p$,

$$Aw \in D\mathcal{U}^p \Leftrightarrow Av \in D\mathcal{U}^p \Leftrightarrow Bv \in \mathcal{U}^p.$$

Using (2), we complete the proof. □

Proposition 4. *There is a canonical exact sequence*

$$0 \to T\mathcal{M} \to Bh(A,B) \to X \to 0.$$

Proof. Consider the homomorphism $\mathcal{M}^q \xrightarrow{A} \mathcal{M}^p$. By the previous proposition, this induces a surjective linear map $Bh(A,B) \to X\delta$.

We have an exact sequence

$$0 \to T \to \mathbb{F}(s)^q \xrightarrow{A} \mathbb{F}(s)^p.$$

Tensoring this by $\mathcal{M}$, we get the exact sequence

$$0 \to T \otimes \mathcal{M} \to \mathcal{M}^q \xrightarrow{A} \mathcal{M}^p.$$

The image of $T \otimes \mathcal{M} \to \mathcal{M}^q$ coincides with $T\mathcal{M}$. It follows that the kernel of $\mathcal{M}^q \xrightarrow{A} \mathcal{M}^p$ is equal to $T\mathcal{M}$. It remains to see that the above kernel is contained in $Bh(A,B)$ and that $X\delta \simeq X$.

The proof is complete. □

Closing the section, we remark that the transfer function of (A,B) can be defined in terms of its behavior; namely, we have

$$T = \{f \in \mathbb{F}(s)^q \mid \forall w \in \mathcal{M}, \quad fw \in Bh(A,B)\}.$$

4. Smooth/impulsive linear systems

A natural question to ask is: What is special about the behaviors of AR-models? For the regular case this question was posed in [21], and the works [9,17] have been devoted to it. We aim to generalize the main result in [9].

Fix a positive integer q. In the present section we define smooth/impulsive linear systems with signal number q as $\mathbb{F}$-linear subspaces of $\mathcal{M}^q$ satisfying certain axioms. We then show that these are exactly those ones that are representable as the behaviors of generalized AR-models with signal number q.

Let $\mathcal{B}$ be an $\mathbb{F}$-linear subspace of $\mathcal{M}^q$. The remark at the end of the previous section suggests to define the transfer function of $\mathcal{B}$ to be

$$T = \{f \in \mathbb{F}(s)^q \mid \forall w \in \mathcal{M}, \quad fw \in \mathcal{B}\}.$$

By definition, $T\mathcal{M} \subseteq \mathcal{B}$. Intuitively, $T\mathcal{M}$ is the set of zero initial condition trajectories in $\mathcal{B}$. The $\mathbb{F}$-linear space $\mathcal{B}/T\mathcal{M}$ should be viewed as the initial condition space of $\mathcal{B}$. We define the McMillan degree (or the relative dimension) of $\mathcal{B}$ as the dimension of the initial condition space.

By a smooth/impulsive (sm/imp) linear system, we shall understand an $\mathbb{F}$-linear subspace $\mathcal{B} \subseteq \mathcal{M}^q$ satisfying the following axioms:

(LS1) $\mathcal{B}$ has finite McMillan degree;
(LS2) $w \in \mathcal{B} \Rightarrow \Pi_+(w) \in \mathcal{B}$ and $\Pi_-(w) \in \mathcal{B}$;
(LS3) $u \in \mathcal{B} \cap \mathcal{U}^q \Rightarrow \sigma(u) \in \mathcal{B} \cap \mathcal{U}^q$ and $v \in \mathcal{B} \cap \Delta^q \Rightarrow \tau(v) \in \mathcal{B} \cap \Delta^q$.

In other words, a sm/imp linear system is an $\mathbb{F}$-linear subspace $\mathcal{B} \subseteq \mathcal{M}^q$ having finite relative dimension and such that

$$\mathcal{B} = \mathcal{B}_+ + \mathcal{B}_-,$$

where $\mathcal{B}_+$ and $\mathcal{B}_-$ are σ- and τ-invariant subspaces in $\mathcal{U}^q$ and Δ^q, respectively.

It immediately follows from the results of the previous section that the behavior of a generalized AR-model is a sm/imp linear system.

Proposition 5. *Let $\mathcal{B}$ be a sm/imp linear system with transfer function T. Then*

$$\mathcal{B} \subseteq T\mathcal{M} + \mathbb{F}(s)^q\hbar.$$

Proof. Take an arbitrary trajectory $w \in \mathcal{B}$, and let $w = u + v$ with $u \in \mathcal{U}^q$ and $v \in \Delta^q$. By the axioms (LS2) and (LS3),

$$u, \sigma u, \sigma^2 u, \ldots \in \mathcal{B} \quad \text{and} \quad v, \tau v, \tau^2 v, \ldots \in \mathcal{B}.$$

By the axiom (LS1), there exist nonzero finite sequences of constants $(a_0, \ldots, a_l)$ and $(b_0, \ldots, b_m)$ such that

$$a_0 u + \cdots + a_l \sigma^l u \in T\mathcal{M} \quad \text{and} \quad b_0 v + \cdots + b_m \tau^m v \in T\mathcal{M}.$$

Putting $a = a_0 + \cdots + a_l s^l$ and $b = b_0 + \cdots + b_m t^m$, we have

$$a_0 u + \cdots + a_l \sigma^l u = au + f\delta \quad \text{and} \quad b_0 v + \cdots + b_m \tau^m v = bv + g\hbar$$

for some $f \in \mathbb{F}[s]^q$ and $g \in \mathbb{F}[t]^q$. It follows that

$$u = a^{-1}(a_0 u + \cdots + a_l \sigma^l u) - a^{-1} f\delta \quad \text{and} \quad v = b^{-1}(b_0 v + \cdots + b_m \tau^m v) - b^{-1} g\hbar.$$

We see that both u and v belong to $T\mathcal{M} + \mathbb{F}(s)^q\hbar$.

The proof is complete. □

Remark. The proposition says that there are sufficiently many exponential-impulsive trajectories in $\mathcal{B}$. More precisely, $\mathcal{B}$ always has an exponential-impulsive trajectory with any given initial condition.

There is an obvious $\mathbb{F}(s)$-bilinear form

$$\mathbb{F}(s)^q \times \mathcal{M}^q \to \mathcal{M}$$

taking (f, w) to $f^{\mathrm{tr}} w$. ("tr" stands for the transpose.) The decomposition

$$\mathcal{M} = t\mathcal{U} \oplus \mathbb{F}\hbar \oplus \Delta$$

determines a canonical $\mathbb{F}$-linear map $\mathcal{M} \to \mathbb{F}$. Composing the above form with this map, we obtain a canonical $\mathbb{F}$-bilinear form

$$\langle -, - \rangle : \mathbb{F}(s)^q \times \mathcal{M}^q \to \mathbb{F}, \tag{4}$$

which will be very helpful in the sequel. (It allows to reduce "analysis" to "algebra".)

We remark that if $f \in \mathbb{F}[s]^q$, $g \in O^q$, $u \in \mathcal{U}^q$ and $v \in \Delta^q$, then

$$\langle f, v \rangle = 0 \quad \text{and} \quad \langle tg, u \rangle = 0; \tag{5}$$

$$\langle sf, u \rangle = \langle f, \sigma(u) \rangle \quad \text{and} \quad \langle t^2 g, v \rangle = \langle tg, \tau(v) \rangle. \tag{6}$$

Lemma 3. *The $\mathbb{F}$-bilinear forms*

$$\mathbb{F}[s]^q \times (O\hbar)^q \to \mathbb{F} \quad \text{and} \quad tO^q \times \Delta^q \to \mathbb{F},$$

induced by (4), are nondegenerate.

Proof. Left to the reader. □

For every transfer function T, we let T° denote the set

$$\{f \in \mathbb{F}(s)^q | f^{tr} g = 0 \quad \forall g \in T\},$$

(T° is the orthogonal of T with respect to the standard $\mathbb{F}(s)$-bilinear form $\mathbb{F}(s)^q \times \mathbb{F}(s)^q \to \mathbb{F}(s)$.) This again is a transfer function, of course.

Lemma 4. *Let T be a transfer function. Then*

$$(T\hbar)^\perp = T^\circ \quad \text{and} \quad (T^\circ)^\perp \cap \mathbb{F}(s)^q \hbar = T\hbar.$$

Proof. Left to the reader. □

Lemma 5. *Let T be a transfer function. Then*

$$(T\mathcal{M})^\perp = T^\circ.$$

Proof. The inclusion "$\supseteq$" is obvious: If $f \in T^\circ$, then $f^{tr} g w = 0$ for all $g \in T$, $w \in \mathcal{M}$.

The inclusion "$\subseteq$" follows from

$$(T\mathcal{M})^\perp \subseteq (T\hbar)^\perp = T^\circ.$$

The proof is complete. □

By the previous lemma, we have $\mathcal{B}^\perp \subseteq (T\mathcal{M})^\perp = T^\circ$.

Lemma 6. *Let $\mathcal{B}$ be a sm/imp linear system, and let T be its transfer function. Then the bilinear form*

$$T^\circ/\mathcal{B}^\perp \times \mathcal{B}/T\mathcal{M} \to \mathbb{F},$$

induced by (4), *is nondegenerate.*

Proof. It is obvious that the form is nondegenerate from the left.

Take any $w \in \mathcal{B}$ such that $\langle g, w\rangle = 0$ for every $g \in T^\circ$. Since modulo $T\mathcal{M}$ the space $\mathcal{B}$ is generated by exponential-impulsive trajectories, we may assume that w is exponential-impulsive. By Lemma 4, then $w \in T\hbar$; hence, the class of w in $\mathcal{B}/T\mathcal{M}$ is zero.

The proof is complete. □

Theorem 1 (Duality Theorem)**.** *Let $\mathcal{B}$ be a sm/imp linear system, and let T be its transfer function. There is a unique subbundle (M, N) of $\mathcal{O}^q$ such that*

$$\mathcal{B}^\perp = M + tN.$$

Moreover, both $Q(M)$ and $Q(N)$ are equal to T°.

Proof. Set

$$M = \{f \in \mathbb{F}[s]^q \mid \langle f, u\rangle = 0 \quad \forall u \in \mathcal{B} \cap \mathcal{U}^q\} \quad \text{and} \quad N = \{g \in O^q \mid \langle tg, v\rangle = 0 \quad \forall v \in \mathcal{B} \cap \Delta^q\}.$$

It is easily seen that M is an $\mathbb{F}[s]$-submodule of $\mathbb{F}[s]^q$ and N is an O-submodule of O^q. (This follows from the relationships (6) and the invariance properties of $\mathcal{B} \cap \mathcal{U}^q$ and $\mathcal{B} \cap \Delta^q$.)

Using (5) and the invariance of $\mathcal{B}$ with respect to Π_+ and Π_-, one can show easily that $\mathcal{B}^\perp = M + tN$.

Further, by Lemma 6, the space

$$T^\circ/(M + tN)$$

has finite dimension over $\mathbb{F}$. Applying Lemma 3 in [9], we conclude that $Q(M) = T^\circ = Q(N)$. Hence, the pair (M, N) is a subbundle of $\mathcal{O}^q$.

The statement about uniqueness is trivial due to the fact that $\mathbb{F}[s]^q \cap tO^q = \{0\}$.

The proof is complete. □

The subbundle, existence of which is asserted in the previous theorem, will be called the annihilator; we shall denote it by $Ann(\mathcal{B})$.

Proposition 6. *If (A, B) is a generalized AR-model, then*

$$Ann(Bh(A, B)) = Ass(A, B).$$

Proof. The proof is very similar to that of Proposition 3 in [10], and is left to the reader. □

The following result was obtained in [10]. In [3,13,14] the reader can found its different versions. For the regular version, the reader is referred to [1,2,6,12,16,21].

Corollary 1 (Equivalence Theorem)**.** *Two generalized AR-models have the same behavior if and only if they are equivalent.*

Proof. The “if" part is easy. The “only if" part follows from the duality theorem and the previous proposition (and the fact that two generalized AR-models are equivalent if and only if their associated vector bundles coincide).

The proof is complete. □

We are ready now to prove our main result.

Theorem 2 (Representation Theorem)**.** *Every sm/imp linear system has an AR-representation.*

Proof. Assume $\mathcal{B}$ is a linear system with transfer function T, and let (M, N) be its annihilator. Take a full rank polynomial matrix A and a full rank proper rational matrix B such that

$$A^{\mathrm{tr}}\mathbb{F}[s]^p = M \quad \text{and} \quad B^{\mathrm{tr}}O^p = N.$$

We claim that (A, B) is an AR-model. Indeed, we have

$$A^{\mathrm{tr}}\mathbb{F}(s)^p = Q(M) = T^\circ \quad \text{and} \quad B^{\mathrm{tr}}\mathbb{F}(s)^p = Q(N) = T^\circ.$$

Next, note that if R is a rational matrix of size $p \times q$, then

$$\{f \in \mathbb{F}(s)^q \mid Rf = 0\} = (R^{\mathrm{tr}}\mathbb{F}(s)^p)^\circ.$$

In view of this, we have

$$\{f \in \mathbb{F}(s)^q \mid Af = 0\} = (A^{\mathrm{tr}}\mathbb{F}(s)^p)^\circ = T = (B^{\mathrm{tr}}\mathbb{F}(s)^p)^\circ = \{f \in \mathbb{F}(s)^q \mid Bf = 0\},$$

which proves the claim.

We are going to show that $\mathcal{B} = Bh(A, B)$.

To show that $\mathcal{B} \subseteq Bh(A, B)$, it suffices to show that $\mathcal{B} \cap \mathbb{F}(s)^q\hbar \subseteq Bh(A, B)$. (This is because $T\mathcal{M} \subseteq Bh(A, B)$ and $\mathcal{B} \subseteq T\mathcal{M} + \mathbb{F}(s)^q\hbar$.)

Take any exponential-impulsive trajectory $w \in \mathcal{B}$, and write $w = u + v$ with exponential u and impulsive v. We have

$$\forall f \in \mathbb{F}[s]^p, \quad \langle A^{\mathrm{tr}}f, u\rangle = 0 \quad \text{and} \quad \forall g \in O^p, \quad \langle tB^{\mathrm{tr}}g, v\rangle = 0.$$

Because

$$\langle A^{\mathrm{tr}}f, u\rangle = \langle f, A(\sigma)u\rangle \quad \text{and} \quad \langle tB^{\mathrm{tr}}g, v\rangle = \langle tg, B(\tau)v\rangle,$$

we have

$$\forall f \in \mathbb{F}[s]^p, \quad \langle f, A(\sigma)u\rangle = 0 \quad \text{and} \quad \forall g \in O^p, \quad \langle tg, B(\tau)v\rangle = 0.$$

In view of Lemma 3, this implies

$$A(\sigma)u = 0 \quad \text{and} \quad B(\tau)v = 0.$$

We conclude that $\mathcal{B} \subseteq Bh(A, B)$.

To complete the proof consider the tower

$$T\mathcal{M} \subseteq \mathcal{B} \subseteq Bh(A, B).$$

By the previous proposition, $Bh(A, B)^{\perp} = M + tN$. Applying Lemma 6 both to $\mathcal{B}$ and $Bh(A, B)$, we get

$$\dim(\mathcal{B}/T\mathcal{M}) = \dim(T^{\circ}/(M + tN)) = \dim(Bh(A, B)/T\mathcal{M}).$$

It is immediate from this that $\mathcal{B} = Bh(A, B)$.

The proof is complete. $\square$

Every subbundle (M, N) of $\mathcal{O}^q$ can be written in the form $(M, N) = Ass(A, B)$, where (A, B) is a generalized AR-model. The latter is uniquely determined up to equivalence. So, we can define $Bh(M, N)$ by the formula

$$Bh(M, N) = Bh(A, B).$$

From the proof of the theorem, we have the following

Corollary 2. *If $\mathcal{B}$ is a linear system, then*

$$Bh(Ann(\mathcal{B})) = \mathcal{B}.$$

References

[1] P.A. Fuhrmann, On behavior homomorphisms and system equivalence,System Contr. Lett. 44 (2001) 127–134.
[2] P.A. Fuhrmann, A study of behaviors, Linear Algebra Appl. 351/352 (2002) 303–380.
[3] A.H.W. Geerts, J.M. Schumacher, Impulsive-smooth behavior in multimode systems. Part I: State-space and polynomial representations, Automatica J. IFAC 32 (1996) 747–758.
[4] A.H.W. Geerts, J.M. Schumacher, Impulsive-smooth behavior in multimode systems. Part II: Minimality and equivalence, Automatica J. IFAC 32 (1996) 819–832.
[5] M. Hazewinkel, On families of linear systems: degenerating phenomena, in: C.I. Byrnes, C.F. Martin (Eds.), Algebraic and Geometric Methods in Linear Systems Theory, Lectures in Appl. Math., vol. 18, AMS, Providence, RI, 1980, pp. 157–189.
[6] D. Hinrichsen, D. Prätzel-Wolters, Solution modules and system equivalence, Int. J. of Control 32 (1980) 777–802.
[7] A. Grothendieck, Les schémas de Hilbert, Seminaire Bourbaki, 13 (1960/1961) Exp. 221.
[8] V. Lomadze, Finite-dimensional time-invariant linear dynamical systems: algebraic theory, Acta Appl. Math. 19 (1990) 149–201.
[9] V. Lomadze, When are linear differentiation-invariant spaces differential? Linear Algebra Appl. 424 (2007) 540–554.
[10] V. Lomadze, (Generalised) autoregressive models and their trajectories, Int. J. Control 82 (2009) 1929–1936.
[11] J. Mikusinski, Operational Calculus, Pergamon Press, London, 1959.
[12] J.W. Polderman, J.C. Willems, Introduction to Mathematical Systems Theory, Springer, New York, 1998.
[13] A.C. Pugh, E.N. Antoniou, N.P. Karampetakis, G.E. Hayton, Fundamental equivalence of non-regular AR-representation. Mathematical Science Report, Number A315, Department of Mathematical Sciences, Loughborough University of Technology, UK, 1998.
[14] A.C. Pugh, E.N. Antoniou, N.P. Karampetakis, Equivalence of AR-representations in the light of the impulsive-smooth behavior, Int. J. Robust Nonlinear Control 17 (2006) 769–785.
[15] M.S. Ravi, J. Rosenthal, J.M. Schumacher, Homogeneous behaviors, Math. Control Signals Systems 10 (1997) 61–75.
[16] J.M. Schumacher, Transformations of linear systems under external equivalence, Linear Algebra Appl. 102 (1988) 1–34.
[17] H. Soethoudt, Introduction to a behavioral approach for continuous-time systems, Ph.D. thesis, Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, 1993.
[18] A.I.G. Vardulakis, Linear Multivariable Control. Algebraic Analysis and Synthesis Methods, Wiley, Chichester, 1991.
[19] A.I.G. Vardulakis, E.N. Antoniou, N. Karampetakis, On the solution and impulsive behavior of polynomial matrix descriptions of free linear multivariable systems, Int. J. Control 22 (1999) 215–228.
[20] G.C. Verghese, Infinite-frequency behavior in generalized dynamical systems, Ph.D. dissertation, Stanford University, Stanford 1979.
[21] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Automat. Control 36 (1991) 259–294.
[22] K. Yosida, Operational Calculus: A Theory of Hyperfunctions, Springer, New-York, 1984.