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Relative completeness and specifiedness properties of continuous linear dynamical systems

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ABSTRACT

The properties of relative completeness and specifiedness are introduced for (continuous) linear dynamical systems. It is shown then that having these two properties and the differentiation-invariance property is necessary and sufficient for a (continuous) linear dynamical system to be represented by means of a linear constant coefficient differential equation.

0. Introduction

In [1,2] Willems introduced for dynamical systems an important new concept of completeness. A dynamical system $\mathcal{B}$, defined on a time axis $\mathbb{T}$, is said to be complete if

$$w \in \mathcal{B} \Leftrightarrow w|_{[t_1,t_2]} \in \mathcal{B}|_{[t_1,t_2]}, \quad \forall t_1, t_2 \in \mathbb{T},\ t_1 \leq t_2.$$

In the discrete-time case this definition leads to an interesting result: A discrete dynamical system is represented as the solution set of a linear constant coefficient difference equation if and only if it is linear, shift-invariant and complete. (See Theorem 5 in [1] and Theorem III.1 in [2].)

In order to extend this result to the continuous-time case, Willems introduced also the notion of local specifiedness. In the discrete-time case one has "complete $\Leftrightarrow$ closed (in the pointwise topology)". And it was conjectured by Willems that the continuous-time analog of "linear, shift-invariant and closed" is "linear, time-invariant, closed and locally specified" (see [2]). However, it turned out that the conjecture failed to be true.

In this paper we suggest a different approach to the problem that is based on different interpretations of "completeness" and "specifiedness".

Let k be the field of real or complex numbers. We shall assume, without loss of generality, that $\mathbb{T}$ contains 0 (which will be regarded as an initial time). By a flat function we shall mean any function $f \in C^\infty(\mathbb{T}, k)$ such that $f^{(n)}(0) = 0$ for all $n \geq 0$. A function $a \in C^\infty(\mathbb{T}, k)$ will be said to be analytic if the series $\sum (a^{(n)}(0)/n!)x^n$ converges uniformly to a on every compact subset of $\mathbb{T}$. (Flat functions are "bad", analytic functions are "good"!)

Let $\mathcal{B} \subseteq C^\infty(\mathbb{T}, k^q)$ be a dynamical system. Let t be an indeterminate, and for each $n \geq 0$, define $\mathcal{B}|_n$ to be the image of $\mathcal{B}$ under the mapping sending every $w \in C^\infty(\mathbb{T}, k^q)$ to its Taylor polynomial of degree n, i.e., the mapping

$$w \mapsto w(0) + w'(0)t + \cdots + w^{(n)}(0)t^n, \quad w \in C^\infty(\mathbb{T}, k^q).$$

We call $\mathcal{B}|_n$ the nth truncation of $\mathcal{B}$. We suggest to consider the spaces $\mathcal{B}|_n$ instead of the spaces $\mathcal{B}|_{[t_1,t_2]}$. The spaces $\mathcal{B}|_n$ seem to be quite interesting invariants, and it should be natural to try to derive information about $\mathcal{B}$ looking at them. A priori is clear however that $(\mathcal{B}|_n)_{n\geq 0}$ is not sufficient to recover $\mathcal{B}$. This is because they contain no information about the flat trajectories of $\mathcal{B}$. Therefore, one can only expect recovering of $\mathcal{B}$ modulo flat trajectories. When this happens, we say that $\mathcal{B}$ is relatively complete.

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We do need an extra property of $\mathcal{S}$ in order to overcome the insufficiency of $(\mathcal{S}|_n)_{n\geq 0}$. Recall that

$(*)$ Once the transfer matrix of a classical linear system is known the set of all its zero initial condition trajectories is known and modulo this set every trajectory of the system is congruent to an analytic (even exponential) one.

One can easily axiomatize $(*)$. Given a linear dynamical system $\mathcal{S}$, say that $w \in \mathcal{S}$ has zero initial condition, if all its integrals belong to $\mathcal{S}$. Further, define a transfer matrix as any left invertible proper rational matrix. (Example of such a matrix is $\begin{bmatrix} I \\ G \end{bmatrix}$, where G is a proper rational matrix.)

Based on $(*)$ we give the following formulation of what we call specifiedness. A linear system is specified if its zero initial condition trajectory set is "generated" by a transfer matrix and if its every trajectory is congruent with an analytic one, modulo the zero initial condition trajectory set. (The meaning of "generated" will be explained, of course.)

The goal of this paper is to demonstrate that the continuous-time analog of "linear, shift-invariant and complete" can be "linear, differentiation-invariant, relatively complete and specified".

The paper is organized into seven sections, of which the first two are preliminary. There are four appendices, where overviews of some basic topics are presented.

Throughout, k and $\mathbb{T}$ are as above, s is an indeterminate and q is a positive integer.

We let $\mathcal{U}$ denote the space $C^\infty(\mathbb{T}, k)$. The symbol ∂ will stand for the differentiation operator and the symbol $\int$ for the integration operator. (For $w \in \mathcal{U}$, $\int w$ is the primitive of w that is 0 at the initial time.)

The space $\mathcal{U}^q$ will be our "universum". (See Willems [2] about this concept.) Following Polderman and Willems [3], by a (continuous) linear dynamical system (with signal number q) we understand a k-linear subspace of $\mathcal{U}^q$. A basic example is a linear (time-invariant) differential system whose trajectories are the solutions of a differential equation of the form

$$R(\partial)w = 0,$$

where $R \in k[s]^{\bullet\times q}$.

We let O be the ring of proper rational functions (in s) and t the "uniformizer" $s^{-1} \in$ O. The ring O is very nice; this is a principal ideal domain with a unique nonzero prime ideal. (In commutative algebra such a ring is called a discrete valuation ring (see [4]).) Every $f \in k(s) \setminus \{0\}$ has the form

$$g = t^n e,$$

where $n \in \mathbb{Z}$ and e is a unit of O. The representation is unique; the number n is the order of f at infinity and is denoted by $\mathrm{ord}_\infty(f)$.

As is known, every proper rational function can be viewed as a formal power series in t. Recall (see [5]) that a formal power series $\sum_{n\geq 0} a_n t^n$ is called convergent if there is a positive number r such that $\sum_{n\geq 0} |a_n| r^n$ converges. We denote by $k\{\{t\}\}$ the ring of convergent formal power series. It is an elementary fact that every formal power series with nonzero free coefficient is invertible. Theorem on units in [5, Chapter 4, Section 4] states that if a formal power series with nonzero free coefficient is convergent, then so is its inverse. Making use of this theorem, one can easily see that O $\subseteq k\{\{t\}\}$. (Indeed, let g be a proper rational function, and let

$$g = \frac{a_0 s^n + \cdots + a_n}{b_0 s^n + \cdots + b_n},$$

with $b_0 \neq 0$. Rewrite this as

$$g = \frac{a_0 + \cdots + a_n t^n}{b_0 + \cdots + b_n t^n}.$$

Certainly, $b_0 + \cdots + b_n t^n \in k\{\{t\}\}$, and therefore $(b_0 + \cdots + b_n t^n)^{-1} \in k\{\{t\}\}$).

There is exactly one (continuous) action of O on $\mathcal{U}$ for which

$$tw = \int w.$$

It is defined as follows. If $g = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n + \cdots \in$ O and $w \in \mathcal{U}$, then

$$gw = b_0 w + b_1 \int w + b_2 \int^2 w + \cdots + b_n \int^n w + \cdots.$$

(The reader can easily prove that the series converges uniformly on every compact neighborhood of 0 (see also [6]).) This action makes $\mathcal{U}$ a module over O. From the fact that $\int$ is an injective operator and the fact that any $g \in$ O has the form $g = t^n e$ with $n \geq 0$ and a unit e, it immediately follows that this module is without torsion.

We let $\hbar$ denote the function that is identically 1 in $\mathbb{T}$, and we let L be the unique homomorphism O $\to \mathcal{U}$ for which

$$L(1) = \hbar.$$

The homomorphism is injective; it should be interpreted as the (inverse) Laplace transform. Notice that the functions $L(g)$ with $g \in O$ are exponential functions.

If R is a full row rank polynomial matrix of size $p \times q$, then clearly RO^q is a torsion free O-submodule in $k(s)^q$ of rank p. Hence, there exists a nonsingular rational matrix D such that $DO^p = RO^q$. (Notice that saying that $DO^p = RO^q$ is equivalent to saying that $D^{-1}R$ is a right invertible proper rational matrix.) The matrix D is uniquely determined up to right multiplication by a biproper rational matrix. Every such a matrix will be called a denominator of R.

For any $R \in k[s]^{\bullet\times q}$, $Bh(R)$ will denote the behavior of R, i.e., the kernel of the operator $R(\partial)$.

We shall denote by $\mathcal{F}$ the space of flat functions, and by $\mathcal{A}$ the space of analytic functions. For every $n \geq 0$, we shall write Ω_n and Γ_n to denote the spaces of polynomials in s and t, respectively, of degree less than or equal to n.

1. Operational calculus

Operational calculus is an algebraic method for solving linear constant coefficient differential equations, which was invented by Heaviside. One rigorous foundation of Heaviside's operational calculus was given by Mikusinski (see [7]). Mikusinski's idea is to use the quotient field construction (from commutative algebra) in order to define generalized functions.

In this section we describe a simplified version of Heaviside–Mkusinski's theory.

We know how to multiply functions by proper rational functions. But we want to be able to multiply functions by arbitrary rational functions as well. Commutative algebra suggests to take the quotient space of $\mathcal{U}$. Denote this quotient space by $\mathcal{M}$ and call its elements generalized (or Mikusinski) functions. By the very definition, a generalized function is a ratio of the form w/g, where $w \in \mathcal{U}$ and $g \in$ O, $g \neq 0$. Two ratios w_1/g_1 and w_2/g_2 represent the same generalized function, if $g_2 w_1 = g_1 w_2$. An example of generalized function is $\delta = \hbar/t$, which may be interpreted as the Dirac delta-function. We identify $\mathcal{U}$ with its image in $\mathcal{M}$ under the canonical map $w \mapsto w/1$. Certainly, every generalized function (and, in particular, every usual one) can be multiplied by a rational function.

It is interesting to note that every generalized function can be written as w/t^n, where $w \in \mathcal{U}$ and $n \geq 0$. (Indeed, if w/g is a generalized function and if $g = t^n e$ with invertible e, then $w/g = (e^{-1}w)/t^n$.) Note also that $w/t^n = s^n w$. (This is due to the identification above.)

The functions $f\delta$ with $f \in k[s]$ will be called (purely) impulsive functions. We denote by Δ the space of all such functions.

Lemma 1. $\mathcal{M} = \mathcal{U} \oplus \Delta$.

Proof. As we said just now, any Mikusinski function has the form $s^n w$ with $n \geq 0$ and $w \in \mathcal{U}$. By the Taylor formula,

$$w = (w(0) + w'(0)t + \cdots + w^{(n-1)}(0)t^{n-1})\hbar + t^n w^{(n)}.$$

Multiplying this by s^n, we get

$$s^n w = w^{(n)} + s^n L(w(0)) + \cdots + sL(w^{(n-1)}(0)).$$

Thus, $\mathcal{M} = \mathcal{U} + \Delta$.

Suppose that $\mathcal{U} \cap \Delta \neq \{0\}$. We then have

$$w = (c_0 s^n + \cdots + c_n)\delta,$$

where $w \in \mathcal{U}$, $n \geq 0$, $c_0, \ldots, c_n \in k$ and $c_0 \neq 0$. Multiplying this by t^{n+1}, we get

$$t^{n+1} w = (c_0 + \cdots + c_n t^n)\hbar.$$

But this is a contradiction: The left function is zero at 0, while the right one is c_0. □

In the behavioral linear system theory one is interested in linear differential equations of the form

$$R(\partial)w = 0,$$

where R is an arbitrary (not necessarily square) polynomial matrix. To treat such "rectangular" differential equations we need the notion of transfer module.

By a transfer module (with signal number q), we mean any submodule $T \subseteq O^q$ for which O^q/T is torsion free. If T is a transfer module, then it has a representation $T = AO^m$, where m is a uniquely determined nonnegative integer and A is a transfer matrix of size $q \times m$. (We remind that a transfer matrix is a left invertible proper rational one.) The integer m is called the rank. Every such a matrix A is called a matrix representation. Two matrix representations are equivalent in the sense that one is obtained from the other by multiplication from the right by a biproper rational matrix.

If A is a transfer matrix of size $q \times m$, then clearly it contains a nonsingular square submatrix of size $m \times m$. It follows that A is equivalent to a transfer matrix containing the identity matrix of size $m \times m$. And we see that a transfer module after possible permutation of the components in k^q can be represented by a matrix of the form $\begin{bmatrix} I \\ G \end{bmatrix}$, where G is a proper rational matrix. (Transfer modules are substitutes in the behavioral framework for classical transfer matrices.)

Given a transfer module T, we let $T\mathcal{U} \subseteq \mathcal{U}^q$ be the set of all finite sums of products gu with $g \in T$ and $u \in \mathcal{U}$. Clearly, $T\mathcal{U}$ is an O-submodule of $\mathcal{U}^q$. If A is a matrix representation of T and m is the rank, then clearly

$$T\mathcal{U} = A\mathcal{U}^m.$$

We shall need the following

Lemma 2. *Let T be a transfer module. Then*

$$T = L^{-1}(T\mathcal{U} \cap L(O^q)).$$

Proof. Without loss of generality, we may assume that T has the form

$$T = \begin{bmatrix} I \\ G \end{bmatrix} O^m,$$

where G is a proper rational matrix. Clearly, $\binom{u}{Gu}$ with $u \in \mathcal{U}^m$ is exponential if and only if u is exponential. Hence

$$\begin{bmatrix} I \\ G \end{bmatrix} \mathcal{U}^m \cap L(O^q) = \begin{bmatrix} I \\ G \end{bmatrix} L(O^m).$$

The assertion follows. □

Corollary 1. *The mapping $T \mapsto T\mathcal{U}$ is one-one.*

Assume now we are given a full row rank polynomial matrix R of size $p \times q$, say. We can consider the operator

$$R : \mathcal{U}^q \to \mathcal{M}^p, \qquad w \mapsto Rw.$$

Composing this with the canonical projection $\Pi : \mathcal{M}^p \to \mathcal{U}^p$, we get the operator

$$\Pi \circ R : \mathcal{U}^q \to \mathcal{U}^p.$$

It is not surprising that $\Pi \circ R = R(\partial)$. From this we get that

$$R(\partial)w = 0 \Leftrightarrow Rw \in \Delta^p, \tag{1}$$

and thus

$$Bh(R) = \{w \in \mathcal{U}^q \mid Rw \in \Delta^p\}.$$

We see that the operator $R : \mathcal{U}^q \to \mathcal{M}^p$ induces a canonical linear map

$$Bh(R) \overset{R}{\to} \Delta^p. \tag{2}$$

Let us find the kernel and the image of this map.

Define T to be the kernel of $O^q \overset{R}{\to} k(s)^p$, i.e.,

$$T = \{g \in O^q \mid Rg = 0\}.$$

Due to the embedding $O^q/T \to k(s)^p$, T is a transfer module. We call it the transfer module of R. Choose a denominator D and consider the exact sequence

$$0 \to T \to O^q \overset{D^{-1}R}{\to} O^p \to 0.$$

Applying $\otimes\mathcal{U}$ to this sequence, we get an exact sequence

$$T \otimes \mathcal{U} \to \mathcal{U}^q \overset{D^{-1}R}{\to} \mathcal{U}^p \to 0.$$

(We remind the reader that tensoring is a right exact functor (see [4]). It should be noted that in fact we have more. The module $\mathcal{U}$, being a torsion free module over a principal ideal domain, is flat, and therefore we have an exact sequence

$$0 \to T \otimes \mathcal{U} \to \mathcal{U}^q \overset{D^{-1}R}{\to} \mathcal{U}^p \to 0.)$$

The image of $T \otimes \mathcal{U} \to \mathcal{U}^q$ is just $T\mathcal{U}$, and so, by the above exact sequence,

$$T\mathcal{U} = \{w \in \mathcal{U}^q \mid D^{-1}Rw = 0\} = \{w \in \mathcal{U}^q \mid Rw = 0\}.$$

Certainly, every w with $Rw = 0$ belongs to $Bh(R)$. And we conclude that the kernel of (2) coincides with $T\mathcal{U}$.

The image of (2) is equal to $\Delta^p \cap R\mathcal{U}^q = \Delta^p \cap D\mathcal{U}^p$. (Because $D^{-1}RO^q = O^p$, we have $D^{-1}R\mathcal{U}^q = \mathcal{U}^p$; whence $R\mathcal{U}^q = D\mathcal{U}^p$.) We find that

$$\begin{aligned} \Delta^p \cap D\mathcal{U}^p &= \Delta^p \cap k(s)^p\hbar \cap D\mathcal{U}^p = \Delta^p \cap D(k(s)^p\hbar \cap \mathcal{U}^p) \\ &= \Delta^p \cap DO^p\hbar = X\delta, \end{aligned}$$

where $X = k[s]^p \cap tDO^p$.

We call X the initial condition space of R.

2. "Algebraic geometry over a projective line"

We have seen that the space $k[s]^p \cap tDO^p$ is relevant. (Long ago this was observed by Fuhrmann [8].) Soon we shall see that $k(s)^p/(sk[s]^p + DO^p)$ also is relevant. These should be given names, and we call them cohomologies (as they indeed are cohomologies).

Remark. One can associate a vector bundle over a projective line with every nonsingular rational matrix D, and every vector bundle (up to isomorphism) can be obtained this way. The spaces above are respectively 0-dimensional and 1-dimensional cohomology spaces of the vector bundle associated with D. (The interested reader is referred to [9], where an algebraic definition of vector bundles is provided and the link between nonsingular rational matrices and vector bundles is described.)

For a nonsingular rational matrix D, we thus make the following definition:

$$H^0(D) = k[s]^p \cap tDO^p \quad \text{and} \quad H^1(D) = k(s)^p/(sk[s]^p + DO^p),$$

where p is the size of D.

Example. If n is an arbitrary integer, then

$$H^0(s^n) = \Omega_{n-1} \quad \text{and} \quad H^1(t^n) = \Gamma_{n-1}.$$

If D_1 and D_2 are two nonsingular rational matrices, then one says that they are Wiener–Hopf equivalent if there exist a unimodular matrix U and a biproper rational matrix V such that $D_2 = UD_1V$. If this is the case, then D_1 and D_2 have isomorphic cohomologies. (The isomorphisms

$$H^0(D_1) \simeq H^0(D_2) \quad \text{and} \quad H^1(D_1) \simeq H^1(D_2)$$

are established via the left multiplication by U.)

The well-known Wiener–Hopf theorem states that every nonsingular rational matrix D is Wiener–Hopf equivalent to a diagonal matrix of the form $\mathrm{diag}\{s^{n_1}, \dots, s^{n_p}\}$. The integers $n_1, \dots, n_p$ are uniquely determined (up to order) and are called the Wiener–Hopf indices.

Lemma 3. *Let D be a nonsingular rational matrix with Wiener–Hopf indices $n_1, \dots, n_p$. Then*

$$\dim H^0(D) = \sum_{i=0}^{p} \max\{n_i, 0\} \quad \text{and}$$

$$\dim H^1(D) = \sum_{i=0}^{p} \max\{-n_i, 0\}.$$

Proof. In view of the example above, this is obvious when $D = diag\{s^{n_1}, \dots, s^{n_p}\}$. The general case follows immediately from this special one by the Wiener–Hopf theorem. □

If D is a nonsingular rational matrix, we define the Chern number to be

$$ch(D) = -\mathrm{ord}_\infty(\det D).$$

It is clear that the Chern number is equal to the sum of the Wiener–Hopf indices.

Lemma 4 (*"Riemann–Roch theorem"*)**.** *If D is a nonsingular rational matrix, then*

$$\dim H^0(D) - \dim H^1(D) = ch(D).$$

Proof. The same as that of the previous lemma. □

Let $p \geq 0$. There is a canonical $k(s)$-bilinear form

$$k(s)^p \times k(s)^p \to k(s)$$

taking a pair (f, g) to $f^{\mathrm{tr}}g$. ("tr" stands for the transpose). Composing this standard bilinear form with the k-linear map $k(s) \to k$ taking $\sum a_i t^i$ to a_0, we get a canonical k-bilinear form

$$\langle -, - \rangle : k(s)^p \times k(s)^p \to k. \tag{3}$$

This clearly is non-degenerate, and we have

$$(k[s]^p)^\perp = sk[s]^p \quad \text{and} \quad (O^p)^\perp = tO^p.$$

Let D be a nonsingular rational matrix of size p. We define the dual of D as $D^* = (D^{-1})^{\mathrm{tr}}$. From the second orthogonality relation above, we get

$$(tD^*O^p)^\perp = DO^p.$$

Using the relation $X^\perp + Y^\perp \subseteq (X \cap Y)^\perp$, we see that

$$sk[s]^p + DO^p \subseteq (k[s]^p \cap tD^*O^p)^\perp.$$

Therefore, (3) induces a canonical pairing

$$(k[s]^p \cap tD^*O^p) \times k(s)^p/(sk[s]^p + DO^p) \to k,$$

which can be rewritten as

$$H^0D^* \times H^1D \to k.$$

Lemma 5 (*"Serre's Duality Theorem"*)**.** *The pairing above is non-degenerate.*

Proof. This is non-degenerate from the left, of course, since so is (3). To prove the non-degeneracy it suffices therefore to show that H^0D^* and H^1D have the same dimension. But this is immediate from Lemma 3. □

Let R be a full row rank polynomial matrix with denominator D. We define the McMillan degree of R to be the Chern number of D, and we define the lag indices as the Wiener–Hopf indices of D. These indices are nonnegative necessarily (because R has full row rank). It follows that the 1-dimensional cohomologies of D are trivial. From this, by the Riemann–Roch formula, it is immediate that $\dim H^0(D) = ch(D)$. In other words, the dimension of the initial condition space of R is equal to its McMillan degree.

3. Three properties of a linear dynamical system

For every $w \in \mathcal{U}^q$ and every $n \geq 0$, we set

$$w|_n = w(0) + w'(0)t + \cdots + w^{(n)}(0)t^n.$$

It is a trivial remark that w is flat if and only if $w_{|n} = 0$ for all $n \geq 0$.

Assume we are given a linear dynamical system $\mathcal{S}$.

For each $n \geq 0$, let $\mathcal{S}|_n$ denote the image of $\mathcal{S}$ under the mapping $w \mapsto w|_n$. We call $\mathcal{S}|_n$ the nth truncation of $\mathcal{S}$.

Definition 1. We say that $\mathcal{S}$ is relatively complete, if

$$w \in \mathcal{S} + \mathcal{F}^q \Leftrightarrow \forall n \geq 0, \quad w|_n \in \mathcal{S}|_n.$$

Comment. The family $(\mathcal{S}_{|n})$ contains no information about the flat trajectories of $\mathcal{S}$. So, $\mathcal{S} + \mathcal{F}^q$ is the maximum that can be recovered from knowledge of $(\mathcal{S}_{|n})$. Relative completeness guarantees this maximum.

We say that $w \in \mathcal{S}$ is a zero initial condition trajectory if all its n-fold integrals also are trajectories of $\mathcal{S}$. Let us denote by $\mathcal{S}_0$ the set of all such trajectories. So,

$$\mathcal{S}_0 = \left\{ w \in \mathcal{S} \mid \forall n \geq 0, \ \int^n w \in \mathcal{S} \right\}.$$

The quotient space $\mathcal{S}/\mathcal{S}_0$ should be thought of as the initial condition space. If x is an initial condition and if $w \in \mathcal{S}$ is a trajectory representing it, we shall say that x is the initial condition of w.

Definition 2. We say that $\mathcal{S}$ is specified, if there is a transfer module T such that $\mathcal{S}_0 = T\mathcal{U}$ and if $\mathcal{S} \subseteq \mathcal{S}_0 + \mathcal{A}^q$. By Corollary 1, the "$T$" is uniquely determined. It is called the transfer module.

Comment. Specifiedness means that the zero initial condition trajectory set has a very special simple structure (that it is generated by a transfer module). It means also that there are sufficiently many analytic trajectories (that there always exists an analytic trajectory with any given initial condition).

Lemma 6. *Suppose that $\mathcal{S}$ is relatively complete and specified. If $w \in \mathcal{A}^q$, then*

$$w \in \mathcal{S} \Leftrightarrow \forall n \geq 0, \quad w|_n \in \mathcal{S}|_n.$$

Proof. Let T be the transfer module. Without loss of generality, we may assume that

$$T = \begin{bmatrix} I \\ G \end{bmatrix} O^m,$$

where G is a proper rational matrix.

Suppose that $w|_n \in \mathcal{S}|_n$ for every $n \geq 0$. By the hypotheses,

$$w \in \mathcal{S} + \mathcal{F}^q \subseteq \mathcal{S}_0 + \mathcal{A}^q + \mathcal{F}^q.$$

Hence, we have

$$w = \begin{bmatrix} u \\ Gu \end{bmatrix} + a + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where $u \in \mathcal{U}^m$, $a \in \mathcal{S} \cap \mathcal{A}^q$, $f_1 \in \mathcal{F}^m$ and $f_2 \in \mathcal{F}^{q-m}$. Rewrite this equality as

$$w = \begin{bmatrix} u + f_1 \\ G(u + f_1) \end{bmatrix} + a + \begin{bmatrix} 0 \\ f_2 - Gf_1 \end{bmatrix}.$$

Replacing $u + f_1$ by u and $f_2 - Gf_1$ by f_2, we get

$$w = \begin{bmatrix} u \\ Gu \end{bmatrix} + a + \begin{bmatrix} 0 \\ f_2 \end{bmatrix}.$$

We can see that u is analytic, and consequently so is Gu. It follows that f_2 must be analytic. On the other hand, f_2 is a flat function. Hence, $f_2 = 0$.

It follows that w is a trajectory of $\mathcal{S}$. □

The following says that

relative completeness + specifiedness ⇒ "completeness".

Lemma 7. *If $\mathcal{S}$ is relatively complete and specified, then it is possible to recover $\mathcal{S}$ from its truncations.*

Proof. Let T denote the transfer module. By Lemma 2,

$$T = \{g \mid L(g) \in \mathcal{S}_0 \cap L(O^q)\}.$$

From this and from the previous lemma, follows that T can be determined by means of $(\mathcal{S}_{|n})$. Namely, we have

$$T = \{g \in O^q \mid (t^l L(g))_{|n} \in \mathcal{S}_{|n} \ \forall l, n \geq 0\}.$$

From T we immediately arrive at $\mathcal{S}_0 = T\mathcal{U}$.

Now, we claim that

$$\mathcal{S} = \{w \in \mathcal{S}_0 + \mathcal{A}^q \mid \forall n \geq 0, \ w|_n \in \mathcal{S}|_n\}.$$

Indeed, let $w = w_0 + a$, where $w_0 \in \mathcal{S}_0$ and $a \in \mathcal{A}^q$, and assume that $w|_n \in \mathcal{S}|_n$ for every $n \geq 0$. Clearly, $a|_n \in \mathcal{S}|_n$ for every $n \geq 0$. By the previous lemma, $a \in \mathcal{S}$. Hence, w also is a trajectory of $\mathcal{S}$.

Thus, knowledge of $(\mathcal{S}_{|n})$ implies knowledge of $\mathcal{S}$.

The proof is complete. □

Closing the section, we introduce the third property, which is standard.

Definition 3. We say that $\mathcal{S}$ is differentiation-invariant, if $\partial(\mathcal{S}) \subseteq \mathcal{S}$.

4. Linear differential systems have the three properties

Let R be a full row rank polynomial matrix of size $p \times q$, and let D be a denominator of R. Put $m = q - p$ and $\mathcal{B} = \mathrm{Bh}(R)$.

Proposition 1. *$\mathcal{B}$ is relatively complete.*

Proof. After reordering the components in k^q (if necessary), our behavior can be described by the familiar equation

$$\begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$$

with state space X and "observable pair" (A, C). Let $G = C(sI - A)^{-1}B + D$. As one knows,

$$(u, x_0) \rightarrow \begin{bmatrix} u \\ Gu + C(I - tA)^{-1}x_0 \end{bmatrix}$$

is a bijective map from $\mathcal{U}^m \times X$ onto $\mathcal{B}$.

Let $w = \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{U}^q$, and suppose that

$$\forall n \geq 0, \quad w|_n \in \mathcal{B}|_n.$$

Because $\begin{bmatrix} u \\ Gu \end{bmatrix}$ is a trajectory of $\mathcal{B}$,

$$w \in \mathcal{B} + \mathcal{F}^q \Leftrightarrow w - \begin{bmatrix} u \\ Gu \end{bmatrix} \in \mathcal{B} + \mathcal{F}^q.$$

Therefore, we may assume that $w = \begin{bmatrix} 0 \\ y \end{bmatrix}$.

Let n be any nonnegative integer. By the hypothesis, there is a pair (u, x_0) with $u \in \mathcal{U}^m$ and $x_0 \in X$ such that

$$\begin{bmatrix} 0 \\ y_{|n} \end{bmatrix} = \begin{bmatrix} u_{|n} \\ (Gu)_{|n} + (C(I - tA)^{-1}x_0)_{|n} \end{bmatrix}.$$

From $u_{|n} = 0$ we get that $u \in t^{n+1}\mathcal{U}^m$. This yields that $Gu \in t^{n+1}\mathcal{U}^p$, and consequently $(Gu)_{|n} = 0$.

It follows that

$$\begin{bmatrix} 0 \\ y_{|n} \end{bmatrix} = \begin{bmatrix} 0 \\ (C(I - tA)^{-1}x_0)_{|n} \end{bmatrix}.$$

Thus, for every $n \geq 0$, there exists $x_0 \in X$ such that

$$y_{|n} = Cx_0 + CAx_0t + \cdots + CA^nx_0t^n.$$

Because the pair (A, C) is observable, the linear map

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^n \end{bmatrix} : X \rightarrow k^{p(n+1)}$$

is injective for every $n \geq \dim(X) - 1$. It follows that for the "n"-s that are large enough the "x_0" must be the same.

Thus, there exists x_0 such that

$$y_{|n} = (C(I - tA)^{-1}x_0)_{|n}$$

for all sufficiently large n, and therefore for all $n \geq 0$. We get

$$y \equiv C(I - tA)^{-1}x_0 \bmod \mathcal{F}^p.$$

This completes the proof. □

Proposition 2. *$\mathcal{B}$ is specified.*

First Proof. We claim that $\mathcal{B}_0 = T\mathcal{U}$, where T is the transfer module of R. Indeed, we saw that $T\mathcal{U} = \{w \in \mathcal{U}^q | \mathrm{Rw} = 0\}$, and hence the inclusion "$\supseteq$" is obvious. To show the inclusion "$\subseteq$", take any $w \in \mathcal{B}_0$. Then

$$\forall n \geq 0, \quad t^n Rw \in \Delta^p.$$

From this it is immediate that $Rw = 0$. But then $w \in T\mathcal{U}$, as desired.

Take now any trajectory $w \in \mathcal{B}$. Then $Rw \in L(sk[s]^p \cap DO^p)$. Because $DO^p = RO^q$, we have $Rw = RL(g)$ for some $g \in O^q$. It follows that $R(w - L(g)) = 0$, and hence $w - L(g) \in T\mathcal{U}$.

The proof is complete.

Second Proof. Let X, A, B, C and D be as in the proof of Proposition 1. For $u \in \mathcal{U}^m$ and $x_0 \in X$, set

$$w_{u,x_0} = \begin{bmatrix} u \\ Gu + C(I - tA)^{-1}x_0 \end{bmatrix}.$$

For $u \in \mathcal{U}^m$ and $n \geq 0$, we clearly have

$$\int^n w_{u,0} = w_{t^n u,0}.$$

Hence, all $w_{u,0}$ belong to $\mathcal{B}_0$.

Conversely, suppose that $w_{u,x_0} \in \mathcal{B}_0$. Take any integer $n \geq 0$. We must have

$$\int^n w_{u,x_0} = w_{u_1,x_1}$$

for some $u_1 \in U^m$ and $x_1 \in X$. This equality can be rewritten as

$$\begin{bmatrix} t^n u \\ Gt^n u + C(I - tA)^{-1}t^n x_0 \end{bmatrix} = \begin{bmatrix} u_1 \\ Gu_1 + C(I - tA)^{-1}x_1 \end{bmatrix},$$

yielding

$$C(I - tA)^{-1}t^n x_0 = C(I - tA)^{-1}x_1.$$

The left side belongs to $t^n\mathcal{U}^p$, and consequently

$$Cx_1 + CAx_1 t + \cdots + CA^{n-1}x_1 t^{n-1} = 0.$$

Require now that n be sufficiently large. Since (A, C) is an observable pair, we get from this that $x_1 = 0$. This in turn implies that

$$C(I - tA)^{-1}x_0 = 0;$$

whence $x_0 = 0$.

Thus,

$$\mathcal{B}_0 = \{w_{u,0} \mid u \in \mathcal{U}^m\} = \begin{bmatrix} I \\ G \end{bmatrix} \mathcal{U}^m.$$

It is obvious that $\mathcal{B}$ contains sufficiently many analytic trajectories. Indeed, for $u \in \mathcal{U}^m$ and $x_0 \in X$, we have

$$w_{u,x_0} = w_{u,0} + w_{0,x_0}.$$

Here $w_{u,0}$ is a zero initial condition trajectory and w_{0,x_0} is an exponential trajectory.

The proof is complete. □

The following is obvious and well-known.

Proposition 3. *$\mathcal{B}$ is differentiation-invariant.*

For later use we shall prove the following

Lemma 8. *Let $n \geq 0$. There is a canonical exact sequence*

$$\mathcal{B} \to \Gamma_n^q \to H^1(t^{n+1}D) \to 0.$$

Proof. Define the first map and the second map to be

$$w \mapsto w_{|n} \quad \text{and} \quad u \mapsto (Ru)\mathrm{mod}(sk[s]^p + t^{n+1}DO^p),$$

respectively.

Let w be any trajectory in $\mathcal{B}$. By the Taylor formula,

$$w = (w_{|n})\hbar + t^{n+1}w^{(n+1)}.$$

We therefore have

$$R(w|_n)\hbar = Rw - t^{n+1}Rw^{(n+1)} \in \Delta^p + t^{n+1}R\mathcal{U}^q.$$

Since $R\mathcal{U}^q = D\mathcal{U}^p$, there is $w_1 \in \mathcal{U}^p$ such that $Dw_1 = Rw^{(n+1)}$. The function w_1 is uniquely determined, and our claim is that it is exponential. Indeed, because $(R(w|_n))\hbar$ and Rw belong to $k(s)^p\hbar$, so is Dw_1. It follows that w_1 itself belongs to $k(s)^p\hbar$. This proves the claim because $\mathcal{U}^p \cap k(s)^p\hbar = O^p\hbar$. Thus $w_1 = g\hbar$ with $g \in O^p$, and we obtain

$$R(w|_n) \in sk[s]^p + t^{n+1}DO^p.$$

So, the map $\Gamma_n^q \to H^1(t^{n+1}D)$ sends $w|_n$ to 0.

Assume now that $u \in \Gamma_n^q$ goes to 0 under the map $\Gamma_n^q \to H^1(t^{n+1}D)$. Then $Ru = f + t^{n+1}Dg$ with $f \in sk[s]^p$ and $g \in O^p$. Since $RO^q = DO^p$, there exists $v \in O^q$ such that $Rv = Dg$. Putting

$$w = u\hbar - t^{n+1}v\hbar,$$

we have

$$Rw = Ru\hbar - t^{n+1}Rv\hbar = Ru\hbar - t^{n+1}Dg\hbar = f\hbar.$$

Hence, by (1), $w \in \mathcal{S}$. It is clear that $w|_n = u$.

Finally, $\Gamma_n^q \to H^1(t^{n+1}D)$ is surjective in view of the following commutative diagram

$$\begin{array}{ccc} k(s)^q & \to & k(s)^p \\ \downarrow & & \downarrow \\ k(s)^q/(sk[s]^q + t^{n+1}O^q) & \to & k(s)^p/(sk[s]^p + t^{n+1}DO^p) \end{array}$$

Indeed, the top arrow $R : k(s)^q \to k(s)^p$ is surjective, so must be the bottom arrow as well. This latter is just our map.

The proof is complete. □

5. Formal languages and convolutional codes

This section deals with two kinds of objects which will prove very helpful in the next section. They will be referred to as "formal languages" and "convolutional codes". (We hope that the reader will be willing to excuse using this terminology.)

By a (linear) formal language (on k^q) we understand a family $(B_n)_{n\geq 0}$ of linear subspaces $B_n \subseteq \Gamma_n^q$ satisfying the following two conditions:

(i) $b_0 + b_1 t + \cdots + b_n t^n \in B_n \Rightarrow b_1 + b_2 t + \cdots + b_n t^{n-1},\ b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} \in B_{n-1}$;
(ii) $b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} \in B_{n-1} \Rightarrow \exists b \in k^q,\ b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + bt^n \in B_n$.

Remark. The polynomials in Γ_n^q can be viewed in an obvious way as words of length $n + 1$ (with letters in k^q). The first condition means that the words of a language are invariant with respect to the left and right "deletion" operators; the second one means that every word of a language has a continuation to the right. Formal languages of the above kind are very special, of course. (By a formal language, we remind, one understands an arbitrary set of words.)

The notion is motivated by the following

Example. Let $\mathcal{S}$ be an arbitrary differentiation-invariant linear dynamical system (in $\mathcal{U}^q$). Then $(\mathcal{S}|_n)$ is a formal language.

By a convolutional code (on k^q) we understand a family $(C_n)_{n\geq 0}$ of linear subspaces $C_n \subseteq \Omega_n^q$ satisfying the following two conditions:

(i) $C_{n-1} \subseteq C_n$ and $sC_{n-1} \subseteq C_n$;
(ii) $C_{n-1} = C_n \cap \Omega_{n-1}^q$.

Remark. In [10] a convolutional code is defined to be a submodule of $k[s]^q$. The two definitions clearly are equivalent: If C is a submodule in $k[s]^q$, then the family $(C \cap \Omega_n^q)$ is a convolutional code; conversely, if (C_n) is a convolutional code, then $C = \bigcup C_n$ is a submodule in $k[s]^q$.

There is an obvious orthogonality relation between formal languages and convolutional codes. Indeed, for each $n \geq 0$, there is a canonical non-degenerate pairing

$$\Omega_n^q \times \Gamma_n^q \ \rightarrow\ k,$$

which is given by

$$\langle a_0 s^n + \cdots + a_n, b_0 + \cdots + b_n t^n\rangle = a_0^{\mathrm{tr}} b_n + \cdots + a_n^{\mathrm{tr}} b_0.$$

(The pairing is a special case of the pairing in Lemma 5.) It is not difficult to see that if (B_n) is a formal language, then $(B_n^{\perp})$ is a convolutional code. Conversely, if (C_n) is a convolutional code, then $(C_n^{\perp})$ is a formal language.

Let now R be a full row rank polynomial matrix of size $p \times q$, and let D be its denominator.

Associated with R there are a canonical formal language and a canonical convolutional code. Indeed, for each $n \geq 0$, we have two canonical maps

$$\gamma_n : \Gamma_n^q \rightarrow H^1(t^{n+1}D) \quad \text{and} \quad \omega_n : H^0(s^{n+1}D^*) \rightarrow \Omega_n^q.$$

The first one is given by R and is surjective; the second is given by R^{tr} and is injective. Let B_n denote the kernel of the first map, and let C_n denote the image of the second one. One can check without difficulty that (B_n) is a formal language and (C_n) is a convolutional code.

There are alternative ways for obtaining the same formal language and the same convolutional code. Indeed, if $\mathcal{B}$ is the behavior of R, then, by Lemma 8, (B_n) is exactly the formal language $(\mathcal{B}|_n)$. Next, it is easily seen that $k[s]^p = \bigcup H^0(s^{n+1}D^*)$, and, letting C denote the submodule $R^{\mathrm{tr}}k[s]^p$, we have $(C_n) = (C \cap \Omega_n^q)$.

Lemma 9. *The formal language and the convolutional code constructed above are orthogonal to each other; that is, for each $n \geq 0$, we have*

$$(C_n)^{\perp} = B_n \quad \text{and} \quad (B_n)^{\perp} = C_n.$$

Proof. It suffices, of course, to show, say, the first relation only.

For any $u \in H^0(s^{n+1}D^*)$ and any $v \in \Gamma_n^q$, we have

$$\langle \omega_n(u), v\rangle = \langle u, \gamma_n(v)\rangle.$$

Using this formula and Lemma 5, we get

$$\begin{aligned}(C_n)^{\perp} &= \{v \mid \forall u,\ \langle \omega_n(u), v\rangle = 0\} = \{v \mid \forall u,\ \langle u, \gamma_n(v)\rangle = 0\}\\ &= \{v \mid \gamma_n(v) = 0\} = B_n. \quad \square\end{aligned}$$

We shall denote by $\mathrm{FL}(R)$ the formal language associated with R.

6. Equivalence theorem and kernel representation theorem

Recall that two full row rank polynomial matrices R_1 and R_2 are said to be equivalent if $R_2 = UR_1$ for some unimodular matrix U.

Theorem 1 (*Equivalence Theorem*)**.** *Let R_1 and R_2 be two full row rank polynomial matrices. Then*

$$R_1 \sim R_2 \Leftrightarrow \mathrm{Bh}(R_1) = \mathrm{Bh}(R_2) \Leftrightarrow \mathrm{FL}(R_1) = \mathrm{FL}(R_2).$$

Proof. With evident notations we have

$$\begin{aligned}R_1 \sim R_2 &\Rightarrow \mathcal{B}_1 = \mathcal{B}_2 \Rightarrow (B_{1,n})_n = (B_{2,n})_n\\ &\Rightarrow (C_{1,n})_n = (C_{2,n})_n \Rightarrow C_1 = C_2 \Rightarrow R_1 \sim R_2.\end{aligned}$$

The proof is complete. $\square$

Theorem 2. *The mapping*

$$\mathcal{S} \mapsto (\mathcal{S}|_n)$$

establishes a bijective correspondence between linear differential systems and formal languages.

Proof. The injectivity is immediate by the equivalence theorem.

To prove the surjectivity, take any formal language (B_n). For each $n \geq 0$, put $C_n = (B_n)^{\perp}$. Define $C = \bigcup C_n$. As already remarked, C is a submodule in $k[s]^q$. We can choose a full row rank polynomial matrix R so that $C = \mathrm{Im} R^{\mathrm{tr}}$. In view of Lemma 9, $\mathrm{FL}(R)$ is orthogonal to (C_n). But (B_n) is orthogonal also to (C_n); hence, $\mathrm{FL}(R) = (B_n)$. Letting $\mathcal{B} = Bh(R)$, we have

$$(\mathcal{B}_{|n}) = (B_n).$$

The proof is complete. $\square$

Corollary 2 (*Kernel Representation Theorem*)**.** *Every relatively complete, specified, differentiation-invariant linear dynamical system is differential.*

Proof. Let $\mathcal{S}$ be a relatively complete, specified, differentiation-invariant linear dynamical system. By Example 2, $(\mathcal{S}|_n)$ is a formal language. And, by the previous theorem, there is a linear differential system $\mathcal{B}$ for which

$$(\mathcal{B}_{|n}) = (\mathcal{S}|_n).$$

Using Lemma 7, we have $\mathcal{S} = \mathcal{B}$. $\square$

7. Concluding remarks

Let σ denote the backward shift operator, and let t be an indeterminate. For every function w defined on $\mathbb{Z}_+$ and for every integer $n \geq 0$, define

$$w_{|n} = w(0) + (\sigma w)(0)t + \cdots + (\sigma^n w)(0)t^n.$$

Given a linear dynamical system $\mathcal{S} \subseteq C(\mathbb{Z}_+, k^q)$, set

$$\mathcal{S}_{|n} = \{w_{|n} \mid w \in \mathcal{S}\}.$$

Identifying $C(\mathbb{Z}_+, k^q)$ with $k^q[[t]]$, we clearly have, $\mathcal{S}_{|n} = \mathcal{S}_{|[0,n]}$. So, Willems' definition of completeness for discrete-time linear dynamical systems can be formulated as follows: A linear dynamical system $\mathcal{S} \subseteq C(\mathbb{Z}_+, k^q)$ is complete if

$$w \in \mathcal{S} \Leftrightarrow \forall n \geq 0, \quad w_{|n} \in \mathcal{S}_{|n}.$$

The continuous-time analog of σ is the differentiation operator, and the definitions of $w_{|n}$ and $\mathcal{S}_{|n}$ can be extended to the continuous-time case in an obvious way. However, the above definition cannot be translated word for word. The reason is that in the continuous-time case taking $(\mathcal{S}_{|n})$ "kills" the flat trajectories of $\mathcal{S}$. We can speak only about relative completeness (completeness modulo flat functions). Letting $\mathcal{F}$ denote the space of flat functions, the definition above is modified as follows: A (continuous-time) linear dynamical system $\mathcal{S}$ is *relatively complete* if

$$w \in \mathcal{S} + \mathcal{F}^q \Leftrightarrow \forall n \geq 0, \quad w_{|n} \in \mathcal{S}_{|n}.$$

In order to be able to learn everything about $\mathcal{B}$ while knowing $(\mathcal{B}_{|n})$, one has to impose some additional condition on $\mathcal{B}$.

Our choice in this paper is *specifiedness*. (The term is borrowed from Willems [2]; however our interpretation is very much different.)

A (continuous-time) linear dynamical system $\mathcal{B}$ is specified if its zero initial trajectory set is determined by a transfer matrix and if it has sufficiently many analytic trajectories. (Remark that taking $(\mathcal{B}_{|n})$ one loses no information about the analytic trajectories.)

The main result of the paper claims that the properties of relative completeness, specifiedness and differentiation-invariance characterize linear differential systems among all linear dynamical systems.

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Appendix A. State representations

We would like to recall here the construction of state representations given in [11], which is very natural (in our opinion).

Assume we have a linear differential system $\mathcal{B}$ with transfer function T. Set $\mathcal{B}_0 = T\mathcal{U}$, which, as we know, is the set of trajectories of $\mathcal{B}$ that have zero initial condition. The ingredients for the construction are also the following two linear spaces

$$\mathcal{B}^a = t\mathcal{B} + k^q \quad \text{and} \quad \mathcal{B}_0^0 = T\mathcal{U} \cap t\mathcal{U}^q.$$

The first is the space of all primitives (antiderivatives) of trajectories in $\mathcal{B}$; the second is the space of all trajectories in $\mathcal{B}$ that have zero initial condition and zero initial value.

There is a canonical short exact sequence

$$0 \to \mathcal{B} \to \mathcal{B} \oplus \mathcal{B} \oplus k^q \to \mathcal{B}^a \to 0,$$

where the second and the third arrows are given by

$$w \mapsto (w', w, w(0)) \quad \text{and} \quad (w_1, w_2, c) \mapsto tw_1 - w_2 + c.$$

(The exactness at the middle term is immediate by the Newton–Leibniz formula; the exactness at the other terms is evident.) There is also the following obvious exact sequence

$$0 \to \mathcal{B}_0^0 \to \mathcal{B}_0 \oplus \mathcal{B}_0 \to \mathcal{B}_0 \to 0,$$

where the second and third arrows are defined respectively by

$$w \mapsto (w', w) \quad \text{and} \quad (w_1, w_2) \mapsto tw_1 - w_2.$$

Combining these two exact sequences, we get the following commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{B}_0^0 & \to & \mathcal{B}_0 \oplus \mathcal{B}_0 \oplus \{0\} & \to & \mathcal{B}_0 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{B} & \to & \mathcal{B} \oplus \mathcal{B} \oplus k^q & \to & \mathcal{B}^a & \to 0 \end{array},$$

where the vertical arrows represent the inclusion maps. Applying the snake lemma (see [4, Proposition 2.10]), we obtain an exact sequence

$$0 \to Y \xrightarrow{\begin{bmatrix} F \\ G \\ H \end{bmatrix}} X \oplus X \oplus k^q \xrightarrow{\begin{bmatrix} K & -L & M \end{bmatrix}} Z \to 0,$$

where

$$Y = \mathcal{B}/\mathcal{B}_0^0, \qquad X = \mathcal{B}/\mathcal{B}_0 \quad \text{and} \quad Z = \mathcal{B}^a/\mathcal{B}_0,$$

the linear maps $F, G : Y \to X, H : Y \to k^q$ are defined by the formulas

$$F(w \bmod \mathcal{B}_0^0) = w' \bmod \mathcal{B}_0, \qquad G(w \bmod \mathcal{B}_0^0) = w \bmod \mathcal{B}_0,$$
$$H(w \bmod \mathcal{B}_0^0) = w(0)$$

and the linear maps $K, L : X \to Z, M : k^q \to Z$ by the formulas

$$K(w \bmod \mathcal{B}_0) = tw \bmod \mathcal{B}_0, \qquad L(w \bmod \mathcal{B}_0) = w \bmod \mathcal{B}_0,$$
$$M(c) = c \bmod \mathcal{B}_0.$$

One can show (see [11]) that the equations

$$\begin{cases} Gy' &= Fy \\ w &= Hy \end{cases} \quad \text{and} \quad Kx' - Lx + Mw = 0$$

represent $\mathcal{B}$.

Thus, $\mathcal{B}$ has two different canonical state representations, which however are closely related.

Appendix B. Controllability

There are several definitions of controllability. One possible definition that was given in [12] is as follows.

A linear differential system $\mathcal{B}$ is controllable if every its trajectory can be obtained from a zero initial condition trajectory using a finite number of differentiations. In other words, $\mathcal{B}$ is controllable if

$$\mathcal{B} = \bigcup_{n \geq 0} (T\mathcal{U})^{(n)},$$

where T is the transfer module of the system.

Put $V = \bigcup s^n T$. This is the least $k(s)$-linear subspace of $k(s)^q$ containing T. By the Taylor formula, $\Pi(s^n T\mathcal{U}) = (T\mathcal{U})^{(n)}$ for each $n \geq 0$. We therefore have

$$\bigcup (T\mathcal{U})^{(n)} = \bigcup \Pi(s^n TU) = \Pi(V\mathcal{U}).$$

The following implies that the above definition agrees with that of Willems [2,3].

Theorem 3. *$\mathcal{B}$ is controllable if and only if it has an image representation, i.e., there is a polynomial matrix M such that*

$$\mathcal{B} = ImM(\partial).$$

Proof. Suppose that the system is controllable. Choose a polynomial matrix M so that $Mk[s]^l = V \cap k[s]^q$, where l is the column number of M. Using the equality above, we have

$$\begin{aligned} \mathcal{B} &= \Pi(V\mathcal{U}) = \Pi(M\mathcal{M}^l) = \Pi(M\mathcal{U}^l + M\Delta^l) = \Pi(M\mathcal{U}^l) \\ &= M(\partial)\mathcal{U}^l = ImM(\partial). \end{aligned}$$

Suppose now that $\mathcal{B}$ has an image representation via M. We then have

$$\begin{aligned} \mathcal{B} &= \mathrm{Im}M(\partial) = \Pi(M\mathcal{U}^l) = \Pi(M\mathcal{U}^l + M\Delta^l) = \Pi(M\mathcal{M}^l) \\ &= \Pi((Mk(s)^l)\mathcal{U}), \end{aligned}$$

where again l is the column number of M. It remains to see that $Mk(s)^l = V$. Letting R be a kernel representation of $\mathcal{B}$, we have an exact sequence

$$\mathcal{U}^l \overset{M(\partial)}{\to} \mathcal{U}^q \overset{R(\partial)}{\to} \mathcal{U}^p,$$

where p is the row rank of R. This yields the following exact sequence

$$k[s]^p \overset{R^{\mathrm{tr}}}{\to} k[s]^q \overset{M^{\mathrm{tr}}}{\to} k[s]^l.$$

It follows that the sequence

$$k(s)^l \overset{M}{\to} k(s)^q \overset{R}{\to} k(s)^l$$

is exact; whence $Mk(s)^l = \{f \in k(s)^q | \mathrm{R}f = 0\} = V$. $\square$

In [12] the reader can found another proof for the "if" part of the theorem.

Appendix C. Integer invariants

Let $\mathcal{S}$ be a linear differential system.

The most important integer invariants of $\mathcal{S}$ (after the signal number) are the input number and the McMillan degree. If T is a transfer module, then the input number of $\mathcal{S}$ is the rank of T and the Mcmillan degree is the dimension of $\mathcal{S}/T\mathcal{U}$.

Of great importance are the lag indices, which are defined to be the lag indices of any minimal kernel representation. (By the equivalence theorem, these indices are well-defined.)

We introduce the complexity of $\mathcal{S}$ in the same manner as in Willems [2, Chapter X]:

$$c(\mathcal{S}) = (c_0(\mathcal{S}), c_1(\mathcal{S}), \dots)$$

with

$$c_n(\mathcal{S}) = \dim \mathcal{S}_{|n}.$$

The following implies that knowledge of the complexity is equivalent to that of the lag indices.

Theorem 4. *Let $l_1, \dots, l_p$ be the lag indices of $\mathcal{S}$. Then*

$$c_n(\mathcal{S}) = q(n+1) - \sum_{i=1}^{p} \max\{n+1-l_i, 0\}.$$

Proof. Let R be a kernel representation of $\mathcal{S}$, and let D be its denominator. By Lemma 8, we have a short exact sequence

$$0 \to \mathcal{S}_{|n} \to \Gamma_n^q \to H^1(t^{n+1}D) \to 0.$$

The Wiener–Hopf indices of $t^{n+1}D$ are $l_i - n - 1$. So that, the formula follows from Lemma 3. □

Corollary 3. *Let d be the McMillan degree of $\mathcal{S}$ and m the input number. For $n \geq d-1$, we have*

$$c_n(\mathcal{S}) = m(n+1) + d.$$

Appendix D. Modelling from data

Here we indicate how the theory of modelling as developed by Willems in [2, Chapter XIV] can be extended to the continuous-time case.

By a model let us mean any linear differential system (with signal number q). The set of models is partially ordered by the inclusion order.

Consider any data set $\mathcal{D} \subseteq \mathcal{U}^q$. A model $\mathcal{S}$ is unfalsified by $\mathcal{D}$ if $\mathcal{D} \subseteq \mathcal{S}$. The set of models unfalsified by $\mathcal{D}$ is nonempty. (This set obviously contains $\mathcal{U}^q$.) An interesting question is whether there exists the most powerful unfalsified model (MPUM).

Theorem 5. *The MPUM exists.*

Proof. Let $\mathcal{H}$ denote the "Hankelization" of $\mathcal{D}$, i.e. the set of all trajectories of the form $w^{(n)}$, where $w \in \mathcal{D}$ and $n \geq 0$. (This is the least differentiation-invariant dynamical system containing $\mathcal{D}$.) For each $n \geq 0$, set

$$C_n = (\mathcal{H}|_n)^{\perp}.$$

Clearly, the collection (C_n) is a convolutional code (notwithstanding $(\mathcal{H}|_n)$ is not, in general, a formal language). For each $n \geq 0$, set $B_n = C_n^{\perp}$. The model corresponding to (B_n) is the MPUM. □

Remark. The algorithm defined in [2, Chapter XIV] can be applied without changes to obtain a kernel representation of the MPUM.

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