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A note on interconnections

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ABSTRACT

The regular interconnection problem is considered in the context of Fliess models defined over an arbitrary noetherian ring. It is shown that the problem always has a solution provided that the plant is strongly controllable.

0. Introduction

We are concerned with the question of whether, given a plant, another system, more powerful than the plant, is regularly implementable, that is, whether it can be obtained as a regular interconnection of the plant with some suitable controller. In the classical case it was proved in [1,2] that if a plant is controllable, then the answer to this question is always affirmative. This result has been extended by Rocha and Wood [3] to the case of *n*D linear systems provided that the plant is strongly controllable. (Certainly, for classical systems, "controllability" = "strong controllability".)

The goal of this short work is to re-examine the result of Rocha and Wood. In our opinion, the problem is essentially algebraic, and we think that the module-theoretic framework for linear systems, introduced by Fliess [4], is very much appropriate for treating it.

In terms of polynomial representations our result can be formulated as follows. Assume that we have a plant Σ_1, represented by a right invertible polynomial matrix R_1, and assume that a "desired" system Σ is given that is more powerful than the plant and that is represented by a polynomial matrix R. Then Σ can be regularly implemented using the controller Σ_2, represented by

$$R_2 = R(I - YR_1),$$

where Y is a right inverse of R_1.

For various results about the regular interconnection problem (in the context of non-classical systems), the interested reader is referred to the papers [5–8].

Throughout, D is a noetherian (commutative) ring, and q is a fixed positive integer.

1. Preliminaries

A Fliess model (with signal number q) is a pair (M, ρ), where M is a finitely generated module over D and ρ is an epimorphism of D^q onto M.

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Given a Fliess model $\Sigma = (M, \rho)$, one can always find a polynomial matrix $R \in D^{\bullet \times q}$ such that the sequence

$$D^g \xrightarrow{R^t} D^q \to M \to 0,$$

where g is the row number of R, is exact. (The superscript "t" stands for the transpose.) Every such a matrix is called a representation of Σ. Elements in $Ker(\rho)$ are called syzygies of Σ. One says that Σ is controllable if M is torsion free. There are various degrees of torsion freeness, and consequently one has various degrees of controllability (see [9]). The ideal is when M is projective, and one then says that the model is strongly controllable. (Fliess and Mounier [10] call this "projective controllability".) Remark that Σ is strongly controllable if and only if R is right invertible. (In Rocha and Wood [3], the reader can found a behavioral characterization of strong controllability.)

Example 1. Let $D = \mathbb{F}[s_1, s_2]$, where $\mathbb{F}$ is a field and s_1, s_2 are indeterminates. Consider the Fliess model corresponding to the polynomial matrix $R = [s_1\ s_2]$. The matrix is not right invertible of course, and so the model is not strongly controllable. However, it is controllable; this is immediate in view of the exact sequence

$$D \xrightarrow{\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}} D^2 \xrightarrow{\begin{bmatrix} -s_2 & s_1 \end{bmatrix}} D.$$

A morphism from (M_1, ρ_1) to (M, ρ) is a homomorphism $\phi : M_1 \to M$ such that $\rho = \phi\rho_1$. Clearly, Fliess models form a category. In particular, we can speak about isomorphisms between Fliess models.

One says that $\Sigma = (M, \rho)$ is more powerful than $\Sigma_1 = (M_1, \rho_1)$ if there exists a morphism from Σ_1 to Σ; we then shall write $\Sigma \succeq \Sigma_1$. Remark that if R and R_1 are representations of Σ and Σ_1, respectively, then

$$\Sigma \succeq \Sigma_1 \Leftrightarrow \exists A, \qquad R_1 = AR \Leftrightarrow Ker(\rho_1) \subseteq Ker(\rho).$$

We now recall the definition of interconnections due to Fliess and Bourlès [11].

Let $\Sigma_1 = (M_1, \rho_1)$ and $\Sigma_2 = (M_2, \rho_2)$ be two Fliess models. Putting

$$N_1 = Ker(\rho_1), \qquad N_2 = Ker(\rho_2) \quad \text{and} \quad M = Coker \begin{bmatrix} \rho_1 \\ -\rho_2 \end{bmatrix},$$

we have an evident exact sequence

$$0 \to N_1 \cap N_2 \to D^q \xrightarrow{\begin{bmatrix} \rho_1 \\ -\rho_2 \end{bmatrix}} M_1 \oplus M_2 \to M \to 0.$$

Certainly, the two compositions

$$D^q \to M_1 \to M \quad \text{and} \quad D^q \to M_2 \to M$$

are equal. Letting ρ denote this common map, we get a Fliess model (M, ρ). (From the surjectivity of ρ_1 and ρ_2, it is clear that the homomorphisms $M_1 \to M$ and $M_2 \to M$ are surjective as well; hence, ρ is an epimorphism.) This is called the interconnection and is denoted by $\Sigma_1 \wedge \Sigma_2$. It is immediate from the definition that $\Sigma_1 \wedge \Sigma_2 \succeq \Sigma_1, \Sigma_2$.

We shall say that Σ_1 and Σ_2 are independent if they have no common syzygies except for 0. The interconnection of two independent Fliess models is said to be regular.

2. The regular interconnection problem

Assume that we have a Fliess model (plant) $\Sigma_1 = (M_1, \rho_1)$, and assume that a ("desired") Fliess model $\Sigma = (M, \rho)$ is given that is more powerful than the plant.

By an admissible controller, we shall mean any Fliess model that is independent from the plant. The regular interconnection problem asks whether there exists an admissible controller $\Sigma_2 = (M_2, \rho_2)$ such that

$$\Sigma \simeq \Sigma_1 \wedge \Sigma_2.$$

Let $N_1 = ker(\rho_1)$ and $N = ker(\rho)$. It is clear that $N_1 \subseteq N$, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & N_1 & \to & D^q & \to & M_1 & \to 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \to & N & \to & D^q & \to & M & \to 0 \end{array}$$

from which we can see that there is a canonical homomorphism $N \to M_1$. (This is just the restriction of ρ_1 on N.) The kernel of this homomorphism clearly is equal to N_1. Setting $N_2 = N/N_1$, we therefore have a canonical injective homomorphism

$$j : N_2 \to M_1.$$

Applying the snake lemma to the above diagram, we see that $Ker(M_1 \to M) \simeq N_2$. Thus we have an exact sequence

$$0 \to N_2 \to M_1 \to M \to 0. \qquad (1)$$

Theorem 1. *The regular interconnection problem has a solution if and only if there exists a homomorphism $N_2 \to D^q$ such that j is equal to the composition*

$$N_2 \to D^q \xrightarrow{\rho_1} M_1.$$

Proof. "If": Let M_2 be the cokernel of $N_2 \to D^q$, and let $\rho_2 : D^q \to M_2$ be the canonical epimorphism. We claim that $\Sigma_2 = (M_2, \rho_2)$ is a solution of the problem.

Indeed, the diagram

$$\begin{array}{ccccccccc} 0 \to & N_2 & \to & D^q & \to & M_2 & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & M_1 & \to & M_1 \oplus M_2 & \to & M_2 & \to 0 \end{array}$$

is commutative. Letting $L = Coker \begin{bmatrix} \rho_1 \\ -\rho_2 \end{bmatrix}$, by the snake lemma, we have an exact sequence

$$0 \to Ker \begin{bmatrix} \rho_1 \\ -\rho_2 \end{bmatrix} \to 0 \to M \to L \to 0.$$

From this, we see that Σ_2 is an admissible controller and that $M \simeq L$. From the first square in the diagram above, we have a commutative square

$$\begin{array}{ccc} M_1 & \to & M_1 \oplus M_2 \\ \downarrow & & \downarrow \\ M & \simeq & L \end{array}.$$

This, in turn, yields the following commutative square:

$$\begin{array}{ccc} M_1 & = & M_1 \\ \downarrow & & \downarrow \\ M & \simeq & L \end{array}.$$

Composing the vertical arrows here with $D^q \to M_1$, we find that the square

$$\begin{array}{ccc} D^q & = & D^q \\ \downarrow & & \downarrow \\ M & \simeq & L \end{array}$$

commutes. Hence, $\Sigma \simeq \Sigma_1 \wedge \Sigma_2$.

"Only if": Let $\Sigma_2 = (M_2, \rho_2)$ be a solution of the problem. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 \to & D^q & \to & M_1 \oplus M_2 & \to & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & M_2 & = & M_2 & \to & 0 & \end{array}.$$

Again applying the snake lemma, we get an exact sequence

$$0 \to Ker(\rho_2) \to M_1 \to M \to 0.$$

In view of (1), this yields an isomorphism $Ker(\rho_2) \simeq N_2$ for which the square

$$\begin{array}{ccc} Ker(\rho_2) & \simeq & N_2 \\ \downarrow & & \downarrow \\ M_1 & = & M_1 \end{array} \qquad (2)$$

is commutative. Further, from the first square of the above diagram, we get a homomorphism $Ker(\rho_2) \to M_1$ such that the square

$$\begin{array}{ccc} Ker(\rho_2) & \to & M_1 \\ \downarrow & & \downarrow \\ D^q & \to & M_1 \oplus M_2 \end{array}$$

commutes. This yields the following commutative square:

$$\begin{array}{ccc} Ker(\rho_2) & \to & M_1 \\ \downarrow & & \| \\ D^q & \to & M_1 \end{array}.$$

This together with (2) completes the proof. □

Remark. Applying the functor $Hom(N_2, -)$ to the exact sequence $0 \to N_1 \to D^q \to M_1 \to 0$, we get the following exact sequence:

$$0 \to Hom(N_2, N_1) \to Hom(N_2, D^q) \to Hom(N_2, M_1).$$

We can see that if there exists an admissible controller implementing Σ, then all such controllers can be parameterized via $Hom(N_2, N_1)$.

3. The case of a strongly controllable plant

We keep the notation of the previous section. The following is a reformulation of a result by Rocha and Wood [3].

Theorem 2. *The following conditions are equivalent:*
(a) The plant Σ_1 is strongly controllable;
(b) Every Fliess model that is more powerful than the plant is regularly implementable;
(c) The zero Fliess model is regularly implementable.

Proof. $(a) \Rightarrow (b)$ Because M_1 is projective, the epimorphism $\rho_1 : D^q \to M_1$ splits, that is, there is a homomorphism $\mu : M_1 \to D^q$ such that $\rho_1 \circ \mu = id$. The homomorphism μj certainly satisfies the condition of Theorem 1.

$(b) \Rightarrow (c)$ Trivial.

$(c) \Rightarrow (a)$ We have $M = 0$. Hence, $N = D^q$, $N_2 = D^q/N_1$ and $j : N_2 \to M_1$ is bijective. By Theorem 1, there is $\phi : N_2 \to D^q$ for which $\rho_1 \circ \phi = j$. Because j is an isomorphism, we get $\rho_1 \circ (\phi j^{-1}) = id$.

Thus the epimorphism $D^q \to M_1$ splits, and consequently M_1 is a direct summand of D^q.

The proof is complete. □

Example 2. Let the plant Σ_1 be the model from Example 1. Because Σ_1 is not strongly controllable, the regular implementability property fails for the zero model.

Closing, we want to present an explicit formula for a particular controller that implements Σ under the assumption that Σ_1 is strongly controllable.

Let R_1 be a representation of Σ_1 and R a representation of Σ, and let p and g be the row numbers of these representations. Let Y be a right inverse of R_1, that is, a polynomial matrix such that $R_1 Y = I$. We have an exact sequence

$$0 \to D^p \stackrel{R_1^t}{\to} D^q \stackrel{\rho_1}{\to} M_1 \to 0,$$

which splits. Hence, there is a homomorphism $\mu : M_1 \to D^q$ such that

$$R_1^t Y^t + \mu\rho_1 = I \quad \text{and} \quad \rho_1\mu = I. \tag{3}$$

Put $\Sigma_2 = (M_2, \rho_2)$, where M_2 is the cokernel of μj and ρ_2 is the canonical epimorphism $D^q \to M_2$. By the proofs of Theorems 1 and 2, this is an admissible controller implementing the desired system.

Consider the diagram

$$\begin{array}{ccccc} D^g & \stackrel{R^t}{\to} & N & & \\ & & \pi \downarrow & \searrow & \\ & & N_2 & \to & M_1 & \to & D^q, \end{array}$$

where π is the canonical homomorphism and the south-east arrow is the restriction of ρ_1 on N. The triangle in this diagram is commutative. Hence,

$$\mu\rho_1 R^t = (\mu j) \circ (\pi R^t).$$

Because $R^t : D^g \to N$ and $\pi : N \to N_2$ are surjective, so is the homomorphism πR^t; because $j : N_2 \to M_1$ and $\mu : M_1 \to D^q$ are injective, so must be their composition μj. It follows that the image of $\mu\rho_1 R^t$ is equal to that of μj. The image of the latter is equal to the kernel of ρ_2, and thus the sequence

$$D^g \stackrel{\mu\rho_1 R^t}{\to} D^q \stackrel{\rho_2}{\to} M_2 \to 0$$

is exact. This means that $R_2 = R\rho_1^t \mu^t$ is a representation of the controller Σ_2. Finally, by (3), we have

$$R_2 = R(I - YR_1).$$

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