BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 54, 2007 (79–96)

A boundary-transmission problem with first and second kind conditions for the Helmholtz equation in Besov and Bessel potential spaces

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Received 19 December 2006   Accepted 7 September 2007

Abstract

The paper is devoted to the analysis of a boundary-transmission value problem for the Helmholtz equation. The conditions under consideration are originated from problems of wave diffraction by a union of strips where first and second kind boundary conditions are assumed. The well-posedness of the problem is obtained in Besov and Bessel potential spaces for a set of integrability and smoothness orders depending on the boundary parameters.

Keywords: Boundary-transmission problem, Helmholtz equation, invertibility, well-posedness, Fredholm property, Bessel potential spaces, Besov spaces

1. Introduction

The physical motivations behind the present study arise from the problem of electromagnetic plane wave diffraction by a union of strips where first and second kind boundary conditions are considered (cf. [13] for a survey about such kind of problems). The problem is here formulated for the real wave number case, and worked out in a framework of Bessel potential and Besov spaces with general integrability and smoothness indices.

Depending on the kind of boundary conditions in use, and on the geometry of the problem, different studies have been made about the type of the spaces which are more appropriate to deal with such kind of problems (cf., e.g., [13] and [17]). In fact, a great part of the mathematical interest in this kind of problems is devoted to the question of finding out the largest set of possible spaces where it is possible to show the existence of a unique solution, and continuous dependence on the known data. Within this goal, it is relevant to mention that for the real wave number case some of the known methods (which work for the complex wave number case) fail. This last peculiarity can be seen, e.g., in the standard techniques of the Wiener-Hopf method (where there is the necessity to use integral representations through Fourier transformations). A combination of this last method with a strong concern about the use of appropriate classes of Bessel potential spaces can be found in the work of Meister and Speck and their collaborators (cf., e.g., [12]–[14] and [19]–[20]). For a general framework about these kind of problems we also refer the reader to [5], [15], and [23].

In the present paper we provide new results on the possible smoothness orders and integrability parameters of Bessel potential and Besov spaces for the well-posedness of the announced problem in the geometrical case of a union of strips with first and second kind boundary conditions on their both faces. Thus, the present work improve the known corresponding results in two directions: generalization of the geometrical situation (several strips instead of only one), and generalization of the spaces in consideration (having Bessel potential and Besov spaces with general smoothness and integrability indices). Therefore, in particular, we improve the results of [3] where the one strip geometry was considered in Hilbert Bessel potential spaces.

2. Formulation of the problem

In this section we establish the general notation which will allow already the mathematical formulation of the problem.

As usual, $\mathcal{S}(\mathbb{R}^m)$ denotes the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^m)$ the dual space of tempered distributions on $\mathbb{R}^m$. The Bessel potential space $H^s_p(\mathbb{R}^m)$, with $s \in \mathbb{R}$ and $1 < p < +\infty$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^m)$ such that

$$\|\varphi\|_{H^s_p(\mathbb{R}^m)} = \left\| \mathcal{F}^{-1}\Big(1+|\xi|^2\Big)^{s/2} \cdot \mathcal{F}\varphi \right\|_{L_p(\mathbb{R}^m)}$$

is finite. As the notation indicates, $\|\cdot\|_{H^s_p(\mathbb{R}^m)}$ is a norm for the space $H^s_p(\mathbb{R}^m)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^m$.

For a given domain, $\mathcal{D}$, on $\mathbb{R}^m$ we denote by $\widetilde{H}^s_p(\mathcal{D})$ the closed subspace of $H^s_p(\mathbb{R}^m)$ whose elements have supports in $\mathcal{D}$, and $H^s_p(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^m$ that belong to $H^s_p(\mathbb{R}^m)$. The space $\widetilde{H}^s_p(\mathcal{D})$ is endowed with the subspace topology, and on $H^s_p(\mathcal{D})$ we put the norm of the quotient space $H^s_p(\mathbb{R}^m)/\widetilde{H}^s_p(\mathbb{R}^m\backslash\mathcal{D})$. Obviously, these definitions are valid for L_p spaces. Note that the spaces $H^0_p(\mathbb{R}^m_+)$ and $\widetilde{H}^0_p(\mathbb{R}^m_+)$ can be identified, and we will denote them by $L_p(\mathbb{R}^m_+)$. We will also make use of Bessel potential spaces defined on (smooth) manifolds which may be defined in a standard way by using partitions of unit and local diffeomorphisms (cf., e.g., [9, §21]).

For defining the Besov spaces we will use the sets

$$\begin{aligned} M_0 &= \{\xi \mid \xi \in \mathbb{R}^m\,,\ |\xi| \le 2\}\,, \\ M_j &= \{\xi \mid \xi \in \mathbb{R}^m\,,\ 2^{j-1} \le |\xi| \le 2^{j+1}\}\,,\ \ j = 1, 2, \dots . \end{aligned}$$

For $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \le q < +\infty$, the Besov spaces $B^s_{p,q}$ are defined by

$$B^s_{p,q}(\mathbb{R}^m) = \left\{ \varphi \mid \varphi \in \mathcal{S}'(\mathbb{R}^m)\,;\ \varphi = \sum_{j=0}^{+\infty} a_j\,;\ \operatorname{supp}\mathcal{F}a_j \subset M_j\,; \right. \\ \left. \|\{a_j\}\| := \left[\sum_{j=0}^{+\infty} \left(2^{sj} \|a_j\|_{L_p(\mathbb{R}^m)} \right)^q \right]^{1/q} < +\infty \right\},$$

and for $s \in \mathbb{R}$, $1 < p < +\infty$, and $q = +\infty$, the definition is changed to

$$B^s_{p,+\infty}(\mathbb{R}^m) = \left\{ \varphi \mid \varphi \in \mathcal{S}'(\mathbb{R}^m)\,;\ \varphi = \sum_{j=0}^{+\infty} a_j\,;\ \operatorname{supp}\mathcal{F}a_j \subset M_j\,; \right. \\ \left. \|\{a_j\}\| := \sup_j 2^{sj} \|a_j\|_{L_p(\mathbb{R}^m)} < +\infty \right\}$$

(where in both cases the convergence of the series $\sum_{j=0}^{+\infty} a_j(x)$ is considered in $\mathcal{S}'(\mathbb{R}^m)$). Additionally, for $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \le q \le +\infty$, we use the norm

$$\|\varphi\|_{B^s_{p,q}(\mathbb{R}^m)} = \inf_{\varphi = \sum_{j=0}^{+\infty} a_j} \|\{a_j\}\|$$

which makes $B^s_{p,q}(\mathbb{R}^m)$ a Banach space. For a given domain $\mathcal{D}$ on $\mathbb{R}^m$, the spaces $B^s_{p,q}(\mathcal{D})$ and $\widetilde{B}^s_{p,q}(\mathcal{D})$ are defined in a similar way to what was done for $H^s_p(\mathcal{D})$ and $\widetilde{H}^s_p(\mathcal{D})$, respectively. We refer to [21] for the general properties of these Bessel potential and Besov spaces.

Use will often be made of the restriction operator

$$r_\Sigma : H^s_p(\mathbb{R}) \to H^s_p(\Sigma) \quad \left[B^s_{p,q}(\mathbb{R}) \to B^s_{p,q}(\Sigma) \right]$$

that, by the definition of $H^s_p(\Sigma)$, can be identified with the quotient map from $H^s_p(\mathbb{R})$ onto $H^s_p(\mathbb{R})/\widetilde{H}^s_p(\mathbb{R}\backslash\Sigma)$, where $\Sigma \subseteq \mathbb{R}_+$ (and in a similar way for the Besov spaces).

Let

$$\Omega := \mathbb{R}^2 \backslash \left\{ (x,0) \in \mathbb{R}^2 \,|\, x \in [a_0,a_1] \cup [a_2,a_3] \cup \cdots \cup [a_{n-1},a_n] \right\}$$

where $0 = a_0 < a_1 < a_2 < \cdots < a_n < +\infty$, and n is some fixed positive integer number. On Ω (and also in some of its subsets) we will make use of the usual local Bessel potential spaces $H^s_{p,\mathrm{loc}}(\Omega)$, and local Besov spaces $B^s_{p,q,\mathrm{loc}}(\Omega)$. From now on we will use the notation

$$\mathcal{U} :=]a_0,a_1[\cup]a_2,a_3[\cup \cdots \cup]a_{n-1},a_n[\,.$$

From the mathematical point of view, for $\epsilon \ge 0$, we are interested in studying the problem of existence and uniqueness of an element $u \in L_p(\mathbb{R}^2)$, with $u_{|\Omega} \in H^{1+\epsilon}_{p,\mathrm{loc}}(\Omega)$

$[B^{1+\epsilon}_{p,q,\mathrm{loc}}(\Omega)]$, such that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)u = 0 \qquad \text{in} \qquad \Omega\,, \tag{1}$$

$$\begin{cases} u_0^+ + d_0 u_0^- = h_0 \\ d_1 u_1^+ + u_1^- = h_1 \end{cases} \qquad \text{on} \qquad \mathcal{U}\,, \tag{2}$$

where the real wave number $k \in \mathbb{R}\backslash\{0\}$ is given, as well as $d_0, d_1 \in \mathbb{C}$, and the Dirichlet and Neumann traces are denoted by $u_0^\pm = u_{|y=\pm 0}$ and $u_1^\pm = (\partial u/\partial y)_{|y=\pm 0}$, respectively. In addition, for $j = 0, 1$, the elements $h_j \in H^{\epsilon-j+1-1/p}(\mathcal{U})$ $[B^{\epsilon-j+1-1/p}(\mathcal{U})]$ are arbitrarily given since the dependence on the data is to be studied for well-posedness. Note that the Dirichlet type condition in (2) can be understood in the trace sense, while the second condition is understood in the distributional sense (cf. [11]).

Due to the fact that the specific parameters d_0 and d_1 may take different values, (2) represents in fact a class of conditions. In particular, for $d_0 = d_1 = 0$ our conditions (2) becomes the fundamental Dirichlet-Neumann conditions. However, since such idealized model of perfect conductance and isolation is unnatural from the physical point of view, it makes sense to consider perturbations ($d_0, d_1 \neq 0$) of the Dirichlet-Neumann problem; cf. [13, 19, 20].

As a strategy for reaching to the final main result, we will begging by analyzing the problem in the Hilbert Bessel potential space setting. Then, in a second stage, we will consider the problem with the full generality of $p \in]1, +\infty[$ in both Bessel potentials and Besov spaces (also with $q \in [1, +\infty]$ in this last case).

We emphasize that we are not dealing with a dissipative medium, reflected in such a case by the condition $\Im\mathrm{m}\, k \neq 0$. Thus, it is natural to require that the eventual solution of (1)–(2) should also satisfy the *Sommerfeld radiation condition at infinity*, $u \in \mathrm{Som}(\Omega)$:

$$\frac{\partial}{\partial |x|}u(x) - i|k|u(x) = \mathcal{O}\left(|x|^{-\frac{3}{2}}\right) \quad \text{for} \quad |x| \to \infty \tag{3}$$

see, e.g., [5].

We will refer to the Problem $\mathcal{P}$ as the one characterized by (1)–(3).

We would like to mention here that the above boundary value problem is equivalent to another formulation where it is only required that the Helmholtz equation is fulfilled in the upper and lower half-plane, and – in addition – it is incorporated the transmission condition

$$\begin{cases} u_0^+ - u_0^- = 0 \\ u_1^+ - u_1^- = 0 \end{cases} \qquad \text{on} \qquad \mathbb{R} \setminus \mathcal{U}\,. \tag{4}$$

This last type of formulation is the most common in several papers that consider such kind of wave diffraction problems (cf., e.g., [13]).

3. Uniqueness of solution

Let us assume that $\mathcal{U}$ is a part of some smooth and simple curve S which separates the space $\mathbb{R}^2$ into two disjoint domains Ω^+ and $\Omega^- = \mathbb{R}^2 \backslash \Omega^+$, such that Ω^+ is a bounded domain and $S = \partial\Omega^\pm$ (cf. Figure 1). In this case, we will denote by $\eta(z) = (n_1(z), n_2(z))$ the outward unit normal vector at the point $z \in S = \partial\Omega^+$ (see Figure 1).

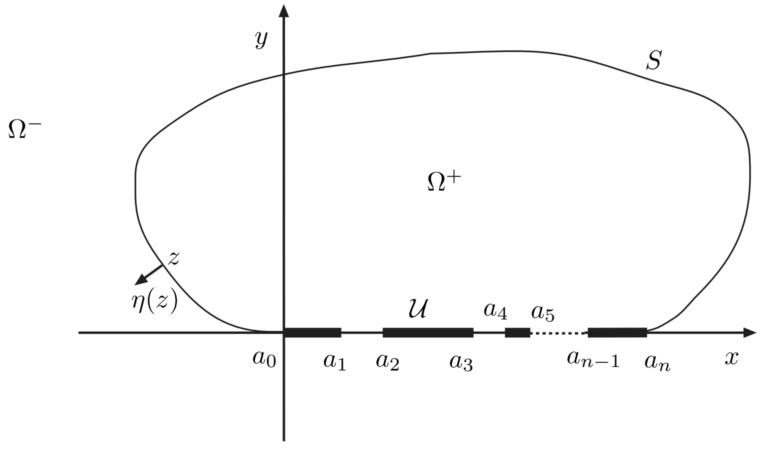

Figure 1: The geometry of the problem.

Theorem 3.1 *If $d_0 = d_1$, $p = 2$ and $\epsilon = 0$, then the homogeneous Problem $\mathcal{P}$ (i.e., Problem $\mathcal{P}$ in the particular case of $h_0 = h_1 = 0$) has only the trivial solution $u = 0$ in the space $H^1_{2,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$.*

Proof. Let R be a sufficiently large positive number and $B(R)$ be the disk centered at the origin with radius R, such that $\Omega_+ \subset B(R)$. Set $\Omega_R^- := \Omega^- \cap B(R)$, and let u be a solution of the homogeneous Problem $\mathcal{P}$. Then the Green formula for u and its complex conjugate u in the domains Ω^+ and Ω_R^- yields

$$\int_{\Omega^+} \left[|\nabla u|^2 - k^2|u|^2\right] dx = \left\langle [\partial_n u]_S^+, [u]_S^+ \right\rangle_S , \tag{5}$$

$$\int_{\Omega_R^-} \left[|\nabla u|^2 - k^2|u|^2\right] dx = -\left\langle [\partial_n u]_S^-, [u]_S^- \right\rangle_S + \int_{\partial B(R)} \partial_n u \, u \, dS . \tag{6}$$

Here the symbols $[\cdot]^\pm$ denote the non-tangential limit values on S from $\Omega^\pm$ and $\langle\cdot,\cdot\rangle_S$, $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ denote the duality brackets between the dual spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$, or $\widetilde{H}^{-\frac{1}{2}}(\mathcal{U})$ and $H^{\frac{1}{2}}(\mathcal{U})$, or $H^{-\frac{1}{2}}(\mathcal{U})$ and $\widetilde{H}^{\frac{1}{2}}(\mathcal{U})$. For regular functions, e.g., $f, g \in L_p(\mathcal{M})$, we have

$$\langle f, g\rangle_{\mathcal{M}} = \int_{\mathcal{M}} f \, g \, d\mathcal{M} ,$$

for $\mathcal{M} = S$ or $\mathcal{M} = \mathcal{U}$.

Note that the interior regularity in Ω of solutions of the Helmholtz equation (1) gives us $[u]^+_{S\setminus\mathcal{U}} = [u]^-_{S\setminus\mathcal{U}}$ and $[\partial_n u]^+_{S\setminus\mathcal{U}} = [\partial_n u]^-_{S\setminus\mathcal{U}}$. Then due to the condition $d_0 = d_1$, by summing up (5) and (6) we obtain

$$
\begin{aligned}
\int_{\Omega^+\cup\Omega^-_R} \Big[|\nabla u|^2 - k^2|u|^2\Big]dx &= \langle u_1^+, u_0^+\rangle_{\mathcal{U}} - \langle u_1^-, u_0^-\rangle_{\mathcal{U}} + \int_{\partial B(R)} \partial_n u\, u\, dS \\
&= -\langle u_1^+, d_0 u_0^-\rangle_{\mathcal{U}} + \langle d_1 u_1^+, u_0^-\rangle_{\mathcal{U}} + \int_{\partial B(R)} \partial_n u\, u\, dS \\
&= \int_{\partial B(R)} \partial_n u\, u\, dS\,. \qquad (7)
\end{aligned}
$$

Since we are assuming R to be sufficiently large, we can apply the Sommerfeld radiation condition on the circle $\partial B(R)$. Let us now separate the imaginary part of the equation (7) and use the fact that $u \in \mathrm{Som}(\Omega)$ implies $u(x) = \mathcal{O}(|x|^{-\frac{1}{2}})$ as $|x| \to \infty$. Then we obtain

$$
k \int_{\partial B(R)} |u|^2\, dS = \mathcal{O}\left(R^{-1}\right)\,,
$$

which yields

$$
\lim_{R\to\infty} \int_{\partial B(R)} |u|^2\, dS = 0\,.
$$

Due to the well-known Rellich-Vekua theorem we finally obtain $u = 0$ in Ω [22].

4. Potential operators for the representation of the solution

In the present section we will introduce potential operators, acting between Bessel potential and Besov spaces. In particular, such potential operators allow a representation of the solutions of Problem $\mathcal{P}$. In addition, in the next section, these potential operators will help us to study the regularity of the solutions of the Problem $\mathcal{P}$. Without lost of generality, we will assume that $k > 0$ (the complementary case of $k < 0$ runs with obvious changes).

Let us denote the standard fundamental solution of the Helmholtz equation by

$$
\Gamma(x,k) \quad := \quad -\frac{i}{4} H_0^{(1)}(k|x|)\,,
$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero. Recall that the fundamental function $\Gamma(x,k)$ satisfies the Sommerfeld radiation condition and it has the following logarithmic singularity in the neighborhood of the origin

$$
\Gamma(x,k) \quad = \quad -\frac{1}{2\pi} \ln \frac{1}{|x|} + \mathcal{O}(|x|^2 \ln|x|), \qquad |x| < \frac{1}{2}
$$

(cf. [5, §3.4]). Then the corresponding single and double layer potentials are of the form

$$\begin{aligned} V(\psi)(x) &= \int_S \Gamma(x-y,k)\psi(y)dS\,, \quad x \notin S\,, \\ W(\varphi)(x) &= \int_S [\partial_{n(y)}\Gamma(x-y,k)]\varphi(y)dS\,, \quad x \notin S\,, \end{aligned}$$

where ψ and φ are density functions.

Now, by the standard arguments of Green identities, we obtain the following integral representation of a radiating solution $u \in H^1_{2,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$ of the homogeneous Helmholtz equation (cf. [22])

$$\pm \int_{\partial\Omega^\pm} \left\{ [\partial_{n(y)}\Gamma(x-y,k)][u(y)]^\pm - \Gamma(x-y,k)[\partial_{n(y)}u(y)]^\pm \right\} dS = \begin{cases} u(x) & \text{for} \quad x \in \Omega^\pm \\ 0 & \text{for} \quad x \in \Omega^\mp \end{cases} . \tag{8}$$

In particular, by summing up we have

$$u(x) = W(u_0^+ - u_0^-)(x) - V(u_1^+ - u_1^-)(x)\,, \quad x \in \Omega\,. \tag{9}$$

Let us now recall some properties of the above introduced potentials. We have the following mapping properties of the single and double layer potentials (cf., e.g., [6]) in Bessel potential and Besov spaces with indices $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \le q \le +\infty$:

$$\begin{aligned} V \ &: \ H_2^s(S) \to H_{2,\mathrm{loc}}^{s+\frac{3}{2}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) & & \left[H_2^s(S) \to H_2^{s+\frac{3}{2}}(\Omega^+)\right], \\ &: \ B_{p,p}^s(S) \to H_{p,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) & & \left[B_{p,p}^s(S) \to H_p^{s+1+\frac{1}{p}}(\Omega^+)\right], \\ &: \ B_{p,q}^s(S) \to B_{p,q,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) & & \left[B_{p,q}^s(S) \to B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)\right], \\ W \ &: \ H_2^s(S) \to H_{2,\mathrm{loc}}^{s+\frac{1}{2}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) & & \left[H_2^s(S) \to H_2^{s+\frac{1}{2}}(\Omega^+)\right] \\ &: \ B_{p,p}^s(S) \to H_{p,\mathrm{loc}}^{s+\frac{1}{p}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) & & \left[B_{p,p}^s(S) \to H_p^{s+\frac{1}{p}}(\Omega^+)\right] \\ &: \ B_{p,q}^s(S) \to B_{p,q,\mathrm{loc}}^{s+\frac{1}{p}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) & & \left[B_{p,q}^s(S) \to B_{p,q}^{s+\frac{1}{p}}(\Omega^+)\right] \end{aligned} \tag{10}$$

where the spaces with the Sommerfeld radiation condition have the topology induced by the corresponding Besov and Bessel potential spaces. The following jump relations are well-known

$$\begin{aligned} &[V(\psi)]_S^+ = [V(\psi)]_S^- =: \mathcal{H}(\psi), \quad [\partial_n V(\psi)]_S^\pm =: [\mp\tfrac{1}{2}I + \mathcal{K}](\psi) \\ &[W(\varphi)]_S^\pm =: [\pm\tfrac{1}{2}I + \mathcal{K}^*](\varphi), \quad [\partial_n W(\varphi)]_S^+ = [\partial_n W(\varphi)]_S^- =: \mathcal{L}(\varphi) \end{aligned} \tag{11}$$

where I denotes the identity operator, and

$$\mathcal{H}(\psi)(z) := \int_S \Gamma(z-y,k)\psi(y)dS\,, \quad z \in S \tag{12}$$

$$\mathcal{K}(\psi)(z) := \int_S [\partial_{n(z)}\Gamma(z-y,k)]\psi(y)dS\,, \quad z \in S \tag{13}$$

$$\mathcal{K}^*(\varphi)(z) := \int_S [\partial_{n(y)}\Gamma(y-z,k)]\varphi(y)dS\,, \quad z \in S \tag{14}$$

$$\mathcal{L}(\varphi)(z) := \lim_{x\to z\in S} \partial_{n(x)} \int_S [\partial_{n(y)}\Gamma(y-x,k)]\varphi(y)dS\,, \quad z \in S\,. \tag{15}$$

Theorem 4.1 *Let $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \leq q \leq +\infty$. The operators* (12)–(15) *considered now on $\mathcal{U}$ are pseudo-differential operators of order* -1, 0, 0, *and* 1 *(respectively) which can be restricted/extended to the following bounded mappings:*

$$r_{\mathcal{U}}\mathcal{H} \;:\; \widetilde{H}^s_p(\mathcal{U}) \to H^{s+1}_p(\mathcal{U}) \qquad \left[\widetilde{B}^s_{p,q}(\mathcal{U}) \to B^{s+1}_{p,q}(\mathcal{U})\right], \tag{16}$$

$$r_{\mathcal{U}}\mathcal{K},\; r_{\mathcal{U}}\mathcal{K}^* \;:\; \widetilde{H}^s_p(\mathcal{U}) \to H^{s}_p(\mathcal{U}) \qquad \left[\widetilde{B}^s_{p,q}(\mathcal{U}) \to B^{s}_{p,q}(\mathcal{U})\right], \tag{17}$$

$$r_{\mathcal{U}}\mathcal{L} \;:\; \widetilde{H}^{s+1}_p(\mathcal{U}) \to H^{s}_p(\mathcal{U}) \qquad \left[\widetilde{B}^{s+1}_{p,q}(\mathcal{U}) \to B^{s}_{p,q}(\mathcal{U})\right]. \tag{18}$$

In addition:

(i) *The operator $r_{\mathcal{U}}\mathcal{H} : \widetilde{H}^s_p(\mathcal{U}) \to H^{s+1}_p(\mathcal{U})$ is Fredholm if and only if*

$$\frac{1}{p} - \frac{3}{2} < s < \frac{1}{p} - \frac{1}{2}\,. \tag{19}$$

(ii) *The operator $r_{\mathcal{U}}\mathcal{L} : \widetilde{H}^{s+1}_p(\mathcal{U}) \to H^{s}_p(\mathcal{U})$ is Fredholm if and only if* (19) *holds true.*

(iii) *The operator $r_{\mathcal{U}}\mathcal{H} : \widetilde{B}^s_{p,q}(\mathcal{U}) \to B^{s+1}_{p,q}(\mathcal{U})$ is Fredholm if* (19) *and $1 \leq q \leq +\infty$ hold true.*

(iv) *The operator $r_{\mathcal{U}}\mathcal{L} : \widetilde{B}^{s+1}_{p,q}(\mathcal{U}) \to B^{s}_{p,q}(\mathcal{U})$ is Fredholm if* (19) *and $1 \leq q \leq +\infty$ hold true.*

(v) *All the operators in* (16) *and* (18) *are invertible provided that* (19) *(and $1 \leq q \leq +\infty$) hold true.*

(vi) *The operators in (17) are compact for every $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \leq q \leq +\infty$.*

The last result was derived using the methods detailed presented in §5 of [7]. The proof is here omitted for a matter of brevity.

Inserting in (8) the single and double layer potentials, and then applying formulas (11), we obtain the identities

$$\mathcal{K}^*\mathcal{H} = \mathcal{H}\mathcal{K}\,, \quad \mathcal{L}\mathcal{K}^* = \mathcal{K}\mathcal{L}\,, \quad \mathcal{H}\mathcal{L} = -\frac{1}{4}I + (\mathcal{K}^*)^2\,, \quad \mathcal{L}\mathcal{H} = -\frac{1}{4}I + (\mathcal{K})^2\,. \tag{20}$$

It can be proved (cf. [6]) that the homogeneous principal symbols

$$\sigma(\mathcal{H})(x,\xi)\,, \quad \sigma(\mathcal{K})(x,\xi)\,, \quad \sigma(\mathcal{K}^*)(x,\xi)\,, \quad \sigma(\mathcal{L})(x,\xi)$$

have the following properties

$$\sigma(\mathcal{K})(x,\xi) =: iK(x,\xi), \quad \sigma(\mathcal{K}^*)(x,\xi) =: -iK(x,\xi), \quad \sigma(\mathcal{K})(x,-\xi) = \sigma(\mathcal{K})(x,\xi), \tag{21}$$

and

$$\sigma(\mathcal{H})(x,\xi) = \sigma(\mathcal{H})(x,-\xi) =: H(x,\xi), \quad \sigma(\mathcal{L})(x,\xi) = \sigma(\mathcal{L})(x,-\xi) =: L(x,\xi), \tag{22}$$

where K, H, L are real valued functions. Recall that $-H$ and L are positive definite on $\mathcal{U}$, i.e., for all $\xi \in \mathbb{R}\backslash\{0\}$, $x \in \mathcal{U}$ and $\eta \in \mathbb{C}$

$$-H(x,\xi)\eta \cdot \eta \geq C_1|\xi|^{-1}|\eta|^2\,, \quad L(x,\xi)\eta \cdot \eta \geq C_2|\xi||\eta|^2\,, \tag{23}$$

for $C_j = \mathrm{const} > 0$, $j = 1, 2$. Moreover, from (20) we easily derive

$$\sigma(\mathcal{K}^*) = \sigma(\mathcal{K}) = 0\,, \quad -\sigma(\mathcal{H})\sigma(\mathcal{L}) = -\sigma(\mathcal{L})\sigma(\mathcal{H}) = \frac{1}{4}\,; \tag{24}$$

(cf. also proposition (vi) of Theorem 4.1).

5. Existence and regularity of solutions on Bessel potential and Besov spaces

In the present section, for the spaces H^s_p or $B^s_{p,q}$, with $1 \leq s = 1+\epsilon < 2$, $2 \leq p < +\infty$, and $1 \leq q \leq +\infty$, we will analyze the existence of a solution of the corresponding Problem $\mathcal{P}$, in the above indicated form

$$u(x) = W(\varphi)(x) - V(\psi)(x)\,, \quad x \in \Omega\,, \tag{25}$$

where the unknown densities φ and ψ are related to the source u and its normal derivative by the following equations (cf. (9)):

$$\varphi = u_0^+ - u_0^-\,, \qquad \psi = u_1^+ - u_1^-\,. \tag{26}$$

If assuming that $d_0 \neq -1$ and $d_1 \neq -1$, then the boundary condition (2) can be equivalently rewritten in the form

$$\begin{cases} u_0^+ - c_0(u_0^+ - u_0^-) &= f_0 \\ c_1(u_1^+ - u_1^-) + u_1^- &= f_1 \end{cases}\,, \tag{27}$$

where $c_j := d_j/(1+d_j)$ and $f_j := h_j/(1+d_j)$ for $j = 0, 1$.

The representation formula (25) together with the jump relations (11) and the boundary conditions (27) lead to the following system of pseudo-differential equations on $\mathcal{U}$ with unknowns φ and ψ

$$\left\{ \begin{array}{lcl} r_{\mathcal{U}}\big[-\mathcal{H}\psi + (c_0 + \frac{1}{2} + \mathcal{K}^*)\varphi\big] & = & f_0 \\ r_{\mathcal{U}}\big[(c_1 - \frac{1}{2} - \mathcal{K})\psi + \mathcal{L}\varphi\big] & = & f_1 \end{array} \right. . \tag{28}$$

This shows us the interest to deal with the operator

$$\mathcal{A} := \begin{pmatrix} -\mathcal{H} & -c_0 + \frac{1}{2} + \mathcal{K}^* \\ c_1 - \frac{1}{2} - \mathcal{K} & \mathcal{L} \end{pmatrix}$$

and with

$$\Phi := (\psi, \varphi)^\top , \quad F := (f_0, f_1)^\top .$$

Then, from (28), we have

$$r_{\mathcal{U}}\mathcal{A}\Phi = F \quad \text{on} \quad \mathcal{U} , \tag{29}$$

where $\Phi \in \widetilde{H}_p^{\epsilon-\frac{1}{p}}(\mathcal{U}) \times \widetilde{H}_p^{\epsilon+1-\frac{1}{p}}(\mathcal{U})$ $[\widetilde{B}_{p,q}^{\epsilon-\frac{1}{p}}(\mathcal{U}) \times \widetilde{B}_{p,q}^{\epsilon+1-\frac{1}{p}}(\mathcal{U})]$ and $F \in H_p^{\epsilon+1-\frac{1}{p}}(\mathcal{U}) \times H_p^{\epsilon-\frac{1}{p}}(\mathcal{U})$ $[B_{p,q}^{\epsilon+1-\frac{1}{p}}(\mathcal{U}) \times B_{p,q}^{\epsilon-\frac{1}{p}}(\mathcal{U})]$.

For convenience we also write

$$\mathcal{A} =: \mathrm{Op}(a(x,\xi)),$$

where $a(x,\xi)$ is the complete (matrix) symbol of the operator $\mathcal{A}$ and Op is a pseudo-differential action based on the Fourier transform, i.e.,

$$\mathrm{Op}(a(x,\xi))u(x) := \frac{1}{2\pi} \iint e^{i(x-y)\xi} a(x,\xi)u(y)\, dy\, d\xi .$$

By $\sigma(\mathcal{A})(x,\xi)$ we denote the homogeneous principal (matrix) symbol of the pseudo-differential operator $\mathcal{A}$; here $x \in \mathcal{U}$, $\xi \in \mathbb{R}\backslash\{0\}$.

Let us also denote by $a \cdot b$ the scalar product of two vectors $a = (a_1, \ldots, a_N)^\top$, $b = (b_1, \ldots, b_N)^\top$ according to $a \cdot b = \sum_{k=1}^N a_k \overline{b_k}$, where the overbar denotes the complex conjugation.

Lemma 5.1 *If $d_0 = d_1 \neq -1$ then for arbitrary $x \in \mathcal{U}$, $|\xi| = 1$, and $\eta \in \mathbb{C}^2$ the inequality*

$$\Re\mathrm{e}\, [\sigma(\mathcal{A})(x,\xi)\eta \cdot \eta] \geq \alpha|\eta|^2 \tag{30}$$

holds for some $\alpha = \mathrm{const} > 0$.

Proof. We have

$$\sigma(\mathcal{A}) = \begin{pmatrix} -H & -c_0 + \frac{1}{2} - iK \\ c_1 - \frac{1}{2} - iK & L \end{pmatrix}$$

for arbitrary $x \in \mathcal{U}$, $|\xi| = 1$. Then, considering $\eta = (\eta_1, \eta_2)^\top$, we obtain

$$\begin{aligned} \sigma(\mathcal{A})\eta \cdot \eta &= -H\eta_1 \cdot \eta_1 + L\eta_2 \cdot \eta_2 - c_0\eta_2 \cdot \eta_1 + c_1\eta_1 \cdot \eta_2 \\ &\quad + \frac{1}{2}(\eta_2 \cdot \eta_1 - \eta_1 \cdot \eta_2) - i(K\eta_1 \cdot \eta_2 + K\eta_2 \cdot \eta_1) \, . \end{aligned}$$

Since $c_0 = c_1$, we get

$$\Re\mathrm{e}\,[\sigma(\mathcal{A})(x,\xi)\eta \cdot \eta] = -H\eta_1 \cdot \eta_1 + L\eta_2 \cdot \eta_2 \, .$$

Applying (23) in the last identity, we obtain (30).

Lemma 5.2 [21, §2.10.3] *Let $s, r \in \mathbb{R}$, $1 < p < +\infty$, $1 \le q \le +\infty$, and consider the operators*

$$\begin{aligned} \Lambda_+^s(D) &:= (D+i)^s \\ \Lambda_-^s(D) &:= r_{\mathbb{R}_+}(D-i)^s \ell^{(r)} \, , \end{aligned}$$

where $(D \pm i)^{\pm s} = \mathcal{F}^{-1}(\xi \pm i)^{\pm s} \cdot \mathcal{F}$, and

$$\ell^{(r)} : H_p^r(\mathbb{R}_+) \to H_p^r(\mathbb{R}) \quad [B_{p,q}^r(\mathbb{R}_+) \to B_{p,q}^r(\mathbb{R})]$$

is any bounded extension operator in these corresponding spaces (which particular choice does not change the definition of $\Lambda_-^s(D)$). These operators arrange isomorphisms in the following space settings

$$\begin{aligned} \Lambda_+^s(D) &: \widetilde{H}_p^r(\mathbb{R}_+) \to \widetilde{H}_p^{r-s}(\mathbb{R}_+) \quad \left[\widetilde{B}_{p,q}^r(\mathbb{R}_+) \to \widetilde{B}_{p,q}^{r-s}(\mathbb{R}_+)\right] , \\ \Lambda_-^s(D) &: H_p^r(\mathbb{R}_+) \to H_p^{r-s}(\mathbb{R}_+) \quad \left[B_{p,q}^r(\mathbb{R}_+) \to B_{p,q}^{r-s}(\mathbb{R}_+)\right] . \end{aligned}$$

Let us define $\Lambda_\pm^s(\xi) := (\xi \pm i)^s = (1+\xi^2)^{\frac{s}{2}} \exp\left\{s\, i \arg(\xi \pm i)\right\}$, for $s \in \mathbb{R}$,

$$\mathcal{E}_+(\xi) := \begin{pmatrix} \Lambda_+^{1/p}(\xi) & 0 \\ 0 & \Lambda_+^{-1+1/p}(\xi) \end{pmatrix} \tag{31}$$

$$\mathcal{E}_-(\xi) := \begin{pmatrix} \Lambda_-^{1-1/p}(\xi) & 0 \\ 0 & \Lambda_-^{-1/p}(\xi) \end{pmatrix} , \tag{32}$$

and assume convenient branches for the power functions $(\xi \pm i)^s$ such that

$$\lim_{\xi \to -\infty} \left(\frac{\xi - i}{\xi + i}\right)^s = \exp\{-2\pi i s\} \, , \quad \lim_{\xi \to +\infty} \left(\frac{\xi - i}{\xi + i}\right)^s = 1 \, . \tag{33}$$

The last elements allow us to consider now the matrix

$$\begin{aligned}
&\boldsymbol{a}(x,\xi) = \sigma(\mathcal{E}_-)(\xi)\;\sigma(\mathcal{A})(x,\xi)\;\sigma(\mathcal{E}_+)(\xi)\\
&= \begin{pmatrix} -\sigma(\Lambda_-^{1-1/p})\,H\,\sigma(\Lambda_+^{1/p}) & \sigma(\Lambda_-^{1-1/p})\big[-c_0+\frac{1}{2}-iK\big]\sigma(\Lambda_+^{-1+1/p}) \\ \sigma(\Lambda_-^{-1/p})\big[c_1-\frac{1}{2}-iK\big]\sigma(\Lambda_+^{1/p}) & \sigma(\Lambda_-^{-1/p})\,L\,\sigma(\Lambda_+^{-1+1/p}) \end{pmatrix}.
\end{aligned} \tag{34}$$

Therefore, due to (33) and from the properties of H, K, L (cf. (21) and (22)), we obtain

$$\boldsymbol{a}(x,+1) = \begin{pmatrix} -H(x,1) & -c_0+\frac{1}{2}-iK(x,1) \\ c_1-\frac{1}{2}-iK(x,1) & L(x,1) \end{pmatrix}, \tag{35}$$

$$\boldsymbol{a}(x,-1) = -\exp\{2\pi i/p\} \begin{pmatrix} -H(x,1) & c_0-\frac{1}{2}+iK(x,1) \\ -c_1+\frac{1}{2}+iK(x,1) & L(x,1) \end{pmatrix}. \tag{36}$$

Lemma 5.3 *If $d_0 = d_1 \neq -1$ and $p = 2$, then the inequality*

$$\Re\mathrm{e}\,\big[\boldsymbol{a}(x,\xi)\eta\cdot\eta\big] \geq \alpha_1|\eta|^2$$

holds for some $\alpha_1 = \mathrm{const} > 0$ and all $x \in \mathcal{U}$, $|\xi| = 1$, $\eta \in \mathbb{C}^2$.

Proof. First note that $\sigma(\mathcal{E}_+)(\xi) = \sigma(\mathcal{E}_-)^{\top}(\xi)$ for $|\xi| = 1$, and $p = 2$. Then, using Lemma 5.1, we derive from (34) that

$$\begin{aligned}
\Re\mathrm{e}\,\big[\boldsymbol{a}(x,\xi)\eta\cdot\eta\big] &= \Re\mathrm{e}\,\big[\sigma(\mathcal{E}_-)(\xi)\sigma(\mathcal{A})(x,\xi)\sigma(\mathcal{E}_+)(\xi)\eta\cdot\eta\big]\\
&= \Re\mathrm{e}\,\big[\sigma(\mathcal{A})(x,\xi)\sigma(\mathcal{E}_+)(\xi)\eta\cdot\sigma(\mathcal{E}_-)^{\top}(\xi)\eta\big]\\
&= \Re\mathrm{e}\,\big[\sigma(\mathcal{A})(x,\xi)\sigma(\mathcal{E}_+)(\xi)\eta\cdot\sigma(\mathcal{E}_+)(\xi)\eta\big]\\
&\geq \alpha|\sigma(\mathcal{E}_+)(\xi)\eta|^2\\
&= \alpha(|\sigma(\Lambda_+^{1/2})(\xi)|^2|\eta_1|^2 + |\sigma(\Lambda_+^{-1/2})(\xi)|^2|\eta_2|^2)\\
&\geq \alpha_1|\eta|^2\,,
\end{aligned}$$

which is fulfilled for arbitrary $x \in \mathcal{U}$, $|\xi| = 1$, $\eta = (\eta_1,\eta_2)^{\top} \in \mathbb{C}^2$.

Theorem 5.1 *Let $d_0 = d_1 \neq -1$, $\epsilon = 0$, and $p = 2$. Then the Problem $\mathcal{P}$ has a unique solution u in the space $H^1_{2,\mathrm{loc}}(\Omega)\cap\mathrm{Som}(\Omega)$, which is representable in the form* (25) *with the densities φ and ψ defined by the uniquely solvable pseudo-differential equation* (29).

Proof. We will apply the local principle (see e.g. [18], [8]) and *freeze the coefficients.* Then, we find out that $\mathcal{A}$ in (29) is a Fredholm operator if and only if its local representatives

$$\mathrm{Op}(a(x_0,\xi)) : H^{-\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}) \to H^{\frac{1}{2}}(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R}) \quad \text{if } x_0 \in \mathcal{U} \tag{37}$$

$$r_{\mathbb{R}_+}\mathrm{Op}(a(x_0,\xi)) : \widetilde{H}^{-\frac{1}{2}}(\mathbb{R}_+) \times \widetilde{H}^{\frac{1}{2}}(\mathbb{R}_+) \to H^{\frac{1}{2}}(\mathbb{R}_+) \times H^{-\frac{1}{2}}(\mathbb{R}_+) \quad \text{if } x_0 \in \partial\mathcal{U} \tag{38}$$

are locally invertible at every $x_0 \in \mathcal{U}$.

Let us first consider the case when $x_0 \in \mathcal{U}$. From (30) we have that the ellipticity condition (in the sense of Douglis-Nirenberg)

$$\inf\{|\det\sigma(\mathcal{A})(x,\xi)| : x \in \mathcal{U}\ , |\xi| = 1\} > 0 \tag{39}$$

holds. Thus the operator $\mathrm{Op}(a(x_0,\xi))$ in (37) is Fredholm. Moreover, due to the well-known properties of the operators $\mathcal{H}$, $\mathcal{K}$, $\mathcal{K}^*$, $\mathcal{L}$ and to the condition $d_0 = d_1$, we conclude that $\mathcal{A}$ is a self-adjoint operator. This fact together with the strong ellipticity condition (30) yields $\mathrm{Ind}\,\mathcal{A} = 0$ and $\mathrm{Ker}\,\mathcal{A} = \{0\}$ (in the space $H^{-\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$), i.e., the operator (37) is invertible. In addition, its inverse is simply given by $\mathrm{Op}(a^{-1}(x_0,\xi))$.

For $x_0 \in \partial\mathcal{U}$, the ellipticity condition (39) is necessary but not sufficient to the invertibility of the operator in (38). Therefore, we lift the operator (38) to the equivalent pseudo-differential (matrix) operator of order zero

$$\boldsymbol{\mathcal{A}} := \mathcal{E}_-(D)\mathrm{Op}(a(x_0,\xi))\mathcal{E}_+(D) : L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+)\ , \tag{40}$$

where the corresponding principal symbol $\boldsymbol{a}(x_0,\xi)$ is defined in (34). From Lemmata 5.1 and 5.3 we have that $\sigma(\mathcal{A})$ is elliptic in the Douglis-Nirenberg sense, and the lifted symbol $\boldsymbol{a}$ is strongly elliptic; then, due to [8, Theorem 3.6], we have that the operator (38) is Fredholm and $\mathrm{Ind}\, r_{\mathbb{R}_+}\mathrm{Op}(a(x_0,\xi)) = 0$. Now let us prove that (38) has a trivial kernel. Consider $r_{\mathbb{R}_+}\mathrm{Op}(a(x_0,\xi))\Phi = 0$, for $\Phi = (\phi_1,\phi_2)^\top$. Arguing analogously as in Lemma 5.1 we have

$$\begin{aligned}
\Re\mathrm{e}\langle r_{\mathbb{R}_+}\mathrm{Op}(a(x_0,\xi))\Phi,\Phi\rangle_{\mathcal{U}} \quad &= \quad \Re\mathrm{e}[-\langle\mathcal{H}\phi_1,\phi_1\rangle_{\mathcal{U}} + \langle\mathcal{L}\phi_2,\phi_2\rangle_{\mathcal{U}} - c_0\langle\phi_2,\phi_1\rangle_{\mathcal{U}} \\
&\qquad + c_1\langle\phi_1,\phi_2\rangle_{\mathcal{U}} + \frac{1}{2}(\langle\phi_2,\phi_1\rangle_{\mathcal{U}} - \langle\phi_1,\phi_2\rangle_{\mathcal{U}}) \\
&\qquad + \langle\mathcal{K}^*\phi_2,\phi_1\rangle_{\mathcal{U}} - \langle\mathcal{K}\phi_1,\phi_2\rangle_{\mathcal{U}}] \\
&\geq \quad M||\Phi||^2
\end{aligned}$$

where $M = \mathrm{const} > 0$, and $||\cdot||$ denotes the norm in $H^{-\frac{1}{2}}(\mathbb{R}_+) \times H^{\frac{1}{2}}(\mathbb{R}_+)$. This yields $\Phi = 0$, and therefore $\mathrm{Ker}\, r_{\mathbb{R}_+}\mathrm{Op}(a(x_0,\xi)) = 0$. Thus we have the local invertibility of the operator (38). As a consequence, the operator

$$\mathcal{A} : \widetilde{H}^{-\frac{1}{2}}(\mathbb{R}_+) \times \widetilde{H}^{\frac{1}{2}}(\mathbb{R}_+) \to H^{\frac{1}{2}}(\mathbb{R}_+) \times H^{-\frac{1}{2}}(\mathbb{R}_+)$$

is Fredholm and $\operatorname{Ind}\mathcal{A} = 0$. Due to Theorem 3.1 and the representation formula (25) (see also (10)), we have that equation (29) is uniquely solvable for all $F \in H^{-\frac{1}{2}}(\mathcal{U}) \times H^{\frac{1}{2}}(\mathcal{U})$.

We are now in conditions to establish the regularity results for the solution of our problem. For this purpose, let us start by considering the matrix

$$Q(x) = \boldsymbol{a}^{-1}(x,+1)\,\boldsymbol{a}(x,-1)\ , \quad x \in \partial\mathcal{U}\ , \tag{41}$$

and their two eigenvalues λ_1 and λ_2. Then we set

$$\delta_j := \delta_j(x) = \frac{1}{2\pi i}\log\lambda_j(x)\ ,$$

where the branch in the logarithmic function is chosen in regard to the inequalities

$$\frac{1}{p} - 1 < -\Re\mathrm{e}\,\delta_j \le \frac{1}{p}\ , \quad j = 1,2\ . \tag{42}$$

Theorem 5.2 *Suppose that $d_0 = d_1 \neq -1$, $1 < p < +\infty$, $1 \le q \le +\infty$ and $\epsilon \in \mathbb{R}$ such that the following inequalities hold*

$$\frac{1}{p} - 1 < -\Re\mathrm{e}\,\delta_j < \frac{1}{p}\,, \qquad \frac{1}{p} - 1 < \epsilon - \Re\mathrm{e}\,\delta_j < \frac{1}{p}$$

for both $j = 1,2$, and $x \in \partial\mathcal{U}$. Then the operators

$$r_{\mathcal{U}}\mathcal{A} \ : \ \widetilde{H}_p^{-\frac{1}{p}+\epsilon}(\mathcal{U}) \times \widetilde{H}_p^{1-\frac{1}{p}+\epsilon}(\mathcal{U}) \to H_p^{1-\frac{1}{p}+\epsilon}(\mathcal{U}) \times H_p^{-\frac{1}{p}+\epsilon}(\mathcal{U}) \tag{43}$$

$$: \ \widetilde{B}_{p,q}^{-\frac{1}{p}+\epsilon}(\mathcal{U}) \times \widetilde{B}_{p,q}^{1-\frac{1}{p}+\epsilon}(\mathcal{U}) \to B_{p,q}^{1-\frac{1}{p}+\epsilon}(\mathcal{U}) \times B_{p,q}^{-\frac{1}{p}+\epsilon}(\mathcal{U}) \tag{44}$$

are Fredholm, $\operatorname{Ind} r_{\mathcal{U}}\mathcal{A} = 0$ and $\operatorname{Ker} r_{\mathcal{U}}\mathcal{A}$ is independent of the smoothness order ϵ and of the integrability parameters p and q.

Proof. The Bessel potential space setting case follows directly as a combination of the above Lemma 5.3 and, e.g., [4, Theorem 1.12]. The Besov potential space setting case follows by interpolation (cf. the interpolation details below).

Note that Theorems 5.2 and 3.1 imply that the operator $r_{\mathcal{U}}\mathcal{A}$ in (43) is invertible provided that

$$\frac{1}{p} - 1 + \Re\mathrm{e}\,\delta_j < \epsilon < \frac{1}{p} + \Re\mathrm{e}\,\delta_j\ , \quad j = 1,2\ . \tag{45}$$

Indeed, for $\epsilon = 0$ we have a uniqueness result and therefore $\operatorname{Ker} r_{\mathcal{U}}\mathcal{A} = 0$ (see also (25) and (10)). Since the kernel is independent of the smoothness ϵ, then $\operatorname{Ker} r_{\mathcal{U}}\mathcal{A} = 0$ for all ϵ satisfying condition (45).

As for the Besov space setting case (due to the present knowledge for Bessel potential space case), we can use the interpolation properties between these spaces.

In fact, we can use the real interpolation method to write the Besov spaces in terms of the Bessel potential spaces in the way that

$$\begin{aligned} B^s_{p,q}(\mathcal{U}) &= \left[H^{s_1}_{p_1}(\mathcal{U}), H^{s_2}_{p_2}(\mathcal{U})\right]_{\theta,q} \\ \widetilde{B}^s_{p,q}(\mathcal{U}) &= \left[\widetilde{H}^{s_1}_{p_1}(\mathcal{U}), \widetilde{H}^{s_2}_{p_2}(\mathcal{U})\right]_{\theta,q} \end{aligned} \tag{46}$$

where $s = (1-\theta)s_1 + \theta s_2$, $0 < \theta < 1$, $s_1, s_2 \in \mathbb{R}$, $1 < p_1, p_2 < +\infty$, $1/p = (1-\theta)/p_1 + \theta/p_2$. In particular, in view of (46), if an operator $T : \widetilde{H}^s_p(\mathcal{U}) \to H^{s-r}_p(\mathcal{U})$ is bounded for $1 < p < +\infty$ and $s_1 < s < s_2$, then $T : \widetilde{B}^s_{p,q}(\mathcal{U}) \to B^{s-r}_{p,q}(\mathcal{U})$ will be also bounded for $1 < p < +\infty$, $1 \le q \le +\infty$ and $s_1 < s < s_2$. Therefore, the assertion for Besov spaces follows from the assertion for Bessel potential spaces and the just presented interpolation property since the operator

$$r_{\mathcal{U}}\mathcal{A} : \widetilde{B}^{\epsilon-\frac{1}{p}}_{p,q}(\mathcal{U}) \times \widetilde{B}^{\epsilon+1-\frac{1}{p}}_{p,q}(\mathcal{U}) \to B^{\epsilon+1-\frac{1}{p}}_{p,q}(\mathcal{U}) \times B^{\epsilon-\frac{1}{p}}_{p,q}(\mathcal{U})$$

and its inverse

$$(r_{\mathcal{U}}\mathcal{A})^{-1} : B^{\epsilon+1-\frac{1}{p}}_{p,q}(\mathcal{U}) \times B^{\epsilon-\frac{1}{p}}_{p,q}(\mathcal{U}) \to \widetilde{B}^{\epsilon-\frac{1}{p}}_{p,q}(\mathcal{U}) \times \widetilde{B}^{\epsilon+1-\frac{1}{p}}_{p,q}(\mathcal{U})$$

are bounded for the same parameters of smoothness and integrability as in the Bessel potential space setting (and for $1 \le q \le +\infty$). Thus, the operator $r_{\mathcal{U}}\mathcal{A}$ in (44) is invertible provided that

$$\frac{1}{p} - 1 + \Re\mathrm{e}\,\delta_j < \epsilon < \frac{1}{p} + \Re\mathrm{e}\,\delta_j \ , \quad j = 1, 2 \ ,$$

and $1 \le q \le +\infty$.

All these are assembled in the next final result, where $1/p + \Re\mathrm{e}\,\delta_j$ is computed and denoted by $\mu_\pm$. In particular, to calculate the value of the elements δ_j (for $j = 1, 2$), cf. (42), we apply (24) and take into account the proposition (vi) of Theorem 4.1. Thus, a direct computation from (35), (36) and (41) yields

$$Q = \frac{-\exp\{2\pi i/p\}}{1/4 + |c_0 - 1/2|^2} \begin{pmatrix} 1/4 - |c_0 - 1/2|^2 & -2L(c_0 - 1/2) \\ -2H(c_0 - 1/2) & 1/4 - |c_0 - 1/2|^2 \end{pmatrix} ,$$

and therefore the equation $\det(Q - \lambda I) = 0$ admits the following two solutions

$$\begin{aligned} \lambda_1 &= \frac{-\exp\{2\pi i/p\}}{1/4 + |c_0 - 1/2|^2}(1/4 - |c_0 - 1/2|^2 - |c_0 - 1/2| i) \ , \\ \lambda_2 &= \frac{-\exp\{2\pi i/p\}}{1/4 + |c_0 - 1/2|^2}(1/4 - |c_0 - 1/2|^2 + |c_0 - 1/2| i) \end{aligned}$$

(which are different if and only if $d_0 \neq 1$).

Theorem 5.3 *Let $d_0 = d_1 \neq -1$, $2 \leq p < +\infty$, and $1 \leq q \leq +\infty$. If the boundary data satisfy the condition*

$$(h_0, h_1) \in H_p^{1-\frac{1}{p}+\epsilon}(\mathcal{U}) \times H_p^{-\frac{1}{p}+\epsilon}(\mathcal{U}) \quad \left[B_{p,q}^{1-\frac{1}{p}+\epsilon}(\mathcal{U}) \times B_{p,q}^{-\frac{1}{p}+\epsilon}(\mathcal{U})\right]$$

for

$$0 \leq \epsilon < \mu_\pm := \frac{1}{p} + \frac{1}{2\pi} \arg \left[\frac{-\exp\{2\pi i/p\} \left(\frac{1}{4} - \left| \frac{d_0 - 1}{2(d_0+1)} \right|^2 \pm \left| \frac{d_0 - 1}{2(d_0+1)} \right| i \right)}{\frac{1}{4} + \left| \frac{d_0 - 1}{2(d_0+1)} \right|^2} \right], \tag{47}$$

with $0 < \mu_\pm < 1$, then the solution u of the Problem $\mathcal{P}$ possesses the following regularity

$$u \in H_{p,\mathrm{loc}}^{1+\epsilon}(\Omega) \cap \mathrm{Som}(\Omega) \quad \left[B_{p,q,\mathrm{loc}}^{1+\epsilon}(\Omega) \cap \mathrm{Som}(\Omega)\right].$$

Note that the condition $p \geq 2$ in the last theorem is due to the use of Green's formula in Section 3. In addition, we also like to mention that the above excluded case of $d_0 = d_1 = -1$ is not important from the physical point of view since it corresponds only to the situation where the differences between the traces in the upper and lower parts of the strip are known (cf. (2)). Anyway, also because of the particular form of (2) for this case and due to the transmission conditions in (4), it is clear that the existence and uniqueness of solution for such special situation require *compatibility conditions* [14]. These are then directly obtained when considering at the same time (2) and (4), which imply the necessity of taking

$$h_j \in r_{\mathcal{U}} \widetilde{H}^{\epsilon-j+1-1/p}(\mathcal{U}) \quad \left[r_{\mathcal{U}} \widetilde{B}^{\epsilon-j+1-1/p}(\mathcal{U})\right], \quad j = 1, 2.$$

Finally, it is also interesting to point out that several particular cases of the general class presented in Section 2 can now be taken in view of the corresponding invertibility and regularity properties. This is the case, as mentioned previously, when we consider $d_0 = d_1 = 0$ for which the class of problems in consideration take the form of the mixed Dirichlet–Neumann problem, or the so-called *Rawlins problem* for the strip. The Rawlins problem for complex wave numbers and a half-plane geometry was considered, e.g., in Bueyuekaksoy [1], Bueyuekaksoy et al [2], Heins [10], Meister [12], Meister et al [13], Rawlins [16] and Speck [19, 20].

Acknowledgements

This work was supported in part by *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro through the Portuguese Science Foundation (*FCT–Fundação para a Ciência e a Tecnologia*). The second author acknowledge the support from the Portuguese Science Foundation (*FCT–Fundação para a Ciência e a Tecnologia*) through the grant number SFRH/BPD/20524/2004.

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