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The impedance boundary-value problem of diffraction by a strip ☆

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Abstract

We consider the impedance boundary-value problem for the Helmholtz equation originated by the problem of wave diffraction by an infinite strip with imperfect conductivity. The two possible different situations of real and complex wave numbers are considered. Bessel potential spaces are used to deal with the problem, and the identification of corresponding operators of single and double layer potentials allow a reformulation of the problem into a system of integral equations. The well-posedness of the problem is obtained for a set of impedance parameters (and wave numbers), after the incorporation of some compatibility conditions on the data. At the end, an improvement of the regularity of the solution is derived for the same set of parameters previously considered.

Keywords: Helmholtz equation; Impedance problem; Boundary-value problem; Potential method; Pseudo-differential equations

1. Introduction

We consider a boundary-transmission problem for the Helmholtz equation which arises within the context of wave diffraction theory (for related works, we refer to [1–10], [12–15], and [17–25]). The problem is formulated in both situations of a real and a complex wave number k, and worked out in a framework of Bessel potential spaces. The boundary under consideration consists of an infinite strip where certain impedance conditions are assumed. Operator theoretical methods are used to deal with the problem and, as a consequence, several integral operators are constructed to help in the characterization of the problem. At the end, the well-posedness of the problem is shown for a range of smoothness orders of the Bessel potential spaces (which includes the finite energy norm space).

As initially mentioned, the physical motivations behind the present study arise from the classical problem of electromagnetic wave diffraction by a strip. Such kind of problems have been extensively discussed in the applied mathematics literature (including also other canonical geometries like the half-plane case originated by the famous Sommerfeld problem). Depending on the kind of boundary conditions in use, and on the geometry of the problem,

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different studies have been made about the type of the spaces which are more appropriate to deal with such kind of problems (cf., e.g., [15] and [19]).

Within this context, a great part of the mathematical interest is naturally devoted to the question of finding out the largest set of possible spaces where it is possible to show the existence of a unique solution, and continuous dependence on the known data. From the physical point of view, the choice of appropriate function spaces is influenced by the physical argument to have local finite scattering energy, and outgoing scattered waves (i.e., radiation condition).

It is also important to mention that in the real (non-complex) wave number case some of the known methods fail. This last peculiarity can be seen, e.g., in the standard techniques of the Wiener–Hopf method (where there is the necessity to use integral representations through Fourier transformations). A combination of this last method with a strong concern about the use of appropriate classes of Bessel potential spaces can be found in the work of Meister and Speck and their collaborators (cf., e.g., [14–16], and [21–23]).

In the present paper we provide new results on the possible smoothness orders of Bessel potential spaces for the well-posedness of the announced problem in the strip geometry case, with impedance boundary conditions on both faces of the strip.

Although we will present below the problem within the context of Hilbert Bessel potential spaces $H^s := H^s_2$ (also known as Sobolev spaces), we would like to mention that the present methods also work for settings where the integrability parameter $p = 2$ assumes values other than 2. For example, without significant changes, we can perform the below study for (Banach) Bessel potential spaces H^s_p (cf. [27]) with a Lebesgue index $p \in (1, +\infty)$. Anyway, from the Physical point of view it makes more sense to study those problems for $p = 2$.

As for other possible geometrical generalizations, it is clear that we can substitute the finite interval Σ by some other C^∞-smooth, non-self-intersecting, open curve, and all the below results will still be valid.

The paper is divided as follows. In Section 2, after introducing the necessary notation, we give the precise formulation of the problem to be dealt with in the next sections. In Section 3 the uniqueness of solution for the homogeneous problem is worked out. This is done by the use of Green's formula, the Sommerfeld radiation condition at infinity, and the Rellich–Vekua theorem. In Section 4 we recall the fundamental solution of the Helmholtz equation, and exhibit some properties of single and double potentials in the context of Bessel potential spaces. In the final section, the problem is reformulated into an equivalent system of integral operators where the previously introduced potentials appear. Then, for a range of impedance parameters, the problem is shown to be uniquely solvable upon a representation formula. In addition, in this Section 5, an improvement of the smoothness orders of the solution is also obtained based on a detailed study of certain pseudo-differential operators derived from the initial problem.

2. Formulation of the problem

In this section we establish the general notation, and present the mathematical formulation of the problem.

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\varphi\|_{H^s(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}\left(1+|\xi|^2\right)^{s/2} \cdot \mathcal{F}\varphi\right\|_{L_2(\mathbb{R}^n)} < \infty.$$

As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For Lipschitz domains $\mathcal{D}$ of $\mathbb{R}^n$ we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$ (note that these spaces are only defined for Lipschitz domains whereas the necessary spaces for the slit domain—which is not Lipschitz—are defined below). The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we put the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n \setminus \overline{\mathcal{D}})$. For unbounded open subsets $\mathcal{D} \subset \mathbb{R}^n$, we are also going to use the corresponding (usual) local Bessel potential spaces $H^s_{\mathrm{loc}}(\mathcal{D})$ (which formal definition can be found, e.g., in [26]).

Use will often be made of the restriction operator

$$r_\Sigma : H^s(\mathbb{R}) \to H^s(\Sigma)$$

that, by the definition of $H^s(\Sigma)$, can be identified with the quotient map from $H^s(\mathbb{R})$ onto $H^s(\mathbb{R})/\widetilde{H}^s(\mathbb{R}\setminus\overline{\Sigma})$, where $\Sigma \subseteq \mathbb{R}_+$. Anyway, in what follows we will not distinguish between elements of $\widetilde{H}^s(\Sigma)$ and $r_\Sigma \widetilde{H}^s(\Sigma)$.

Let us adopt the Cartesian axes $Oxyz$, and consider a differential formulation of the diffraction problem. We will deal with a boundary value problem originated by the diffraction of plane waves by an infinite strip located in the Oxz-plane (perpendicular to the y-axis), and with an edge chosen to coincide with the z-axis. We assume throughout this work that the material is invariant in the z-direction. Thus, in effect, the geometry of the problem is two-dimensional, leading us from $\mathbb{R}^3$ to $\mathbb{R}^2$, and consequently from the mentioned infinite strip to a finite interval $\Sigma =]0, a[$ (where $0 < a < \infty$).

Set $\Omega := \mathbb{R}^2 \setminus \overline{\Sigma}$. Besides $H^0(\Omega) = L_2(\Omega)$, we will also consider the Bessel potential spaces on Ω for smoothness orders $s > 0$, in the following way: first, for $s \in \mathbb{N}$, let $H^s(\Omega)$ be the completion of $\{u \in C^\infty(\Omega)\colon \|u\|_s < \infty\}$ with respect to the norm $\|u\|_s := \{\sum_{|\alpha|\leqslant s} \|\partial^\alpha u\|^2_{L_2(\Omega)}\}^{1/2}$; second, by using the complex $(\cdot,\cdot)_\theta$ interpolation functor, for non-integer orders $s = (1-\theta)s_0 + \theta s_1$ (with $s_0, s_1 \in \mathbb{N}_0$ and $0 < \theta < 1$), the space $H^s(\Omega)$ is defined by the complex interpolation method

$$H^s(\Omega) := \big(H^{s_0}(\Omega), H^{s_1}(\Omega)\big)_\theta.$$

Moreover, we say that $u \in H^s_{\mathrm{loc}(\infty)}(\Omega)$ if and only if $\psi|_\Omega u \in H^s(\Omega)$ for every $\psi \in C_0^\infty(\mathbb{R}^2)$.

We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\epsilon}_{\mathrm{loc}(\infty)}(\Omega)$ for a given wave number $k \in \mathbb{R}\setminus\{0\}$ or $u \in H^{1+\epsilon}(\Omega)$ for the case of $k \in \mathbb{C}\setminus\mathbb{R}$ (with $0 \leqslant \epsilon < 1$), subject to the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u = 0 \quad \text{in } \Omega, \tag{2.1}$$

and satisfying the boundary condition

$$\begin{cases} u_1^+ - ip^+u_0^+ = h^+, \\ u_1^- + ip^-u_0^- = h^- \end{cases} \quad \text{on } \Sigma, \tag{2.2}$$

where the impedance parameters $p^\pm \in \mathbb{C}$ are given, and the Dirichlet and Neumann traces are denoted by $u_0^\pm = u|_{y=\pm 0}$ and $u_1^\pm = -(\partial u/\partial y)|_{y=\pm 0}$, respectively. In addition, the elements $h^\pm \in H^{\epsilon-1/2}(\Sigma)$ are arbitrarily given since the dependence on the data is to be studied for well-posedness.

Note that the conditions in (2.2) are understood in the distributional sense (cf. [13]). The Dirichlet traces on Σ are due to the Trace Theorem, and the Neumann traces can be in fact defined by means of Green's formula and the duality arguments.

We emphasize that we will deal with both situations of a dissipative and a non-dissipative medium reflected by the conditions $\Im\mathrm{m}\, k \neq 0$ and $\Im\mathrm{m}\, k = 0$, respectively. Thus, for the real wave number case ($k \in \mathbb{R}$) it is natural to require that the eventual solution of (2.1)–(2.2) should also satisfy the *Sommerfeld radiation condition at infinity*, $u \in \mathrm{Som}(\Omega)$:

$$\frac{\partial}{\partial |x|}u(x) - i|k|u(x) = \mathcal{O}\big(|x|^{-\frac{3}{2}}\big) \quad \text{for } |x| \to \infty \tag{2.3}$$

see, e.g., [9]. Therefore, in what follows it is assumed that for a real k we have a radiating u (2.3), while for a complex k it is assumed that u exponentially decays at infinity.

In particular, conditions (2.1)–(2.2) lead to

$$\begin{cases} u_0^+ - u_0^- = 0, \\ u_1^+ - u_1^- = 0 \end{cases} \quad \text{on } \mathbb{R}\setminus\overline{\Sigma}. \tag{2.4}$$

Thus, when computing

$$h^+ - h^- = \big(u_1^+ - u_1^-\big) - \big(ip^+u_0^+ + ip^-u_0^-\big),$$

we realize that we necessarily need to have

$$h^+ - h^- \in \widetilde{H}^{\epsilon-\frac{1}{2}}(\Sigma). \tag{2.5}$$

This occurs because from (2.4) and (2.2) it follows $u_1^+ - u_1^- \in \widetilde{H}^{\epsilon-1/2}(\Sigma)$, and also due to $ip^+u_0^+ + ip^-u_0^- \in H^{\epsilon+1/2}(\Sigma)$ (which does not change the space in (2.5) because $H^{\epsilon+1/2}(\Sigma) \hookrightarrow \widetilde{H}^{\epsilon-1/2}(\Sigma)$ is continuously embedded). Anyway, since $\widetilde{H}^s(\Sigma) = H^s(\Sigma)$ if and only if $s \in (-1/2, 1/2)$ (cf., e.g., [27]), it follows that for the proposed values of ϵ the condition (2.5) is an additional restriction only for $\epsilon = 0$. Therefore, in the case of $\epsilon = 0$, (2.5) is a necessary condition for the above problem to be solvable and we may recognize it as a compatibility condition between the data [16].

From now on we will refer to *Problem* $\mathcal{P}_{Imp}$ as the one characterized by (2.1)–(2.3) and (2.5) in the case of $k \in \mathbb{R} \setminus \{0\}$, and by (2.1), (2.2) and (2.5) in the case of $k \notin \mathbb{R}$.

3. Uniqueness of solution

In view of the proof of the next uniqueness result, let us now assume that Σ is a part of some smooth and simple curve S which separates the space $\mathbb{R}^2$ into two disjoint domains Ω^+ and $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$, such that Ω^+ is a bounded domain and $S = \partial\Omega^\pm$. In this case, let us denote by $n(x) = (n_1(x), n_2(x))$ the outward unit normal vector at the point $x \in S = \partial\Omega^+$.

Theorem 1.

(i) *If* $\Im\mathrm{m}\, k = 0$, $\Re\mathrm{e}\, p^\pm \geqslant 0$ *and* $\epsilon = 0$, *then the homogeneous Problem* $\mathcal{P}_{Imp}$ (*i.e., Problem* $\mathcal{P}_{Imp}$ *in the particular case of* $h^\pm = 0$) *has only the trivial solution* $u = 0$ *in the space* $H^1_{\mathrm{loc}(\infty)}(\Omega) \cap \mathrm{Som}(\Omega)$.

(ii) *If* $\Im\mathrm{m}\, k \neq 0$, $\epsilon = 0$, *and one of the following situations holds*:

(a) $(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) > 0, \quad \Re\mathrm{e}\, p^\pm \geqslant 0,$

(b) $(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) < 0, \quad \Re\mathrm{e}\, p^\pm \leqslant 0,$

(c) $|\Im\mathrm{m}\, k| \geqslant |\Re\mathrm{e}\, k|, \quad \Im\mathrm{m}\, p^\pm \geqslant 0,$

(d) $\Re\mathrm{e}\, k = 0, \quad \Re\mathrm{e}\, p^+ \neq 0, \quad \Im\mathrm{m}\, p^- \geqslant \Im\mathrm{m}\, p^+ \dfrac{\Re\mathrm{e}\, p^-}{\Re\mathrm{e}\, p^+},$

(e) $\Re\mathrm{e}\, k = 0, \quad \Re\mathrm{e}\, p^- \neq 0, \quad \Im\mathrm{m}\, p^+ \geqslant \Im\mathrm{m}\, p^- \dfrac{\Re\mathrm{e}\, p^+}{\Re\mathrm{e}\, p^-},$

(f) $\Re\mathrm{e}\, k \neq 0, \quad p^- = 0, \quad \Re\mathrm{e}\, p^+ \neq 0, \quad \Im\mathrm{m}\, p^+ \neq 0,$
$(\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 - 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\dfrac{\Im\mathrm{m}\, p^+}{\Re\mathrm{e}\, p^+} \geqslant 0,$

(g) $\Re\mathrm{e}\, k \neq 0, \quad p^+ = 0, \quad \Re\mathrm{e}\, p^- \neq 0, \quad \Im\mathrm{m}\, p^- \neq 0,$
$(\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 - 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\dfrac{\Im\mathrm{m}\, p^-}{\Re\mathrm{e}\, p^-} \geqslant 0,$

then the homogeneous Problem $\mathcal{P}_{Imp}$ *has only the trivial solution* $u = 0$ *in the space* $H^1(\Omega)$.

Proof. (i) We start by considering the case of $\Im\mathrm{m}\, k = 0$, $\Re\mathrm{e}\, p^\pm \geqslant 0$ (and $\epsilon = 0$). Let R be a sufficiently large positive number and $B(R)$ be the disk centered at the origin with radius R, such that $\overline{\Omega_+} \subset B(R)$. Set $\Omega_R^- := \Omega^- \cap B(R)$, and let u be a solution of the homogeneous Problem $\mathcal{P}_{Imp}$. Then, Green's formula for u and its complex conjugate $\bar{u}$ (in the domains Ω^+ and Ω_R^-) yields

$$\int_{\Omega^+} \left[|\nabla u|^2 - k^2|u|^2\right] dx = \left\langle [\partial_n u]_S^+, [u]_S^+ \right\rangle_S, \tag{3.1}$$

$$\int_{\Omega_R^-} \left[|\nabla u|^2 - k^2|u|^2\right] dx = -\left\langle [\partial_n u]_S^-, [u]_S^- \right\rangle_S + \int_{\partial B(R)} \partial_n u \bar{u}\, dS. \tag{3.2}$$

Here, the symbols $[\cdot]^{\pm}$ denote the non-tangential limit values on S from $\Omega^{\pm}$ and $\langle\cdot,\cdot\rangle_S$, $\langle\cdot,\cdot\rangle_\Sigma$ denote the duality brackets between the dual spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$, or $\widetilde{H}^{-\frac{1}{2}}(\Sigma)$ and $H^{\frac{1}{2}}(\Sigma)$, or $H^{-\frac{1}{2}}(\Sigma)$ and $\widetilde{H}^{\frac{1}{2}}(\Sigma)$. For regular functions, e.g., $f, g \in L_2(\mathcal{M})$, we have

$$\langle f, g\rangle_{\mathcal{M}} = \int\limits_{\mathcal{M}} f\bar{g}\, d\mathcal{M},$$

for $\mathcal{M} = S$ or $\mathcal{M} = \Sigma$.

Note that the interior regularity in Ω of solutions of the Helmholtz equation (2.1) gives us $[u]^+_{S\setminus\overline{\Sigma}} = [u]^-_{S\setminus\overline{\Sigma}}$ and $[\partial_n u]^+_{S\setminus\overline{\Sigma}} = [\partial_n u]^-_{S\setminus\overline{\Sigma}}$. Then, by summing up (3.1) and (3.2), we obtain

$$\int\limits_{\Omega^+\cup\Omega^-_R} \big[|\nabla u|^2 - k^2|u|^2\big]\, dx = \big\langle u_1^+, u_0^+\big\rangle_\Sigma - \big\langle u_1^-, u_0^-\big\rangle_\Sigma + \int\limits_{\partial B(R)} \partial_n u\bar{u}\, dS$$

$$= ip^+\big\langle u_0^+, u_0^+\big\rangle_\Sigma + ip^-\big\langle u_0^-, u_0^-\big\rangle_\Sigma + \int\limits_{\partial B(R)} \partial_n u\bar{u}\, dS. \tag{3.3}$$

Since we are assuming R to be sufficiently large, we can apply the Sommerfeld radiation condition on the circle $\partial B(R)$. Let us now separate the imaginary part of Eq. (3.3), and use the fact that $u \in \mathrm{Som}(\Omega)$ implies $u(x) = \mathcal{O}(|x|^{-\frac{1}{2}})$ as $|x| \to \infty$. Then we obtain

$$\Re\mathrm{e}\, p^+ \int\limits_\Sigma \big|u_0^+\big|^2\, d\Sigma + \Re\mathrm{e}\, p^- \int\limits_\Sigma \big|u_0^-\big|^2 d\Sigma + |k| \int\limits_{\partial B(R)} |u|^2\, dS = \mathcal{O}\big(R^{-1}\big),$$

which yields

$$\lim_{R\to\infty} \int\limits_{\partial B(R)} |u|^2\, dS = 0,$$

due to the conditions $\Re\mathrm{e}\, p^{\pm} \geqslant 0$. Therefore, from the *Rellich–Vekua Theorem* [28], it follows that $u = 0$ in Ω.

(ii) For the second case, we can repeat the same reasoning as in the case (i) up to the step of formula (3.3). Additionally, since $\Im\mathrm{m}\, k \neq 0$, the function u exponentially decays at infinity and so (passing to the limit as $R \to \infty$ in (3.3)) it follows

$$\int\limits_\Omega \big[|\nabla u|^2 - k^2|u|^2\big]\, dx = ip^+\big\langle u_0^+, u_0^+\big\rangle_\Sigma + ip^-\big\langle u_0^-, u_0^-\big\rangle_\Sigma.$$

From to the real and imaginary parts of the last identity, we obtain

$$\int\limits_\Omega \big[|\nabla u|^2 + \big((\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2\big)|u|^2\big]\, dx$$

$$= -\Im\mathrm{m}\, p^+\big\langle u_0^+, u_0^+\big\rangle_\Sigma - \Im\mathrm{m}\, p^-\big\langle u_0^-, u_0^-\big\rangle_\Sigma - 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) \int\limits_\Omega |u|^2\, dx$$

$$= \Re\mathrm{e}\, p^+\big\langle u_0^+, u_0^+\big\rangle_\Sigma + \Re\mathrm{e}\, p^-\big\langle u_0^-, u_0^-\big\rangle_\Sigma.$$

Thus, for each of the conditions (a)–(g), it follows from the last two identities that $u = 0$ in Ω. □

4. The fundamental solution and potentials

In the present section (without lost of generality) we will assume that $k > 0$ when in the case of a real wave number (the complementer case of $k < 0$ runs with obvious changes).

Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\Gamma(x,k) := -\frac{i}{4} H_0^{(1)}\big(k|x|\big),$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero. Recall that the fundamental function $\Gamma(x,k)$ satisfies the Sommerfeld radiation condition (for a real positive k), and it has the following logarithmic singularity in the neighborhood of the origin

$$\Gamma(x,k) = -\frac{1}{2\pi} \ln \frac{1}{|x|} + \mathcal{O}\big(|x|^2 \ln|x|\big), \quad |x| < \frac{1}{2}$$

(cf. [9, §3.4]). Then the corresponding single and double layer potentials are of the form

$$V(\psi)(x) = \int_S \Gamma(x-y,k)\psi(y)\,dS, \quad x \notin S,$$

$$W(\varphi)(x) = \int_S \big[\partial_{n(y)}\Gamma(x-y,k)\big]\varphi(y)\,dS, \quad x \notin S,$$

where ψ and φ are density functions.

Note that by the standard arguments of the Green identities we obtain the following integral representation of a radiating solution $u \in H^1_{\mathrm{loc}(\infty)}(\Omega) \cap \mathrm{Som}(\Omega)$ of the homogeneous Helmholtz equation (cf. [28])

$$\pm \int_{\Omega^\pm} \big\{\big[\partial_{n(y)}\Gamma(x-y,k)\big]\big[u(y)\big]^\pm - \Gamma(x-y,k)\big[\partial_{n(y)}u(y)\big]^\pm\big\}\,dS = \begin{cases} u(x) & \text{for } x \in \Omega^\pm, \\ 0 & \text{for } x \in \Omega^\mp. \end{cases} \tag{4.1}$$

In particular, by summing up we have

$$u(x) = W\big(u_0^+ - u_0^-\big)(x) - V\big(u_1^+ - u_1^-\big)(x), \quad x \in \Omega. \tag{4.2}$$

Thus, formula (4.2) holds for a solution $u \in H^1_{\mathrm{loc}(\infty)}(\Omega) \cap \mathrm{Som}(\Omega)$ when k is a positive real number, as well as for a solution $u \in H^1(\Omega)$ of the Helmholtz equation when k is a complex (non-real) wave number.

Let us now recall some mapping properties of the above introduced single and double layer potentials (cf., e.g., [10] and [11]):

$$\begin{aligned} &V : H^s(S) \to H^{s+1+\frac{1}{2}}_{\mathrm{loc}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) \quad \big[H^s(S) \to H^{s+1+\frac{1}{2}}\big(\Omega^+\big)\big], \\ &W : H^s(S) \to H^{s+\frac{1}{2}}_{\mathrm{loc}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) \quad \big[H^s(S) \to H^{s+\frac{1}{2}}\big(\Omega^+\big)\big] \end{aligned} \tag{4.3}$$

for a real positive wave number k, and

$$V : H^s(S) \to H^{s+1+\frac{1}{2}}\big(\Omega^\pm\big), \qquad W : H^s(S) \to H^{s+\frac{1}{2}}\big(\Omega^\pm\big), \tag{4.4}$$

for a complex (non-real) k.

The following jump relations are also well-known

$$\begin{aligned} &\big[V(\psi)\big]_S^+ = \big[V(\psi)\big]_S^- =: \mathcal{H}(\psi), \qquad \big[\partial_n V(\psi)\big]_S^\pm =: \left[\mp\frac{1}{2}I + \mathcal{K}\right](\psi), \\ &\big[W(\varphi)\big]_S^\pm =: \left[\pm\frac{1}{2}I + \mathcal{K}^*\right](\varphi), \qquad \big[\partial_n W(\varphi)\big]_S^+ = \big[\partial_n W(\varphi)\big]_S^- =: \mathcal{L}(\varphi), \end{aligned} \tag{4.5}$$

where

$$\mathcal{H}(\psi)(z) := \int_S \Gamma(z-y,k)\psi(y)\,dS, \quad z \in S, \tag{4.6}$$

$$\mathcal{K}(\psi)(z) := \int_S \big[\partial_{n(z)}\Gamma(z-y,k)\big]\psi(y)\,dS, \quad z \in S, \tag{4.7}$$

$$\mathcal{K}^*(\varphi)(z) := \int_S \big[\partial_{n(y)}\Gamma(y-z,k)\big]\varphi(y)\,dS, \quad z \in S, \tag{4.8}$$

$$\mathcal{L}(\varphi)(z) := \lim_{x\to z\in S} \partial_{n(x)} \int_S \big[\partial_{n(y)}\Gamma(y-x,k)\big]\varphi(y)\,dS, \quad z \in S, \tag{4.9}$$

and I denotes the identity operator.

Theorem 2. *(See [11].) The operators* (4.6)–(4.9) *can be extended to the following bounded mappings*:

$$r_\Sigma\mathcal{H} : \widetilde{H}^s(\Sigma) \to H^{s+1}(\Sigma), \tag{4.10}$$

$$r_\Sigma\mathcal{K}, r_\Sigma\mathcal{K}^* : \widetilde{H}^s(\Sigma) \to H^s(\Sigma), \tag{4.11}$$

$$r_\Sigma\mathcal{L} : \widetilde{H}^{s+1}(\Sigma) \to H^s(\Sigma), \tag{4.12}$$

for arbitrary $s \in \mathbb{R}$.

Moreover, they are pseudo-differential operators of order -1, 0, 0, *and* 1, *respectively. The operators* (4.10) *and* (4.12) *are invertible provided that* $-1 < s < 0$.

Note that for the finite interval geometry the operators $r_\Sigma\mathcal{K}$ and $r_\Sigma\mathcal{K}^*$ turn out to be zero. However, we prefer to keep them in the reasoning in view of the announced possible generalization to other C^∞-smooth, non-self-intersecting, open curve, instead of Σ (cf. the introduction) in which case those operators may not be zero.

5. Existence and regularity of solutions

We look for a solution of Problem $\mathcal{P}_{Imp}$ in the form

$$u(x) = W(\varphi)(x) - V(\psi)(x), \quad x \in \Omega, \tag{5.1}$$

where the unknown densities φ and ψ are related to the source u and its normal derivative by the following equations (cf. (4.2)):

$$\varphi = u_0^+ - u_0^-, \qquad \psi = u_1^+ - u_1^-. \tag{5.2}$$

The boundary conditions (2.2) (together with (2.5)) can be equivalently rewritten in the form

$$\begin{cases} u_1^+ - u_1^- - ip^+u_0^+ - ip^-u_0^- = f_0, \\ u_1^+ - ip^+u_0^+ = f_1, \end{cases} \tag{5.3}$$

where $f_0 := h^+ - h^- \in \widetilde{H}^{-\frac{1}{2}}(\Sigma)$ and $f_1 := h^+ \in H^{-\frac{1}{2}}(\Sigma)$.

The representation formula (5.1) together with the jump relations (4.5) and the boundary conditions (5.3) lead to the following system of pseudo-differential equations on Σ with unknown φ and ψ

$$\begin{cases} r_\Sigma\{\psi I - ip^+[(\frac{1}{2}I + \mathcal{K}^*)\varphi - \mathcal{H}\psi] - ip^-[(-\frac{1}{2}I + \mathcal{K}^*)\varphi - \mathcal{H}\psi]\} = f_0, \\ r_\Sigma\{\mathcal{L}\varphi - (-\frac{1}{2}I + \mathcal{K})\psi - ip^+[(\frac{1}{2}I + \mathcal{K}^*)\varphi - \mathcal{H}\psi]\} = f_1. \end{cases} \tag{5.4}$$

Let us introduce the notations

$$\mathcal{A} := \begin{pmatrix} I + ip^+\mathcal{H} + ip^-\mathcal{H} & -ip^+(\frac{1}{2}I + \mathcal{K}^*) - ip^-(-\frac{1}{2}I + \mathcal{K}^*) \\ \frac{1}{2}I - \mathcal{K} + ip^+\mathcal{H} & \mathcal{L} - ip^+(\frac{1}{2}I + \mathcal{K}^*) \end{pmatrix}$$

and

$$\Phi := (\psi, \varphi)^\top, \qquad F := (f_0, f_1)^\top.$$

Then, from (5.4), we have

$$r_\Sigma\mathcal{A}\Phi = F \quad \text{on } \Sigma, \tag{5.5}$$

where $\Phi \in \widetilde{H}^{-\frac{1}{2}}(\Sigma) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$ and $F \in \widetilde{H}^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma)$.

Theorem 3. (i) *If* $\Im\mathrm{m}\, k = 0$ *and* $\Re\mathrm{e}\, p^{\pm} \geqslant 0$*, then Problem* $\mathcal{P}_{Imp}$ *has a unique solution in the space* $H^1_{\mathrm{loc}(\infty)}(\Omega) \cap \mathrm{Som}(\Omega)$*, which is representable in the form* (5.1) *with the densities* φ *and* ψ *defined by the uniquely solvable pseudo-differential equation* (5.5).

(ii) *If* $\Im\mathrm{m}\, k \neq 0$ *and one of the conditions* (a)–(g) *of Theorem* 1 *is satisfied, then Problem* $\mathcal{P}_{Imp}$ *has a unique solution in the space* $H^1(\Omega)$*, which is representable in the form* (5.1) *with the densities* φ *and* ψ *defined by the uniquely solvable pseudo-differential equation* (5.5).

Proof. We will analyse the invertibility of the matrix operator

$$r_\Sigma \mathcal{A} : \widetilde{H}^{-\frac{1}{2}}(\Sigma) \times \widetilde{H}^{\frac{1}{2}}(\Sigma) \to \widetilde{H}^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma). \tag{5.6}$$

Due to the mapping properties (4.10)–(4.11), and the continuous embeddings $\widetilde{H}^{1/2}(\Sigma) \hookrightarrow \widetilde{H}^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma) \hookrightarrow \widetilde{H}^{-1/2}(\Sigma)$, the operators

$$r_\Sigma I, r_\Sigma \mathcal{K}^* : \widetilde{H}^{\frac{1}{2}}(\Sigma) \to \widetilde{H}^{-\frac{1}{2}}(\Sigma), \qquad r_\Sigma I, r_\Sigma \mathcal{K}^* : \widetilde{H}^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma),$$

and

$$r_\Sigma \mathcal{H} : \widetilde{H}^{-\frac{1}{2}}(\Sigma) \to \widetilde{H}^{-\frac{1}{2}}(\Sigma)$$

are compact. Moreover, the operators in (4.11) are also compact, since their kernel have a weak singularity. Therefore, (5.6) is a compact perturbation of the triangular matrix operator with the invertible operators $r_\Sigma I$ and $r_\Sigma \mathcal{L}$ in the main diagonal. Consequently, (5.6) is a Fredholm operator with zero index. Then Theorem 1 implies that $\mathrm{Ker}\, r_\Sigma \mathcal{A} = \{0\}$, and therefore Eq. (5.5) is uniquely solvable for arbitrary $F \in \widetilde{H}^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma)$. □

Note that $p^+ = p^- = 0$ gives us so called crack type (Neumann) conditions on Σ:

$$\begin{cases} u_1^+ = h^+, \\ u_1^- = h^- \end{cases} \quad \text{on } \Sigma.$$

Then we easily obtain $\psi = h^+ - h^- \in \widetilde{H}^{-\frac{1}{2}}(\Sigma)$ and

$$\varphi = \mathcal{L}^{-1}\left\{\frac{1}{2}\left(h^- - h^+\right) + \mathcal{K}\left(h^+ - h^-\right)\right\} \in \widetilde{H}^{\frac{1}{2}}(\Sigma).$$

Theorem 4. *Let*

(i) $\Im\mathrm{m}\, k = 0$ *and* $\Re\mathrm{e}\, p^{\pm} \geqslant 0$*, or*
(ii) $\Im\mathrm{m}\, k \neq 0$ *and one of the conditions* (a)–(g) *of Theorem* 1 *be satisfied.*

If the boundary data satisfy the condition

$$\left(h^+, h^-\right) \in H^{-\frac{1}{2}+\epsilon}(\Sigma) \times H^{-\frac{1}{2}+\epsilon}(\Sigma), \quad \text{for } 0 \leqslant \epsilon < \frac{1}{2},$$

and additionally

$$h^+ - h^- \in \widetilde{H}^{-\frac{1}{2}}(\Sigma), \quad \text{if } \epsilon = 0,$$

then the solution u of Problem $\mathcal{P}_{Imp}$ *possesses the following regularity*

$$u \in H^{1+\epsilon}_{\mathrm{loc}(\infty)}(\Omega) \cap \mathrm{Som}(\Omega), \quad \text{in the case (i)},$$
$$u \in H^{1+\epsilon}(\Omega), \quad \text{in the case (ii)}.$$

Proof. The solvability result follows from Theorem 3. Due to the continuity results in (4.3)–(4.4), and the representation formula (5.1), it is sufficient to show that $(\psi, \varphi) \in \widetilde{H}^{-\frac{1}{2}+\epsilon}(\Sigma) \times \widetilde{H}^{\frac{1}{2}+\epsilon}(\Sigma)$.

The operator $\mathcal{A}$ can be written as

$$\mathcal{A} = \mathcal{B} + T,$$

where

$$\mathcal{B} := \begin{pmatrix} I & 0 \\ \frac{1}{2}I & \mathcal{L} \end{pmatrix}$$

and

$$T = (T_{jl})_{j,l=1,2} := r_\Sigma \begin{pmatrix} i(p^+ + p^-)\mathcal{H} & \frac{i}{2}(p^- - p^+) - i(p^+ + p^-)\mathcal{K}^* \\ -\mathcal{K} + ip^+\mathcal{H} & -ip^+(\frac{1}{2} + \mathcal{K}^*) \end{pmatrix}.$$

Due to Theorem 2, we have that

$$r_\Sigma \mathcal{B} : \widetilde{H}^{-\frac{1}{2}+\epsilon}(\Sigma) \times \widetilde{H}^{\frac{1}{2}+\epsilon}(\Sigma) \to \widetilde{H}^{-\frac{1}{2}+\epsilon}(\Sigma) \times H^{-\frac{1}{2}+\epsilon}(\Sigma)$$

is an invertible operator for every $-\frac{1}{2} < \epsilon < \frac{1}{2}$. The inverse is provided by the formula

$$r_\Sigma \mathcal{B}^{-1} = r_\Sigma \begin{pmatrix} I & 0 \\ -\frac{1}{2}\mathcal{L}^{-1} & \mathcal{L}^{-1} \end{pmatrix}.$$

The operator $r_\Sigma T$ is a compact operator between the spaces $\widetilde{H}^{-\frac{1}{2}}(\Sigma) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$ and $\widetilde{H}^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma)$. Moreover, the entries T_{11}, T_{12}, T_{21}, and T_{22} are pseudo-differential operators of orders -1, 0, 0 and 0, respectively (cf. Theorem 2).

Then, from $r_\Sigma \mathcal{A}\Phi = F$, we have $\Phi = r_\Sigma \mathcal{B}^{-1} F - r_\Sigma \mathcal{B}^{-1} T\Phi$, i.e.

$$\psi = f_0 - T_{11}\psi - T_{12}\varphi, \tag{5.7}$$

and

$$\varphi = r_\Sigma \mathcal{L}^{-1}\left(-\frac{1}{2}f_0 + f_1\right) - r_\Sigma \mathcal{L}^{-1}\left(T_{21} - \frac{1}{2}T_{11}\right)\psi - r_\Sigma \mathcal{L}^{-1}\left(T_{22} - \frac{1}{2}T_{11}\right)\varphi. \tag{5.8}$$

The equality (5.7) together with $(\psi, \varphi) \in \widetilde{H}^{-\frac{1}{2}}(\Sigma) \times \widetilde{H}^{\frac{1}{2}}(\Sigma)$, and the continuity properties of T_{11} and T_{12}, imply $\psi \in \widetilde{H}^{-\frac{1}{2}+\epsilon}(\Sigma)$. Taking into account this result for the equality (5.8), and noting that $\mathcal{L}^{-1}$ is a pseudo-differential operator of order -1, we finally obtain that $\varphi \in \widetilde{H}^{\frac{1}{2}+\epsilon}(\Sigma)$. □

To end up, we would like to point out that the above seven conditions (a)–(g) (cf. Theorem 1), as well as the conditions in proposition (i) of Theorem 1, include several possibilities of concrete wave numbers k and impedance parameters $p^\pm$ which are relevant from the physical point of view. In particular, it is pertinent to mention that the possible situations of $\Im m\, k \lesseqgtr 0$ correspond to a surrounding medium which is slightly conducting/lossy. As for the case $\Im m\, k = 0$, it corresponds to the situation of a non-dissipative medium. Additionally, the parameters $p^\pm$ incorporate already the contribution of the wave number k in their values and the factors $p^\pm/k$ describe the surface material impedances (being also functions of the angular frequency). Thus, from the physical point of view $p^\pm/k$ may assume a great variety of values reflecting each type of material (see [20, §4.1.1] and references cited there). For example, it is known [7] that if Σ models a rigid frame porous then $p^\pm/k$ should take values strictly in the sector of the complex plane defined by $\pm\Re e(p^\pm/k) \geqslant 0$, $\pm\Im m(p^\pm/k) \leqslant 0$, and $|p^\pm/k| \leqslant 1$. In any case, the condition $\pm\Re e(p^\pm/k) \geqslant 0$ must be satisfied when the material located at Σ is to absorb rather than to emit energy. In a stronger way, if $\pm\Re e(p^\pm/k) > 0$ is fulfilled then the corresponding boundary condition (2.2) is describing an absorbing boundary where the magnitude of the reflection coefficient for plane waves is less than 1. All these examples turn clear that the above conditions (a)–(g) reflect a great variety of possible combinations of the parameters k and $p^\pm$ with physical significance.

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