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TWISTING ELEMENTS $da = a \smile_1 a$

Let $(A^*, d : A^* \to A^{*+1}, \smile : A^* \otimes A^* \to A^*)$ be a dg algebra with differential d and multiplication $a \cdot b = a \smile b$. A *twisting element* (Ed. Brown [3]) is defined as $a \in A^1,\ da = a \cdot a$. Later N. Berikashvili [2] has introduced the notion of *perturbation* of twisting elements: for an invertible element $g \in A^0$ the combination $a' = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}$ is a twisting element too. Actually this is the action of the group of units $G = \{g \in A^0,\ \exists g^{-1}\}$ on the set of all twisting elements of A.

If $(M, d_M : M \to M)$ is A-module: $A \otimes M \to M$, and $a \in A$ is twisting then $d_a(m) = d_M(m) + a \cdot m$ is a differential: the Brown's condition guarantees that $d_a d_a = 0$. If $a' \sim a$ then $g : (M, d_a) \to (M, d_{a'})$ given by $g(m) = g \cdot m$ is an isomorphism of dg modules.

These notions have applications in homology theory of fibrations, as well as in differential geometry and in physics. Let us touch this shortly. A *connection* $a \in A^1$ determines the *curvature* $\Omega = da - a \cdot a$, so a twisting element is a *flat* ($\Omega = 0$) connection. Take an invertible $g \in A^0$ and perturb the connection a as $a' = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}$ (gauge transformation). Then it is easy to see that $\Omega' = g \cdot \Omega \cdot g^{-1}$.

Our aim is to modify the notions of twisting element and perturbation for Steenrod's $\smile_1$ product instead of $a \cdot b = a \smile b$. It is easy to formulate the notion of $\smile_1$-twisting element, this is $a \in A^2,\ da = a \smile_1 a$. But since $\smile_1$ is not associative and has some more sophisticated properties than $\smile$ the concept of perturbation of such twisting elements requires some additional structure, namely the structure of homotopy G-algebra, which in fact is a dg algebra with "good" $\smile_1$-product and some follow up higher operations.

The generalization of the notion of twisting element to the case of $\smile_1$ product is aimed to some particular problems, namely $\smile_1$-twisting elements control Satasheff's $A(\infty)$-algebras from one hand side, and Gerstenhaber's deformations of algebras from another, see [9] for detailes.

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1. Homotopy G-algebras

A *homotopy G-algebra* (hGa in short) is a dg algebra with "good" $\smile_1$ product. The general notion was introduced in [5].

Generally multiplication $a \cdot b$ in a dg algebra $(A, d, \cdot)$ is not commutative, for example in the cochain complex $C^*(X)$, but there exists Steenrod's $\smile_1$-product $a^p \smile_1 b^q \in C^{p+q-1}(X)$ which controls this noncommutativity

$$d(a \smile_1 b) = d(a) \smile_1 b + a \smile_1 (b) + a \cdot b - b \cdot a. \tag{1}$$

For our purposes some further properties of Steenrod's $\smile_1$ are needed. First of all, the "left" Hirsch formula

$$(a \cdot b) \smile_1 c - a \cdot (b \smile_1 c) - (a \smile_1 c) \cdot b = 0. \tag{2}$$

As for the "right" Hirsch formula, the similar expression is just homotopical to zero, that is there exists a 3-fold operation $E_{1,2} : A^p \otimes A^q \otimes A^r \to A^{p+q+r-2}$ which satisfies

$$\begin{aligned} c \smile_1 (a \cdot b) - a \cdot (c \smile_1 b) - (c \smile_1 a) \cdot b = \\ = dE_{1,2}(c; a, b) + E_{1,2}(dc; a, b) + E_{1,2}(c; da, b) + E_{1,2}(c; a, db). \end{aligned} \tag{3}$$

A hGa $(A, d, \cdot, \{E_{1,k}\})$ is a dg algebra $(A, d, \cdot)$ equipped with a sequence of multilinear operations $E_{1,k}(a^p; b_1^{q_1}, \ \dots \ , b_k^{q_k}) \in A^{p+q_1+ \ \cdots \ +q_k-k}$, $k = 1, 2, 3, \dots$ which satisfy certain coherency conditions (see for example [9]). Particularly $E_{1,1}(a, b) = a \smile_1 b$ satisfies the above mentioned conditions 1, 2, 3 and

$$(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b), \tag{4}$$

this means that the same operation $E_{1,2}$ measures also the deviation from the associativity of the operation $E_{1,1} = \smile_1$.

Operations $E_{1,k}(a; b_1, \dots, b_k)$ in some papers are called *brace operations* and are denoted as $a\{b_1, \dots, b_k\}$).

The remarkable examples of homotopy G-algebras are: 1. The cochain complex of 1-reduced simplicial set $C^*(X)$, [1]. 2. The Hochschild cochain complex $C^*(U, U)$ of an associative algebra U, [7], [6]. 3. The cobar construction ΩC of a dg *bialgebra* C, [8]. In All three cases starting operations $E_{1,1}$ are classical $\smile_1$ products.

Two main aspects of this notion are (see [9] for more details):

1. A hGa $(A, d, \cdot, \{E_{1,k}\})$ is a B_∞*-algebra*: it defines on the bar construction $B(A)$ a good multiplication $\mu_E : B(A) \otimes B(A) \to B(A)$.

2. A structure of a hGa on A induces on the homology $H(A)$ a structure of *Gerstenhaber algebra* $(H(A), \cdot, [\ .\ ])$ which consists of commutative multiplication $\cdot$ and a Lie bracket of degree -1 $[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$ which is a biderivation: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$. This bracket is induced by the structure of dg Lie algebra on the desuspension $s^{-1}A$ given by $[a, b] = a \smile_1 b + b \smile_1 a$.

Below we will need the bigraded version of the notion of hGa $(C^{*,*}, d, \cdot, \{E_{1,k}\})$. This is a bigraded algebra $(C^{*,*}, \cdot)$, $C^{m,n} \cdot C^{p,q} \subset C^{m+p,n+q}$, together with a differential (derivation) $d : C^{m,n} \to C^{m+1,n}$ and with a sequence of operations

$$E_{1,k} : C^{m,n} \otimes C^{p_1,q_1} \otimes \cdots \otimes C^{p_k,q_k} \to C^{m+p_1+\cdots+p_k-k,n+q_1+\cdots+q_k}$$

so that the *total complex* (the total degree of $C^{p,q}$ is p) is a hGa.

2. $\smile_1$-twisting Elements

Let $(C^{*,*}, d, \cdot, \{E_{1,k}\})$ be a bigraded hGa. A $\smile_1$-*twisting element* we define as $a = \sum_{k=1}^{\infty} a_k,\ a_k \in C^{2,k}$ such that $da = a \smile_1 a$, that is $da_k = \sum_{i=1}^{k-1} a_i \smile_1 a_{k-i}$. We remark here that such a $\smile_1$-twisting element $a \in A$ is a Lie twisting element in the dg Lie algebra $(s^{-1}A, d, [\ ,\ ])$, i.e. satisfies $da = \frac{1}{2}[a, a]$.

We introduce the following perturbation of $\smile_1$-twisting elements: for an arbitrary $g = \sum_{k=1}^{\infty} g_k,\ g_k \in C^{1,k}$ let us define

$$\bar{a} = a + dg + g \cdot g + g \smile_1 a + \sum_{k=1}^{\infty} E_{1,k}(\bar{a}; g, \ \ldots\ , g).$$

Particularly,

$$\begin{aligned}
\bar{a}_1 = \ & a_1 + dg_1;\\
\bar{a}_2 = \ & a_2 + dg_1 + g_1 \cdot g_1 + g_1 \smile_1 a_1 + \bar{a}_1 \smile_1 g_1;\\
\bar{a}_3 = \ & a_3 + dg_2 + g_1 \cdot g_2 + g_2 \cdot g_1 + g_1 \smile_1 a_2 + g_2 \smile_1 a_1 +\\
& +\bar{a}_1 \smile_1 g_2 + \bar{a}_2 \smile_1 g_1;\\
\bar{a}_4 = \ & a_4 + dg_3 + g_1 \cdot g_3 + g_2 \cdot g_2 + g_3 \cdot g_1 +\\
& +g_1 \smile_1 a_3 + g_2 \smile_1 a_2 + g_3 \smile_1 a_1 +\\
& +\bar{a}_1 \smile_1 g_3 + \bar{a}_2 \smile_1 g_2 + \bar{a}_3 \smile_1 g_1 + E_{1,2}(\bar{a}_1; g_1, g_1, g_1);
\end{aligned}$$

so this is a recurrent definition.

Theorem 1. *$\overline{a}$ satisfies $d\overline{a} = \overline{a} \smile_1 \overline{a}$, i.e. is a $\smile_1$-twisting element.*

Actually, this perturbation of $\smile_1$-twisting elements is the action of the group $G = \{g = \sum_{k=1}^{\infty} g_k\ ;\quad g_k \in B^{1,k}\}$ with operation $g' * g = g' + g + \sum_{k=1}^{\infty} E_{1,k}(g'; g, .., g)$ on the set of all $\smile_1$-twisting elements $Tw(C^{*,*})$ by the rule $g * b = b'$ where $b' = b + dg + g \cdot g + E_{1,1}(g; b) + \sum_{k=1}^{\infty} E_{1,k}(b'; g, \ldots, g)$. By $D(C^{*,*})$ we denote the set of orbits $Tw(C^{*,*})/G$.

In particular, for $g = 0 + \cdots + 0 + g_n + 0 + \cdots$ the twisting element $\overline{a} = g * a$ looks as $\overline{a} = a_1 + \cdots + a_n + (a_{n+1} + dg_n) + \overline{a}_{n+2} + \overline{a}_{n+3} + \cdots$, so the components $a_1, \ldots, a_n$ remain unchanged and $\overline{a}_{n+1} = a_{n+1} + dg_n$.

The perturbations allow us to introduce *obstructions* for the following two problems.

1. Quantization. Let us first mention that for a twisting element $a = \sum_{k=1}^{\infty} a_k$ the first component $a_1 \in C^{2,1}$ is a cycle and any perturbation does not change its homology class $[a_1] \in H^{2,1}(C^{*,*})$. Thus, we have the correct map $\phi : D(C^{*,*}) \to H^{2,1}(C^{*,*})$.

A *quantization* of a homology class $\alpha \in H^{2,1}(C^{*,*})$ we define as a twisting element $a = \sum_{k=1}^{\infty} a_k$ such that $[a_1] = \alpha$. Thus, α is quantizable if $\alpha \in Im\phi$.

The obstructions for quatizability lay in homologies $H^{3,n}(C^{*,*}),\ n \geq 2$. Indeed, let $a_1 \in C^{2,1}$ be a cycle from α. The first step to quantize α is to construct a_2 such that $da_2 = a_1 \smile_1 a_1$. The necessary and sufficient condition for this is $[a_1 \smile_1 a_1] = 0 \in H^{3,2}(C^{*,*})$, so this homology class is the first obstruction $O(a_1)$. Suppose it vanishes, so there exists a_2. Then it is easy to see that $a_1 \smile_1 a_2 + a_2 \smile_1 a_1$ is a cycle and its class $O(a_1, a_2) \in H^{3,3}(C^{*,*})$ is the second obstruction. If $O(a_1, a_2) = 0$ then there exists a_3 such that $da_3 = a_1 \smile_1 a_2 + a_2 \smile_1 a_1$. If not, then we take another a_2 and try a new second obstruction. The nth obstruction is $O(a_1, a_2, \ldots, a_n) = [\sum_{k=1}^{n} a_k \smile_1 a_{n-k+1}] \in H^{3,n+1}(C^{*,*})$.

2. Rigidity. A twisting element $a = a_1 + a_2 + \cdots$ we call *trivial* if it is equivalent to 0. A bigraded hGa $C^{*,*}$ is *rigid* if each twisting element is trivial, i.e. if $D(C^{*,*}) = \{0\}$. Arguments similar to above show that obstructions to triviality of a twisting element lies in homologies $H^{2,n}(C^{*,*}),\ n \geq 1$. This, in particular, implies that if $H^{2,n}(C^{*,*}) = 0,\ n \geq 1$, then $C^{*,*}$ is rigid.

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