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ON COCHAIN OPERATIONS FORMING HIRSCH ALGEBRAS

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Abstract. Cochain operations that determine the structure of Hirsch algebras are characterized.

Many constructions that work successfully for commutative differential graded (dg) algebras fail in the noncommutative case. There exists a classical tool that measures the noncommutativity of a dg algebra $(A, d, \cdot)$, namely, the Steenrod $\smile_1$ product satisfying the condition

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b - b \cdot a \tag{1}$$

(the signs are ignored in the whole text). The existence of $\smile_1$ guarantees the commutativity of $H(A)$, but the $\smile_1$ product satisfying just this condition is too pure for most applications. In many constructions, some deeper properties of $\smile_1$ are needed, for example, compatibility with the product of A (the Hirsch formula)

$$a \smile_1 (b \cdot c) = b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c. \tag{2}$$

Perhaps, the following structure is a good notion of a "good" $\smile_1$ product.

Definition 1. A Hirsch algebra is defined as a dg-algebra $(A, d, \cdot)$ equipped with a sequence of operations

$$\Big\{E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A,\ p, q = 0, 1, 2, \dots,\ \deg E_{p,q} = -(p + q - 1)\Big\},$$

which satisfies the conditions

$$E_{0,0} = 0, \quad E_{0,q>1} = 0, \quad E_{0,1} = \mathrm{id}, \quad E_{p>1,0} = 0, \quad E_{1,0} = \mathrm{id}, \tag{3}$$

$$\begin{aligned} dE_{p,q}&(a_1, \dots, a_p; b_1, \dots, b_q) \\ &+ \sum_i E_{p,q}(a_1, \dots, da_i, \dots, a_p; b_1, \dots, b_q) + \sum_i E_{p,q}(a_1, \dots, a_p; b_1, \dots, db_i, \dots, b_q) \\ =& \sum_i E_{p-1,q}(a_1, \dots, a_i \cdot a_{i+1}, \dots, a_p; b_1, \dots, b_q) + \sum_i E_{p,q-1}(a_1, \dots, a_p; b_1, \dots, b_i \cdot b_{i+1}, \dots, b_q) \\ &+ \sum_{i=0}^{p} \sum_{j=0}^{q} E_{i,j}(a_1, \dots, a_i; b_1, \dots, b_j) \cdot E_{m-p,n-q}(a_{i+1}, \dots, a_p; b_{j+1}, \dots, b_q). \end{aligned} \tag{4}$$

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A Hirsch algebra $(A, d, \cdot, \{E_{p,q}\})$ is called associative if, in addition,

$$\sum_{r=1}^{l+m} \sum_{\substack{l_1+\dots+l_r=l,\\ m_1+\dots+m_r=m|}} E_{k,r}(a_1,\dots,a_k; E_{l_1,m_1}(b_1,\dots,b_{l_1}; c_1,\dots,c_{m_1}),\dots,$$
$$E_{l_r,m_r}(b_{l_1+\dots+l_{r-1}+1},\dots,b_l; c_{m_1+\dots+m_{r-1}+1},\dots,c_m)$$
$$= \sum_{s+1}^{k+l} \sum_{\substack{k_1+\dots+k_s=k,\\ l_1+\dots+l_s=l}} E_{s,m}(E_{k_1l_1}(a_1,\dots,a_{k_1}; b_1,\dots,b_{l_1}),\dots,$$
$$E_{k_s,l_s}(a_{k_1+\dots+k_{s-1}+1},\dots,a_k; b_{l_1+\dots+l_{s-1}+1},\dots,b_l); c_1,\dots,c_m). \quad (5)$$

On the bar construction BA, this structure determines a multiplication $\mu_E : BA \otimes BA \to BA$, which turns BA into a dg-bialgebra (see [3–5]). An associative Hirsch algebra with $E_{p>1,q} = 0$ is called a homotopy G-algebra (see [2, 7]). The component $E_{1,1}$ plays the role of a "good" $\smile_1$ product. Conditions (1) and (2) are a part of (4).

To define such a structure on a dg-algebra A, one must solve the "differential equation" (4) with the "initial value" (3).

Condition (4) can be reformulated in operadic terms as follows.

Assume that P is a dg operad with an element $\mu \in P(2)_0$ satisfying the conditions $d\mu = 0$ and $\mu \circ_1 \mu = \mu \circ_2 \mu$. If A is an algebra over P, then it is a dg algebra with multiplication μ. The above differential equation in operadic terms has the form

$$dE_{p,q} = \sum_i E_{p-1,q} \circ_i \mu + \sum_i E_{p,q-1} \circ_{p+i} \mu$$
$$+ \sum_{i=0}^{p} \sum_{j=0}^{q} \Big[(1,\dots,i,p+1,\dots,p+j,i+1,\dots,p,p+j+1,\dots,q)\Big]\mu$$
$$\circ (E_{i,j}, E_{p-i,q-j}). \quad (6)$$

Theorem 1. *Assume that P is an acyclic dg operad with an element $\mu \in P(2)_0$ satisfying the conditions $d\mu = 0$ and $\mu \circ_1 \mu = \mu \circ_2 \mu$, and with a given contraction homotopy*

$$s : P(n)_k \to P(n)_{k+1}, \quad ds(x) + s(dx) = x - \eta\epsilon x.$$

Then there is an explicit solution of (6) *satisfying* (3).

Sketch of the proof. Rewrite Eq. (6) in the form $dE_{p,q} = U_{p,q}$. It is easy to prove by induction that $dU_{p,q} = 0$. Then we define $E_{p,q} = sU_{p,q}$.

Corollary 1. *An algebra over an E_∞ operad is a Hirsch algebra.*

In particular, for the surjection operad $\mathcal{X}$ this process gives the solution

$$E_{1,k} = (1,2,1,3,1,\dots,1,k,1,k+1), \quad E_{p>1,q} = 0,$$

the McClure–Smith elements (see [1, 6]), which automatically satisfies the associativity condition (5); thus, this is the homotopy G-algebra structure.

As for the Barrat–Eccles operad $\mathcal{E}$, we have the following components of solution:

$$E_{1,k} = \Big((\mathbf{1},2,\dots,k+1),\ \dots,\ (2,3,\dots,i,\mathbf{1},i+1,\dots,k+1),\ \dots,\ (2,3,\dots,k+1,\mathbf{1})\Big);$$
$$E_{k,1} = \Big((1,2,\dots,\mathbf{k+1}),\ \dots,\ (1,2,\dots,i,\mathbf{k+1},i+1,\dots,k),\ \dots,\ (\mathbf{k+1},1,2,\dots,k)\Big)$$

and

$$E_{2,2} = \Big((1,2,3,4),(1,3,4,2),(3,1,4,2),(3,4,1,2)\Big) + \Big((1,2,3,4),(3,1,2,4),(3,1,4,2),(3,4,1,2)\Big) + \Big((1,2,3,4),(1,3,2,4),(3,1,2,4),(3,1,4,2)\Big) + \Big((1,2,3,4),(1,3,2,4),(1,3,4,2),(3,1,4,2)\Big).$$

$\square$

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REFERENCES

1. C. Berger and B. Fresse, *Combinatorial operad actions on cochains*, Preprint, `arXi:math.AT/0109158v1` (2001).
2. M. Gerstenhaber and A. A. Voronov, "Higher-order operations on the Hochschild complex," *Funkts. Anal. Prilozh.*, **29**, No. 1, 1–6 (1995).
3. T. Kadeishvili, "Measuring the noncommutativity of DG-algebras," *J. Math. Sci.*, **119**, No. 4, 494–512 (2004).
4. T. Kadeishvili, *Twisting elements in homotopy G-algebras*, `arXiv:math/0709.3130`.
5. T. Kadeishvili and S. Saneblidze, "On a multiplicative model of a fibration," *Bull. Georgian Acad. Sci.*, **153**, No. 3, 345–346 (1996).
6. ddd J. McClure and J. H. Smith, *A solution of Deligne's conjecture*," Preprint, `arXiv:math,QA/9910126` (1999).
7. A. Voronov, "Homotopy Gerstenhaber algebras," in: *Conference Moshe Flato 1999, Dijon, Vol.* II, Math. Phys. Stud., **22**, Kluwer Acad. Publ., Dordrecht (2000), pp. 307–331.

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