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BERIKASHVILI'S FUNCTOR D: GENERALIZATION AND APPLICATION

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Abstract. We present the notion of Berikashvili's functor D, its generalization for the $A(\infty)$ case and the corresponding algebraic model of a fibre bundle. It is a pleasure to dedicate this paper to my teacher Nodar Berikashvili on his 90th anniversary.

1. Twisting Cochains for DG-algebras

1.1. **Browns twisting cochains.** Let $(K, d_K : K_* \to K_{*-1}, \nabla_K : K \to K \otimes K)$ be a DG-coalgebra and $(A, d_A : A_* \to A_{*-1}, \mu_A : A \otimes A \to A)$ be a DG-algebra. Then the *cochain complex* $C^*(K, A) = Hom(K, A)$ with a differential $\delta\alpha = \alpha d_K + d_A\alpha$ and multiplication $\alpha \smile \beta = \mu_A(\alpha \otimes \beta)\nabla_K$ is a DG-algebra.

A *Brown's twisting cochain* [3] is a homomorphism $\phi : K_* \to A_{*-1}$, i.e., $deg\ \phi = -1$, satisfying $\delta\phi = \phi \smile \phi$.

Twisted tensor product. Let $(P, d_P, \nu : A \otimes P \to P)$ be a DG A-module. Then any twisting cochain $\phi : K \to A$ determines a homomorphism $d_\phi : K \otimes P \to K \otimes P$ by $d_\phi(k \otimes p) = d_K k \otimes p + k \otimes d_P m + (k \otimes p) \cap \phi$ where $(k \otimes p) \cap \phi = (id_K \otimes \nu)(id_K \otimes \phi \otimes id_P)(\nabla_K \otimes id_P)(k \otimes p)$. The Brown's condition $d\phi = \phi \smile \phi$ implies that $d_\phi d_\phi = 0$. The obtained chain complex $(K \otimes M,\ d_\phi)$ is called *twisted tensor product* and is denoted as $K \otimes_\phi M$.

Using these notions, Edgar Brown constructed an algebraic model of a fibre bundle (see below).

For a morphism of DG-algebras $f : A \to A'$, a morphism of modules $g : P \to P'$, $g(a \cdot p) = f(a) \cdot g(p)$ and a twisting cochain $\phi : K \to A$ the map $id_K \otimes g : K \otimes_\phi P \to K \otimes_{f\phi} P'$ is a chain map.

1.2. **Berikashvili's equivalence of twisting cochains.** Two twisting cochains $\phi, \psi : K \to A$ are equivalent (Berikashvili [2]) if there exists $c : K \to A,\ deg\ c = 0$, such that $\psi = \phi + \delta c + \psi \smile c + c \smile \phi$, notation $\phi \sim_c \psi$. This equivalence allows one to *perturb* twisting cochains.

Essential applications of Berikashvili's equivalnce give the following.

Theorem 1. *If $\phi \sim_c \psi$, then $K \otimes_\phi P \xrightarrow{F_c} K \otimes_\psi P$ given by $F_c(k \otimes p) = (k \otimes p) \cap c$ is an isomorphism of DG-comodules.*

Berikashvili's functor D. Let $Tw(K, A) = \{\phi : K \to A,\ \delta\phi = \phi \circ \phi\}$ be the set of all twisting cochains. *Berikasvili's functor* $D(K, A)$ is defined as the factorset $D(K, A) = \frac{Tw(K,A)}{\sim}$.

The following property of D plays an essential role in some constructions.

Theorem 2 (Berikashvili [2])**.** *Let (K, d_K, ∇_K) be a DG-colagebra with free $K_i s$ and (A, d_A, μ_A) be a connected DG-algebra. If $f : A \to A'$ is a weak equivalence of connected DG-algebras (i.e., homology isomorphism), then $D(f) : D(K, A) \to D(K, A')$ is a bijection.*

1.3. **Bar interpretation.** The notions of twisting cochain and their equivalence have useful interpretation in terms of Adams's bar construction [1].

Twisting Cochains and the Bar Construction. Any twisting cochain $\phi : K \to A$ induces a map of DG-coalgebras $f_\phi : K \to B(A)$ given by $f_\phi = \sum_i (\phi \otimes\ \cdots\ \otimes \phi)\nabla^i_K$.

Bar interpretation of equivalence of twisting Cochains. In the category of DG-coalgebras there is the following notion of homotopy: two DG-coalgebra maps $f, g : (K, d_K, \nabla_K) \to (K', d_{K'}, \nabla_{K'})$ are

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homotopic, if there exists chain homotopy $D : K \to K'$, $d_{K'}D + Dd_k = f - g$, which, in addition, is a $f - g$-coderivation, that is, $\nabla_{K'}D = (f \otimes D + D \otimes g)\nabla_K$.

If $\phi \sim_c \psi$, then f_ϕ and f_ψ are homotopic by $D(c) : K \to BA$ given by $D(c) = \sum_{i,j}(\psi \otimes \cdots (j - times) \cdots \otimes \psi \otimes c \otimes \phi \otimes \cdots \otimes \phi)\nabla^i_K$.

Bar interpretation of a functor D. Asigning to a twisting cohain $\phi : K \to A$ the DG-coalgebra map $f_\phi : K \to BA$ and having in mind that $\phi \sim_c \psi$ implies $f_\phi \sim_{D(c)} f_\psi$, we obtain a bijection $D(K, A) \leftrightarrow [K, BA]$ where $[K, BA]$ denotes the set of chain homotopy classes in the category of DG-coalgebras.

We remark here that the theorem 2 means that for a weak equivalence of DG-algebras $A \to A'$ the induced map $[K, BA] \to [K, BA']$ is a bijection.

2. Twisting Cochains for $A(\infty)$-algebras

Here we are going to step from the DG-algebra (A, d, μ) to an $A(\infty)$-algebra $(M, \{m_i\})$, this notion was introduced by James Stasheff in [9].

2.1. Category of $A(\infty)$-algebras.

Definition 1 (Stasheff [9]). *An $A(\infty)$ algebra $(A, \{m_i\})$ is a graded module A equipped with a sequence of operations $\{m_i : A^{\otimes i} \to A,\ i = 1, 2, 3, 4, \dots\}$ which satisfies the following conditions: deg $m_i = 2-i$ and*

$$\sum_{i+j=n+1}\sum_{k=0}^{n-j} m_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \cdots \otimes a_n) = 0.$$

Stasheff's defining condition for $n = 1$ gives $m_1m_1 = 0$, i.e., m_1 is a differential, for $n = 2$, m_1 is a derivation with respect to the multiplication m_2, and for $n = 3$, m_2 is homotopy associative, and the appropriate homotopy is m_3. So, $(A, \{m_i\})$ is a *strong homotopy associative* (sha) algebra.

Bar interpretation. The Stasheff's condition guarantees that the coderivation $d_m : B(A) \to B(A)$ given by

$$\begin{aligned} &d_m(a_1 \otimes \cdots \otimes a_n) \\ &= \textstyle\sum_{k=0}^{n}\sum_{j=1}^{n-k} a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \cdots \otimes a_n \end{aligned}$$

satisfies $d_md_m = 0$: the Stasheff's condition is the projection of this equality on the cogenerating module A. So, the bar construction $(B(A, \{m_i\}), d_m)$ with this perturbed differential is a DG-coalgebra.

Particular case. The notion of an $A(\infty)$ algebra generalizes the notion of DG-algebra: an $A(\infty)$-algebra of type $(A, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\})$ is a DG-algebra with the differential m_1 and associative multiplication m_2.

Morphism of A_∞-algebras. This notion was introduced in [5]. A morphism of $A(\infty)$-algebras $(A, \{m_i\}) \to (A', \{m'_i\})$ is defined as a sequence of homomorphisms $\{f_i : A^{\otimes i} \to A',\ i = 1, 2, \dots\}$, which satisfy the following conditions: deg $f_i = 1 - i$ and

$$\begin{aligned} &\textstyle\sum_{i+j=n+1}\sum_{k=0}^{n-j} f_i(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) \\ &= \textstyle\sum_{t=1}^{n}\sum_{k_1+\cdots+k_t=n} m'_t(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes f_{k_2}(a_{k_1+1} \otimes \cdots \otimes a_{k_1+k_2}) \otimes \\ &\qquad \cdots \otimes f_{k_t}(a_{k_1+\cdots+k_{t-1}+1} \otimes \cdots \otimes a_n)). \end{aligned}$$

Particulary, for $n = 1$, this gives $f_1m_1 = m'_1f_1$, that is, $f_1 : (A, m_1) \to (A', m'_1)$ is a chain map. We call $\{f_i\}$ *a weak equivalence* if f_1 induces isomorphism of homologies.

Bar interpretation. A morphism $\{f_i\}$ defines the DG-coalgebra map of the bar constructions $B(\{f_i\}) : B(A, \{m_i\}) \to B(A', \{m'_i\})$, given by

$$\begin{aligned} &B(\{f_i\})(a_1 \otimes \cdots \otimes a_n) \\ &= \textstyle\sum_{t=1}^{n}\sum_{k_1+\cdots+k_t=n} f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{k_1+\cdots+k_{t-1}+1} \otimes \cdots \otimes a_n). \end{aligned}$$

Particularly, a morphism of A_∞-algebras

$$\{f_1, f_2 = 0, f_3 = 0, \dots\} :$$
$$(A, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\}) \to (A', \{m'_1, m'_2, m'_3 = 0, m'_4 = 0, \dots\})$$

is an ordinary map of DG-algebras.

2.2. **Category of $A(\infty)$-modules.** This definition from [5] generalizes the notion of a DG-module over a DG-algebra. An $A(\infty)$-module over an $A(\infty)$-algebra $(A,\{m_i\})$ is a graded module P equipped with a sequence of "actions" $\{p_i : P\otimes A^{\otimes i}\to A,\ i=0,1,2,3,\dots\}$ satisfying the conditions: $\deg p_i = 1-i$ and, for $a_k\in A,\ x\in P$,

$$\sum_{i=0}^{n} p_{n-i}(p_i(x\otimes a_1\otimes\cdots\otimes a_i)\otimes a_{i+1}\otimes\cdots\otimes a_n)+\sum_{k=0}^{n}\sum_{i=1}^{n-k} p_{n-i+1}(x\otimes a_1\otimes\cdots\otimes a_k\otimes m_i(a_{k+1},\otimes\cdots\otimes a_{k+i})\otimes a_{k+i+1}\otimes\cdots\otimes a_n)=0.$$

Bar interpretation. This structure induces $d_p : P\otimes B(A)\to P\otimes B(A)$ by

$$d_p(x\otimes a_1\otimes\cdots\otimes a_n)=\sum_{i=0}^{n} p_i(x\otimes a_1\otimes\cdots\otimes a_i)\otimes a_{i+1}\otimes\cdots\otimes a_n + \sum_{k=0}^{n}\sum_{i=1}^{n-k} x\otimes a_1\otimes\cdots\otimes a_k\otimes m_i(a_{k+1}\otimes\cdots\otimes a_{k+i})\otimes a_{k+i+1}\otimes\cdots\otimes a_n$$

which satisfies $d_pd_p=0$, thus $(P\otimes B(A),d_p)$ is a DG-comodule over DG-coalgebra $(B(A,\{m_i\}),d_m)$.
Particular cases. (1) An $A(\infty)$-module $(P,\{p_1,p_2,0,0,\dots\})$ over an $A(\infty)$-algebra $(A,\{m_1,m_2,0,0,\dots\})$ is a DG-module over DG-algebra (A,m_1,m_2) with a differential $p_0 : P\to P$ and strictly associative action $p_1 : P\otimes A\to A$. (2) An $A(\infty)$-algebra $(A,\{m_i\})$ is an $A(\infty)$-module over itself with structure maps $p_n(x\otimes a_1\otimes\cdots\otimes a_n)=m_{n+1}(x\otimes a_1\otimes\cdots\otimes a_n)$.
Morphism of $A(\infty)$-modules. Let $(P,\{p_i\})$ be an $A(\infty)$-module over an $A(\infty)$-algebra $(A,\{m_i\})$, and let $(P',\{p_i'\})$ be an $A(\infty)$-module over an $A(\infty)$-algebra $(A',\{m_i'\})$. A morphism of the couples

$$(\{g_i\},\{f_i\}) : ((P,\{p_i\}),(A,\{m_i\}))\to((P',\{p_i'\}),(A',\{m_i'\}))$$

is defined in [5] as: a morphism of $A(\infty)$-algebras $\{f_i\} : (A,\{m_i\})\to(A',\{m_i'\})$ and a sequence of homomorphisms $\{g_i : P\otimes A^{\otimes i}\to P',\ i=0,1,2,3,\dots\}$ such that $deg\ g_i=-i$ and

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{j=1}^{n-k} g_{n-j+1}(x\otimes a_1\otimes\cdots\otimes a_k\otimes m_j(a_{k+1}\otimes\cdots\otimes a_{k+j})\cdots\otimes a_n)\\ &\quad+\sum_{k=0}^{n} g_{n-k}(p_k(x\otimes a_1\otimes\cdots\otimes a_k)\otimes a_{k+1}\otimes\cdots\otimes a_n)\\ &=\sum_{t=1}^{n+1}\sum_{k_1+\cdots+k_t=n+1} p_t(g_{k_1}(x\otimes a_1\otimes\cdots\otimes a_{k_1})\otimes f_{k_2}(a_{k_1+1}\otimes\cdots\otimes a_{k_1+k_2})\\ &\quad\otimes f_{k_3}(a_{k_1+k_2+1}\otimes\cdots\otimes a_{k_1+k_2+k_3})\otimes\cdots\otimes f_{k_t}(a_{k+k_1+\cdots+k_{t-1}+1}\otimes\cdots\otimes a_n)).\end{aligned}$$

Bar interpretation. Such a morphism induces the chain map $G : (P\otimes BA,d_p)\to(P'\otimes BA',d_{p'})$ by

$$G(x\otimes a_1\otimes\cdots\otimes a_n)=\sum_{t=1}^{n+1}\sum_{k_1+\cdots+k_t=n+1} g_{k_1}(x\otimes a_1\otimes\cdots\otimes a_{k_1}) \otimes f_{k_2}(a_{k_1+1}\otimes\cdots\otimes a_{k_1+k_2})\otimes\cdots\otimes f_{k_t}(a_{k+k_1+\cdots+k_{t-1}+1}\otimes\cdots\otimes a_n).$$

2.3. **A_∞-twisting Cochains.** Now we have to replace in the definition of a twisting cochain a DG-algebra (A,d_A,μ) by an A_∞-algebra $(A,\{m_i\})$, see [6], [7].

An $A(\infty)$-*twisting cochain* we define as a homomorphism $\phi : K\to A$ of degree -1 satisfying the condition $\sum_{k=1}^{\infty} m_k(\phi\otimes\ \cdots\ \otimes\phi)\nabla_K^k=\phi d_K$.

The set of all $A(\infty)$-twisting cochains $Tw_\infty(K,A)$ is a bifunctor: for a morphism of DG-coalgebras $h : K'\to K$, the composition $\phi\circ h$ belongs to $Tw_\infty(K,A)$, similarly, for a morphism of A_∞-algebras $f=\{f_i\} : (A,\{m_i\})\to(A',\{m_i'\})$, the composition $f(\phi)=\sum f_i(\phi\otimes\cdots\otimes\phi)\nabla^i$ belongs to $Tw_\infty(K,A')$.
Bar Interpretation. An A_∞-twisting cochain $\phi : K\to A$ induces the DG-coalgebra morphism $f_\phi : K\to B(A)$ by $f_\phi=\sum_i(\phi\otimes\ \cdots\ \otimes\phi)\nabla_K^i$. Conversely, any DG-coalgebra map $f : K\to B(A)$ is f_ϕ for the $A(\infty)$-twisting cochain $\phi=p\circ f : K\to B(A)\to A$. So, $Mor_{dgcoalg}(K,B(A))\leftrightarrow T_\infty(K,A)$.
Equivalence of A_∞-twisting Cochains [6]. Two A_∞-twisting cochains $\phi,\psi : K\to A$ are equivalent if there exists $c : K\to A,\ \ deg\ c=0$, such that

$$\psi-\phi=cd_K+\sum_{k,j} m_k(\psi\otimes\ \cdots\ (j)\ \cdots\ \otimes\psi\otimes c\otimes\phi\otimes\ \cdots\ \otimes\phi)\nabla^k,$$

notation $\phi\sim_c\psi$.
Bar interpretation. If $\phi\sim_c\psi$, then f_ϕ and f_ψ are homotopic in the category of DG-coalgebras: chain homotopy $D_\infty(c) : K\to B(A)$ is given by $D_\infty(c)=\sum_{i,j}(\psi\otimes\ \cdots\ (j-times)\ \cdots\ \otimes\psi\otimes c\otimes\phi\otimes\ \cdots\ \otimes\phi)\nabla_K^i$.

FunctorD_∞. By $D_\infty(K,A)$ we denote the factorset $D_\infty(K,A) = \frac{T_\infty(K,A)}{\sim}$. Thus we have a bijection $[K,B(A)] \leftrightarrow D_\infty(K,A)$.

Suppose $f = \{f_i\} : (A,\{m_i\}) \to (A',\{m'_i\})$ is a morphism of A_∞-algebras and $\phi : K \to A$ is an A_∞-twisting cochain.

From the bar construction interpretation it follows that $f(\phi) : K \to A'$ given by $f(\phi) = \sum_i f_i(\phi \otimes \cdots \otimes \phi)\nabla^i_K$ is an A_∞-twisting cochain, too. Moreover, if $\phi \sim_c \psi$, then $f(\phi) \sim_{c'} f(\psi)$ with $c' : K \to A'$ given by

$$c' = \sum_{i,j} f_i(\psi \otimes \ \cdots \ (j - times) \ \cdots \ \otimes \ \psi \otimes c \otimes \phi \ \otimes \ \cdots \ \otimes \phi)\nabla^i_K,$$

thus we have a map $D_\infty(f) : D_\infty(K,A) \to D_\infty(K,A')$.

The following theorem is an analogue of Berikashvilis's theorem 2 for A_∞-algebras proved in [6].

Theorem 3. *Let (K, d_K, ∇_K) be a DG-colagebra and $(A,\{m_i\})$ be a connected DG-algebra. If $f = \{f_i\} : (A,\{m_i\}) \to (A',\{m'_i\})$ is a weak equivalence of A_∞-algebras, then $D_\infty(f) : D_\infty(K,A) \to D_\infty(K,A')$ is a bijection, consequently, $[K,B(A)] \leftrightarrow [K,B(A)']$.*

2.4. **Twisted tensor product, the A_∞-case.** Let (K, d_K, ∇_K) be a DG-coalgebra, $(A,\{m_i\})$ be an A_∞-algebra, $(P,\{p_i\})$ be an A_∞-module over an $(A,\{m_i\})$, and $\phi : K \to A$ be an A_∞-twisting cochain. It defines on the tensor product $K \otimes P$ a differential $\partial_\phi : K \otimes P \to K \otimes P$ given by

$$\partial_\phi = d \otimes id_P + \sum_{i=1}^{\infty} (\hat{id} \otimes p_i)(id_K \otimes \phi \otimes ... \otimes \phi \otimes id_P)(\Delta^i \otimes id_P)$$

which turns $K \otimes_\phi P = (K \otimes P, \partial_\phi)$ into a differential comodule over (K,d).

Particular case. If A is an A_∞-algebra of the form $(A,\{m_1, m_2, 0, 0, ...\})$, and P is an A_∞-module of the form $(P,\{p_1, p_2, 0, 0, ...\})$, then ϕ is the usual twisting cochain, and $K \otimes_\phi P$ coincides with the usual twisted tensor product.

As in the DG-algebra case, equivalent A_∞-twisting cochais $\phi \sim_c \psi$ produce isomorphic twisted tensor products.

Theorem 4. *If $\phi \sim_c \psi$, then $K \otimes_\phi P \xrightarrow{F_c} K \otimes_\psi P$ given by $F_c(k \otimes p) = (k \otimes p) \cap c$ is an isomorphism of DG-comodules.*

Functoriality. For a morphism of couples

$$\{g_i\},\{f_i\} : ((P,\{p_i\}),(A,\{m_i\})) \to ((P',\{p'_i\}),(A',\{m'_i\}))$$

and an A_∞-twisting cochain $\phi : K \to A$ there exists the chain map $K \otimes_\phi P \to K \otimes_{f(\phi)} P'$.

3. Minimality Theorems

Minimal A_∞-algebras. An A_∞-algebra $(M,\{m_i\})$ we call *minimal* if $m_1 = 0$, in this case (M, m_2) is *strictly* associative graded algebra. Suppose $f : (M,\{m_i\}) \to (M',\{m'_i\})$ is a weak equivalence of minimal A_∞-algebras, then $f_1 : (M, m_1 = 0) \to (M', m'_1 = 0)$, which by definition should induce isomorphism of homology, is automatically an isomorphism. It is not hard to check that in this case $\{f_i\}$ is an isomorphism of A_∞-algebras, thus each *weak equivalence of minimal A_∞-algebras is an isomorphism.* This fact motivates the word *minimal* in this notion: the Sullivan's minimal model has similar property - a weak equivalence of minimal DG algebras is an isomorphism.

Let us present here the minimality theorem from [5].

Theorem 5. *For a DG algebra (A, d, μ) its homology $H(A)$ (all $H_i(X)$-s are assumed to be free) can be equipped with a sequence of multi-operations*

$$m_i : H(A)^{\otimes i} \to H(A), \ i = 1, 2, 3, \ldots \ ; \ m_1 = 0, \ m_2 = \mu^*$$

turning $(H(A),\{m_i\})$ into a minimal *A_∞-algebra for which $m_2 = \mu^*$ and there exists a weak equivalence of A_∞-algebras*

$$f = \{f_i\} : (H(A),\{m_i\}) \to (A,\{m_1 = d, m_2 = \mu, m_3 = 0, m_4 = 0, \ldots\}).$$

This structure is unique up to an isomorphism in the category of A_∞-algebras.

Analogous results hold for the modules: if $(P, d_P, \nu_P : A \otimes P \to P)$ is a DG-module over A, then there exist on $H(P, d_P)$ a structure of A_∞-module $(H(P), \{p_i\})$ over $(H(A), \{m_i\})$ and a morhism of couples

$$(\{g_i\}, \{f_i\}) : ((H(P), \{p_i\}), (H(A), \{m_i\})) \to (P, A)$$

such that $p_1 = 0,\ p_2 = \nu^*$ and $g_1^* = id_{H(P)} : H(P) \to H(P)$.

Furthermore, by (3), there is a bijection $D_\infty(f) : D_\infty(K, A) \to D_\infty(K, A')$, this implies the following

Theorem 6. *For a twisting cochain $\phi : K \to A$, there exists an A_∞-cochain $\psi : K \to (H(A), \{m_i\})$ such that $\phi \sim f(\psi)$, consequently, there exists a chain map inducing an isomorphism in the homologies $K \otimes_\psi H(P) \to K \otimes_{f(\psi)} P \overset{\approx}{\to} K \otimes_\phi P$.*

4. Application: A_∞-model of a Fibre Bundle

The minimality theorem (5) and the theorem (6) about the lifting of twisting cochains allow one to construct an effective model of a fibre bundle. Actually, this model and higher operations $\{m_i\}$ and $\{p_i\}$ were constructed in [4]. Later, we have recognized that they form Stasheff's A_∞ structures, and the model in these terms was presented in [5]. Similar model was also presented in [8]. **Topological level.** Let $\xi = (X, p, B, G)$ be a principal G-fibration. If F is a G-space, then the action $G \times F \to F$ determines the *associated fibre bundle* $\xi(F) = (E, p, B, F, G)$ with fiber F. Thus, ξ and the action $G \times F \to F$ on the topological level determine E.

Chain level. Let $K = C_*(B),\ A = C_*(G),\ P = C_*(F)$. The classical result of E. Brown [3] states that the principal fibration ξ determines a twisting cochain $\phi : K = C_*(B) \to A = C_*(G)$ and the action on chain level $C_*(G) \otimes C_*(F) \to C_*(F)$ defines the twisted tensor product $K \otimes_\phi P = C_*(B) \otimes_\phi C_*(F)$ which gives homology modules of the total space $H_*(E)$. Thus, ξ and the action on the chain level $C_*(G) \otimes C_*(F) \to C_*(F)$ determine $H_*(E)$.

The twisting cochain ψ is not uniquely determined and it can be perturbed by the above equivalence relations for computational reasons.

Homology level. Nodar Berikashvili stated the problem to lift the previous "chain level" model of associated fibration to "homology level", i.e., to construct "twisted differential" on $C_*(B) \otimes H_*(F)$. Investigation has shown that the principal fibration ξ and the action of Pontriagin's ring $H_*(G)$ on $H_*(F)$, that is, the pairing $H_*(G) \otimes H_*(F) \to H_*(F)$ *do not determine* $H_*(E)$. But by the minimality theorem it appeared that $H_*(G)$ carries not only Pontriagin's product $H_*(G) \otimes H_*(G) \to H_*(G)$, but also a richer algebraic structure, namely, the structure of minimal A_∞-algebra $(H_*(G), \{m_i\})$, furthermore, the action $G \times F \to F$ induces not only the pairing $H_*(G) \otimes H_*(F) \to H_*(F)$, but also the structure of a minimal A_∞-module $(H_*(F), \{p_i\})$, and all these operations allow one to define correct differential on $C_*(B) \otimes H_*(F)$: according to the theorem (6), there is a weak equivalence, a homology isomorphism

$$C_*(B) \otimes_\psi H_*(F) = K \otimes_\psi H(P) \to K \otimes_\phi P = C_*(B) \otimes_\phi C_*(F) \sim C_*(E).$$

References

1. J. Adams, On the non-existence of elements of Hopf invariant one. *Ann. of Math. (2)* **72** (1960), 20–104.
2. N. Berikashvili, On the differentials of spectral sequence. *Trudy Tbiliss. Mat. Inst. Razmadze* **51** (1976), 1–105.
3. E. Brown, Twisted tensor products. I. *Ann. of Math. (2)* **69** (1959), 223–246.
4. T. Kadeishvili, On the homology theory of fibrations. *Bull. Georg. Acad. Sci.* **77** (1975), no. 2, 309–312.
5. T. Kadeishvili, On the homology theory of fibre spaces. *Russian Math. Surveys* **35** (1980), no. 3, 231–238.
6. T. Kadeishvili, Predifferential of a Fiber Bundle. *Russian Math. Surveys* **41** (1986), 135–147.
7. T. Kadeishvili, Measuring the noncommutativity of DG-algebras. Topology and noncommutative geometry. *J. Math. Sci. (N.Y.)* **119** (2004), no. 4, 494–512.
8. V. Smirnov, Homology of fiber spaces. *Russian Math. Surveys* **35** (1980), no. 3, 294–298.

9. J. D. Stasheff, Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc.* **108** (1963), 275–312.

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