Applied Categorical Structures **12:** 109–126, 2004.

The Steenrod Algebra and Theories Associated to Hopf Algebras

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Abstract. We show that the homotopy category of products of $\mathbb{Z}/p$-Eilenberg–Mac Lane spaces is an Ω-algebra which algebraically is determined by the Steenrod algebra considered as a Hopf algebra with unstable structure.

Mathematics Subject Classifications (2000): 55S05, 55S10, 55S20.

Key words: Steenrod algebra, algebraic theory, Hopf algebra.

A theory $\mathbf{T}$ is a small category with products and a model of $\mathbf{T}$ is a functor $M : \mathbf{T} \to \mathbf{Set}$ which carries products in $\mathbf{T}$ to products of sets. Let $\mathbf{model}(\mathbf{T})$ be the category of models; morphisms are natural transformations. Since the work of Lawvere [5], it is well known that many algebraic categories (like the categories of groups, algebras, Lie algebras, etc.) are such categories of models of a theory $\mathbf{T}$.

In this paper we are interested in theories arising in topology. Let R be a ring and let $\mathbf{K}(R)$ be the homotopy category consisting of products $X = K(n_1) \times \cdots \times K(n_r)$ of Eilenberg–Mac Lane spaces $K(n_i) = K(R, n_i)$ with $n_1, \ldots, n_r \geq 1$ and $r \geq 0$. Then products obviously exist in $\mathbf{K}(R)$ and hence $\mathbf{K}(R)$ is a theory for which the category of models $\mathbf{model}(\mathbf{K}(R))$ is defined. What kind of algebraic category is $\mathbf{model}(\mathbf{K}(R))$? We show for the field $R = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p$:

THEOREM A. *Let p be a prime. Then the category* $\mathbf{model}(\mathbf{K}(\mathbb{Z}/p))$ *is equivalent to the category of connected unstable algebras over the Steenrod algebra* $\mathcal{A}_{(p)}$.

For $R = \mathbb{Z}$ the algebraic category equivalent to $\mathbf{model}(\mathbf{K}(Z))$ is not known.

We observe that the theory $\mathbf{K}(R)$ has additional structure; namely it is an Ω-theory as studied in [3]. In addition multiplication maps in $\mathbf{K}(R)$ give $\mathbf{K}(R)$ the structure of an "Ω-algebra".

THEOREM B. *The Steenrod algebra* $\mathcal{A}_{(p)}$ *has an unstable structure* (B, D) *which determines algebraically the* Ω*-algebra* $\mathbf{K}(\mathbb{Z}/p)$.

In fact, we show that a "Hopf algebra $\mathcal{A}$ with unstable structure (B, D)" determines algebraically an Ω-algebra $\mathcal{K}(A, B, D)$ which for $\mathcal{A} = \mathcal{A}_{(p)}$ coincides with the topological Ω-algebra $\mathbf{K}(\mathbb{Z}/p)$.

We have seen in [3] that the universal Toda bracket $\langle \mathbf{K}(\mathbb{Z}/p) \rangle$ is an element in the third Ω-cohomology of the Ω-theory $\mathbf{K}(\mathbb{Z}/p)$. Hence we obtain by Theorem B:

COROLLARY. *The universal Toda bracket is an element*

$$\langle \mathbf{K}(\mathbb{Z}/p) \rangle \in H^3_\Omega(\mathcal{K}(\mathcal{A}_{(p)}, B, D)),$$

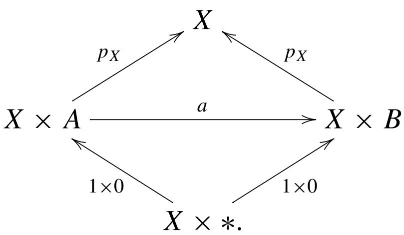
where the right-hand side is defined algebraically in terms of the Steenrod algebra $\mathcal{A}_{(p)}$.

The universal Toda bracket $\langle \mathbf{K}(\mathbb{Z}/p) \rangle$ contains all information on stable and unstable secondary cohomology operations mod p. In the book [2] we study algebraic properties of the class $\langle \mathbf{K}(\mathbb{Z}/p) \rangle$. The main result of this paper is the fact that this class is an element in the cohomology $H^3_\Omega(\mathcal{K}(\mathcal{A}_{(p)}, B, D))$ which is completely determined by the Steenrod algebra $\mathcal{A}_{(p)}$.

1. Graded Ω-theories

A *theory* $\mathbf{T}$ *with zero object* $*$ is a small category with products $A \times B$ for which the final object $*$ is also initial. Hence we have the *zero map* $0 : A \to * \to B$. An Abelian group object A in $\mathbf{T}$ is given by maps $m_A : A \times A \to A$, $n_A : A \to A$ satisfying the usual identities. A map $f : A \to B$ between Abelian group objects is *linear* if f commutes with m_A, m_B and n_A, n_B.

For an object X in $\mathbf{T}$ let $\mathbf{T}(X)$ be the category with the same objects as in $\mathbf{T}$ and with morphisms $a : A \to B$ given by commutative diagrams in $\mathbf{T}$ where px is the projection:



Each map $f : Y \to X$ induces a functor $\mathbf{T}(f) : \mathbf{T}(X) \to \mathbf{T}(Y)$ which is the identity on objects and carries $a : X \times A \to X \times B$ to $\mathbf{T}(f)(a) : Y \times A \to Y \times B$ given by the coordinates p_Y and $p_B a(f \times A)$. In particular $0 : Y \to *$ induces the functor $\mathbf{T} = \mathbf{T}(*) \to \mathbf{T}(Y)$ which preserves products. This functor carries an Abelian group object in $\mathbf{T}$ to an Abelian group object in $\mathbf{T}(Y)$.

Motivated by infinite loop spaces in topology (see Adams [1]) we introduce the following algebraic concept which describes a special case of an Ω-theory studied in [3]; see Example 1.2 below.

DEFINITION 1.1. A *graded* Ω*-theory* $\mathbf{T}$ is a theory with zero object $*$ with the following properties. A product preserving functor

(1) $\Omega : \mathbf{T} \to \mathbf{T}$ termed *loop functor*

and a sequence of Abelian group objects $K(n)$, $n \in \mathbb{Z}$, are given with

(2) $\Omega K(n) = K(n-1)$

such that all objects X of $\mathbf{T}$ are finite products of objects in this sequence. The product preserving functor $\Omega : \mathbf{T} \to \mathbf{T}$ carries the structure maps of the Abelian group object $K(n)$ to the structure maps of the Abelian group object $K(n-1)$. Hence for objects A, B in $\mathbf{T}$ the set $[A, B]$ of morphisms $A \to B$ in $\mathbf{T}$ is an Abelian group since the product B of Abelian group objects $K(n_i)$ is canonically an Abelian group object in $\mathbf{T}$. We denote the group structure in $[A, B]$ by $+$. In addition $\mathbf{T}$ is an Ω-theory; that is, a system of functors

(3) $L_X : \mathbf{T}(X) \to \mathbf{T}(X)$

is given such that L_X preserves finite products and carries each map to a linear map. These functors are natural in the sense that for all $f : Y \to X$ the diagram of functors

(4)
$$\begin{array}{ccc} \mathbf{T}(X) & \xrightarrow{L_X} & \mathbf{T}(X) \\ {\scriptstyle \mathbf{T}(f)}\downarrow & & \downarrow{\scriptstyle \mathbf{T}(f)} \\ \mathbf{T}(Y) & \xrightarrow{L_Y} & \mathbf{T}(Y) \end{array}$$

commutes. Moreover for $X = *$ the functor L_* coincides with Ω in (1). This completes the definition of a graded Ω-theory.

Let $[X \times A, B]_X$ be the group of morphisms $\alpha : X \times A \to B$ for which the composite morphism $X = X \times * \xrightarrow[1\times 0]{} X \times A \xrightarrow[\alpha]{} B$ trivial. Then α determines the map $\alpha = (p_X, \alpha) : X \times A \to Y \times B$ in $\mathbf{T}(X)$ for which via the functor L_X in (3) the induced map $L_X(p_X, \alpha) = (p_X, L_X\alpha) : X \times \Omega A \to X \times \Omega B$ in $\mathbf{T}(X)$ is defined. Hence $\alpha \mapsto L_X\alpha$ yields the homomorphism, termed *partial loop operation*,

(5) $L = L_X : [X \times A, B]_X \to [X \times \Omega A, \Omega B]_X.$

One can check that the following rules are satisfied. We consider the composite

$$Y \times T \xrightarrow{\binom{fp_Y}{a}} X \times A \xrightarrow{b} B.$$

Then we have the formula

(6) $L\left(b\begin{pmatrix} fp_Y \\ a \end{pmatrix}\right) = L(b)\begin{pmatrix} fp_Y \\ La \end{pmatrix}$

with $f : Y \to X$, $a \in [Y \times T, A]_Y$ and $b \in [X \times A, B]_X$. The projection $p_A : X \times A \to A$ is an element of $[X \times A, A]_X$ with

(7) $L(p_A) = p_{\Omega A}$, the projection $X \times \Omega A \to \Omega A$.

Finally for $b \in [X \times A, B]_X$ the map $L_X b : X \times \Omega A \to \Omega B$ is *linear* in ΩA, that is for $\alpha, \beta \in [Y, \Omega A]$ we have

(8) $(L_X b)(f, \alpha + \beta) = (L_X b)(f, \alpha) + (L_X b)(f, \beta).$

In particular maps $\Omega f : \Omega Y \to \Omega X$ are *linear*, that is, they are homomorphisms of Abelian group objects. In general, maps $Y \to X$ in $\mathbf{T}$ need not be linear.

EXAMPLE 1.2. Recall that an *infinite loop space* X is a sequence of pointed CW-spaces $X(n)$, $n \in \mathbb{Z}$, together with homotopy equivalences

$$\Omega X(n) \simeq X(n-1), \tag{$*$}$$

where $\Omega X(n)$ is the path component of the base point in the loop space of $X(n)$. We use $(*)$ as an identification in the homotopy category $(\mathbf{Top}_0^*)_\simeq$. Now let $\mathbf{T}$ be the full subcategory of $(\mathbf{Top}^*)_\simeq$ consisting of finite products $X(n_1) \times \cdots \times X(n_k)$ of spaces $X(n_i)$ given by the infinite loop space X. Since the track category $\mathcal{T}$ in $\mathbf{Top}_0^*$ with $\mathcal{T}_\simeq = \mathbf{T}$ is Ω-representable, we see by Baues and Jibladze [3] that $\mathbf{T}$ is an Ω-theory and hence $\mathbf{T}$ is a graded Ω-theory with the properties in Definition 1.1.

Of course for each Abelian group A we have the infinite loop space $K(A)$ given by *Eilenberg–Mac Lane spaces* $K(A)(n) = K(A, n)$, $n \in \mathbb{Z}$, with $K(A)(n) = *$ for $n \leq 0$. Let $\mathbf{K}(A)$ be the associated graded Ω-theory consisting of products of Eilenberg–Mac Lane spaces $K(A, n)$, $n \geq 1$.

Now let $\mathbf{T}$ be a graded Ω-theory as in Definition 1.1. Then we obtain the *algebra* $\mathcal{A}$ *of stable operations* in $\mathbf{T}$ as follows. A stable operation of degree $k \in \mathbb{Z}$ is a sequence of maps α,

$$\alpha = (\alpha_n : K(n) \to K(n+k))_{n \in \mathbb{Z}}$$

with the property $\Omega \alpha_n = \alpha_{n-1}$. Hence all α_n are linear and therefore the set $\mathcal{A}_k$ of all stable operations of degree k is an Abelian group. Moreover for k, $r \in \mathbb{Z}$ one has by composition the multiplication

(1.3) $\mu : \mathcal{A}_k \otimes_{\mathbb{Z}} \mathcal{A}_r \to \mathcal{A}_{k+r}$

carrying $\alpha \otimes \beta$ to $\alpha \circ \beta$. Here μ is a homomorphism of Abelian groups which yields an associative multiplication for the graded Abelian group $\mathcal{A} = (\mathcal{A}_k)_{k \in \mathbb{Z}}$.

The stable operations of degree 0 yield the ring $\mathcal{A}_0$ so that $\mathcal{A}_k$ is an $\mathcal{A}_0$-bimodule and the multiplication μ of $\mathcal{A}$ is actually defined on the tensor product over $\mathcal{A}_0$

(1.4) $\mu : \mathcal{A}_k \otimes_{\mathcal{A}_0} \mathcal{A}_r \to \mathcal{A}_{k+r}.$

In this sense $\mathcal{A}$ is a Z-graded algebra over $\mathcal{A}_0$.

For example if the graded Ω-theory $\mathbf{T} = \mathbf{K}(\mathbb{Z}/p)$ is given as in Example 1.2 by Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p, n)$, $n \geq 1$, then $\mathcal{A} = \mathcal{A}(p)$ is the *Steenrod algebra*; see [7], with $\mathcal{A}_k = 0$ for $k < 0$.

2. Ω-algebras

Let R be a commutative ring. Then the graded Ω-theory $\mathbf{K}(R)$ given by product of Eilenberg–Mac Lane spaces $K(R, n)$ has canonically the structure of an Ω-algebra. This is the reason why we introduce and study in this section the algebraic concept of an Ω-algebra.

Let $\mathbf{Mod}_*(R)$ be the category of Z-graded R-modules $X = (X_i)_{i\in\mathbb{Z}}$ with the graded tensor product $X \otimes Y$ given by

(2.1) $$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes_R Y_j.$$

Here X is non-negative if $X_i = 0$ for $i < 0$. A non-negative graded R-module A which has the structure of a monoid in $(\mathbf{Mod}_*(R), \otimes)$ is termed a *graded algebra*. Let R be the graded R-module concentrated in degree 0 which is R in that degree. Then of course R is also a graded algebra; in fact, the initial object in the category of graded algebras. A map $\varepsilon : A \to R$ between graded algebras is termed an *augmentation* of A and A is a *connected* algebra if ε is an isomorphism in degree 0, $A^0 = R$. The *interchange map*

(1) $$T : X \otimes Y \to Y \otimes X$$

is defined by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$, where $|x|$ is the degree of a homogeneous $x \in X$, i.e. $|x| = n$ if and only if $x \in X_n$. The graded algebra A is *commutative* if the multiplication $m : A \otimes A \to A$ satisfies $mT = m$. We also write

(2) $$m(x \otimes y) = x \cdot y.$$

The tensor product $A \otimes_R B$ of graded algebras A and B is canonically a graded algebra with the multiplication

(3) $$(x \otimes y)(x' \otimes y') = (-1)^{|y||x'|}(x \cdot x') \otimes (y \cdot y'),$$

where $|y|$ is the degree of $y \in B$. One readily checks that $A \otimes B$ is commutative if A and B are commutative.

Let $\mathbf{Alg}_0$ be the category of connected commutative graded algebras over R. This is a category in which R is the initial and the final object (zero object). Moreover the sum in the category $\mathbf{Alg}_0$ is given by the tensor product with the inclusions

(4) $$\begin{cases} A = A \otimes R \xrightarrow{1\otimes !} A \otimes B, \\ B = R \otimes B \xrightarrow{!\otimes 1} A \otimes B. \end{cases}$$

Here $! : R \to A$ is the unit. Let

(5) $$\tilde{A} = \text{kernel}(\varepsilon : A \to R)$$

be the *augmentation ideal* of A, i.e. the kernel of the augmentation ε.

We introduce the algebraic concept of an Ω-algebra over R as follows.

DEFINITION 2.2. Let $\mathbf{T}$ be a graded Ω-theory as in (1.1) with

(1) $K(n) = *$ for $n \leq 0$.

Assume a ring homomorphism $R \to \mathcal{A}(\mathbf{T})_0$ is given, where $\mathcal{A}(\mathbf{T})_0$ is the ring of degree 0 stable operations of $\mathbf{T}$. Then we have for any object X in $\mathbf{T}$ the graded R-module $\tilde{H}^*(X)$ with

(2) $\tilde{H}^n(X) = [X, K(n)]$ for $n \in \mathbb{Z}$.

Clearly by (1) we have $\tilde{H}^n(X) = 0$ for $n \leq 0$ and we have a canonical element $[n] \in \tilde{H}^*(K(n))$ given by the identity of $K(n)$. We define the graded R-module $H^*(X)$ by the direct sum

(3) $H^*(X) = R \otimes \tilde{H}^*(X)$,

where R is concentrated in degree 0. Then $\mathbf{T}$ is an Ω-*algebra* (over R) if multiplication maps

(4) $\mu = \mu_{i,j} : K(i) \times K(j) \to K(i+j)$

are given for $i, j \geq 1$ which induce an R-linear map

(5) $$\begin{cases} \mu_* : H^*(X) \otimes_R H^*(X) \to H^*(X) \quad \text{by} \\ \mu_*(\xi \otimes \eta) = \mu_{i,j}(\xi, \eta) \end{cases}$$

for $\xi \in \tilde{H}^i(X)$, $\eta \in \tilde{H}^j(X)$ such that $(H^*(X), \mu)$ is a connected commutative graded algebra (hence an object in $\mathbf{Alg}_0$) with $R \to H^*(X) \to R$ given by the inclusion and projection respectively. The trivial map $K(i) \to * \to K(j)$ represents the zero element in $H^j(K(i))$ and the composite

$$K(i) \xrightarrow{([i],0)} K(i) \times K(j) \xrightarrow{\mu} K(i+j)$$

is trivial, where $[i]$ is the identity of $K(i)$, since $\mu([i], 0) = [i] \cdot 0 = 0$. Hence with the notation in Definition 1.1(3) we have

$$\mu_{i,j} \in [K(i) \times K(j), K(i+j)]_{K(i)},$$

so that the partial loop operation L is defined on $\mu_{i,j}$. In addition let

(6) $$L\mu_{i,j} = \begin{cases} \mu_{i,j-1} & \text{for } j > 1, \\ 0 & \text{for } j = 1. \end{cases}$$

This completes the definition of an Ω-algebra (over R).

In an Ω-algebra $\mathbf{T}$ over R we have for objects $X, Y \in \mathbf{T}$ the canonical map in $\mathbf{Alg}_0$

(2.3) $\tau : H^*(X) \otimes_R H^*(Y) \to H^*(X \times Y)$

which carries $\xi \otimes \eta$ to the product $(p_X^*\xi) \cdot (p_Y^*\eta)$, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. We say that the Ω-algebra $\mathbf{T}$ over R has the *Künneth property* if this τ is an isomorphism for all objects X, Y.

LEMMA 2.4. *For any Ω-algebra over R the induced map*

$$\mu_{i,j}^* : H^*K(i+j) \to H^*(K(i) \times K(j))$$

carries $[i+j]$ *to* $\tau([i] \otimes [j])$.

Proof. In fact $\mu_{i,j}^*[i+j]$ is represented by $\mu_{i,j} : K(i) \times K(i) \to K(i+j)$ and $\tau([i] \otimes [j])$ by definition is represented by the composite

$$K(i) \times K(j) \xrightarrow{(p_1,p_2)} K(i) \times K(j) \xrightarrow{\mu} K(i+j),$$

where p_1 and p_2 are the projections, so that (p_1, p_2) is the identity. □

EXAMPLE 2.5. Let R be a commutative ring and let R_{ab} be the underlying Abelian group of R. Then the graded Ω-theory $\mathbf{K}(R)$ associated to the Eilenberg–Mac Lane spaces $K(n) = K(R_{\text{ab}}, n)$ (see Example 1.2) is an Ω-algebra with multiplication maps $(i, j > 1)$

$$\mu_{i,j} : K(i) \times K(j) \xrightarrow{q} K(i) \wedge K(j) \xrightarrow{\tilde{\mu}} K(i+j).$$

Here q is the quotient map for the smash product $X \wedge Y = X \times Y / X \vee Y$ and $\tilde{\mu}$ is up to homotopy the unique map for which the following diagram of homotopy groups commutes with $n = i + j$.

$$\begin{array}{ccc} \pi_n(K(i) \wedge K(j)) & \xrightarrow{\tilde{\mu}_*} & \pi_n K(i+j) \\ \| & & \| \\ R_{\text{ab}} \otimes_{\mathbb{Z}} R_{\text{ab}} & \xrightarrow{m_R} & R_{\text{ab}}. \end{array}$$

Here m_R is the multiplication map of the ring R.

We call $\mathbf{K}(R)$ the *Eilenberg–Mac Lane Ω-algebra* of the commutative ring R. It is well known that $\mathbf{K}(R)$ has the Künneth property if R is a finite field. This is an obvious consequence of the Künneth theorem.

Now let $\mathbf{T}$ be an Ω-algebra over R which has the Künneth property. Then H^* in Definition 2.2 yields a contravariant functor

(2.6) $\quad H^* : \mathbf{T} \to \mathbf{Alg}_0$

which is faithful and carries products to coproducts. Hence the opposite category $\mathbf{T}^{\text{op}}$ can be considered to be asubcategory of $\mathbf{Alg}_0$. This leads to the following concept of Ω-sequence of algebras with $H(n) = H^*(K(n))$.

DEFINITION 2.7. An Ω-*sequence of algebras* (over R) consists of a sequence $H(n)$ with $n \in \mathbb{Z}$ and the structure (χ, Ω, μ) with the following properties.

The object $H(n)$ is an Abelian cogroup object in $\mathbf{Alg}_0$ with the structure maps in $\mathbf{Alg}_0$

$$(1)\quad \begin{cases} + : H(n) \to H(n) \otimes H(n), \\ - : H(n) \to H(n) \end{cases}$$

and $H(n) = R$ for $n \leq 0$. This shows that $\mathrm{Hom}_{\mathbf{Alg}_0}(H(n), X)$ is an Abelian group with addition induced by $+$ in (1) where X is an object in $\mathbf{Alg}_0$. For $n, m \geq 0$ the structure χ is a homomorphism of Abelian groups

$$(2)\quad \chi = \chi_n^m : \tilde{H}^m(n) \to \mathrm{Hom}_{\mathbf{Alg}_0}(H(m), H(n))$$

and we write $\chi(\alpha) = \bar{\alpha}$. There is an element $[n] \in \tilde{H}^n(n)$ with

(a) $\overline{[n]} = 1_{H(n)}$ (identity),
(b) $\bar{\alpha}([m]) = \alpha$,
(c) $\overline{\bar{\beta}(\alpha)} = \bar{\beta} \circ \bar{\alpha}$ for $\alpha \in \tilde{H}^m(n)$, $\beta \in \tilde{H}^n(k)$,
(d) $r[n] : H(n) \to H(n)$ is linear (i.e. a morphism of cogroups) for $r \in R$.

Now (d) together with (1) implies that for each object X in $\mathbf{Alg}_0$ the set $\mathrm{Hom}_{\mathbf{Alg}_0}(H(n), X)$ is an R-module. See (2.8) below. Next the structure

$$(3)\quad \Omega = \Omega_n : \tilde{H}(n) \to \tilde{H}(n-1)$$

is a degree (-1) homomorphism of graded R-modules satisfying

(a) $\Omega[n] = [n-1]$,
(b) $\Omega(\bar{\beta}(\alpha)) = \overline{\Omega\beta}(\Omega\alpha)$,
(c) $\Omega(\xi\eta) = 0$ for $\xi, \eta \in \tilde{H}(n)$ where $\xi\eta$ is the product in the algebra $H(n)$,
(d) $\overline{\Omega\beta} : H(n-1) \to H(k-1)$ is linear.

Finally for $i, j \geq 1$ the structure

$$(4)\quad \mu = \mu^{i,j} : H(i+j) \to H(i) \otimes H(j)$$

is a morphism in $\mathbf{Alg}_0$ satisfying

(a) $\mu^{i,j}([i+j]) = [i] \otimes [j]$,
(b) $T\mu^{i,j} = (-1)^{ij}\mu^{j,i}$ (see (2.1)(1)),
(c) $(\mu^{i,j} \otimes 1)\mu^{i+j,t} = (1 \otimes \mu^{j,t})\mu^{i,j+t}$,
(d) $(\bar{\alpha_1}, \bar{\alpha}_2)\mu^{i,j} = \overline{\alpha_1 \cdot \alpha_2}$ for $\alpha_1 \in H^i(k)$, $\alpha_2 \in H^j(k)$.

Moreover $\mu^{i,j}$ is linear in $H(j)$, that is, for all X in $\mathbf{Alg}_0$ and $\alpha : H(i) \to X$, $\beta_1, \beta_2 : H(j) \to X$ we have

(e) $(\alpha, \beta_1 + \beta_2)\mu^{i,j} = (\alpha, \beta_1)\mu^{i,j} + (\alpha, \beta_2)\mu^{i,j}$,
(f) $(1 \otimes \overline{r[j]})\mu^{i,j} = \mu^{i,j}\overline{r[i+j]}$ for $r \in R$.

Next we introduce for $X = H(i) \otimes H(j)$ the homomorphism of R- modules

$$\chi : \tilde{X}^m \to \mathrm{Hom}_{\mathbf{Alg}_0}(H(m), X)$$

which carries $\xi \otimes \eta$ with $\xi \in \tilde{H}^a(i)$ and $\eta \in \tilde{H}^b(j)$, $a + b = m$, to the composite

$$\chi(\xi \otimes \eta) : H(m) \xrightarrow{\mu} H(a) \otimes H(b) \xrightarrow{\bar{\xi}\otimes\bar{\eta}} H(i) \otimes H(j).$$

This homomorphism is well defined by (e) and (f). Moreover χ satisfies for $\alpha \in \tilde{H}^m(i + j)$ the equation

(g) $\quad \mu \circ \bar{\alpha} = \chi(\mu\alpha) : H(m) \to H(i) \otimes H(j), \quad$ where $\mu = \mu^{i,j}$.

This completes the definition of an Ω-sequence of algebras.

The condition (2)(b) shows that χ is injective so that $[n]$ is uniquely determined. We define an *R-module structure of* $\mathrm{Hom}_{\mathbf{Alg}_0}(H(m), X)$ by the Abelian group structure in (2) and by defining for $r \in R$ and $\alpha : H(m) \to X$ the element $r \cdot \alpha$ via the composite

(2.8) $\quad r \cdot \alpha = \alpha \circ \overline{r[m]}$.

Then we have for $r, s \in R$

$$\begin{aligned}
(r+s)\alpha &= \alpha \circ \overline{(r+s)[m]} \\
&= \alpha \circ (\overline{r[m]} + \overline{s[m]}) \quad \text{see (2)} \\
&= \alpha \circ (\overline{r[m]}, \overline{s[m]})(+) \\
&= (\alpha\overline{r[m]}, \alpha\overline{s[m]})(+) \\
&= \alpha\overline{r[m]}, \alpha\overline{s[m]} \\
&= r\alpha + s\alpha, \\
r(\alpha+\beta) &= (\alpha+\beta)\overline{r[m]} \\
&= (\alpha, \beta)(+)\overline{r[m]} \\
&= (\alpha, \beta)(\overline{r[m]} \otimes \overline{r[m]})(+) \quad \text{see (2)(d)} \\
&= (\alpha\overline{r[m]}, \beta\overline{r[m]})(+) \\
&= \alpha\overline{r[m]} + \beta\overline{r[m]} \\
&= r\alpha + r\beta.
\end{aligned}$$

The homomorphism χ in (2) is actually an injective homomorphism of R-modules since we have for $\alpha \in \tilde{H}^m(n)$

$$\begin{aligned}
r \cdot \bar{\alpha} &= \bar{\alpha} \circ \overline{r[m]} \\
&= \overline{\bar{\alpha}(r[m])} \\
&= \overline{r\bar{\alpha}([m])} \qquad (1) \\
&= \overline{r\alpha}.
\end{aligned}$$

The condition (4)(c) shows that μ yields a canonical map

(2) $\mu = \mu^{n_1,\dots,n_\nu} : H(n_1 + \cdots + n_\nu) \to H(n_1) \otimes \cdots \otimes H(n_\nu)$

for $n_i \geq 1$ and $\nu \geq 1$. For example $\mu^{i,j,t} = (\mu^{i,j} \otimes 1)\mu^{i+j,t}$. For $\nu = 1$ let μ be the identity.

LEMMA 2.9. *Each* Ω*-algebra* $\mathbf{T}$ (*over* R) *with the Künneth property yields the associated* Ω*-sequence of algebras* (*over* R) *by defining*

$H(n) = H^*K(n)$,
$\chi(\alpha) = \alpha^*$,
$\Omega(\alpha)$ *defined by the loop functor* Ω,
$\mu^{i,j} = \tau^{-1}(\mu_{i,j})^*$, *see* (2.3).

One readily checks that the properties in Definition 2.7 are satisfied.
On the other hand we get

THEOREM 2.10. *Each* Ω*-sequence* H *of algebras* (*over* R) *yields an associated* Ω*-algebra* $\mathbf{T}_H$ (*over* R) *which satisfies the Künneth property. This yields a* 1-1-*correspondence between isomorphism classes of* Ω*-algebras with Künneth property and* Ω*-sequences of algebras respectively. The inverse of this correspondence is given by Lemma* 2.9.

Proof. Let an Ω-sequence of algebras be given as in Definition 2.7. We construct the associated Ω-algebra $\mathbf{T} = \mathbf{T}_H$ as follows. We first define a subcategory with coproducts

(1) $\mathbf{T}_0 \subset \mathbf{Alg}_0$

and $\mathbf{T}$ as a category with products is the opposite category of $\mathbf{T}_0$, that is $\mathbf{T} = \mathbf{T}_0^{\mathrm{op}}$. Of course $\mathbf{T}_0$ corresponds to the image category of the functor (2.6). Objects in $\mathbf{T}_0$ are tensor products

(2) $X = H(n_1) \otimes \cdots \otimes H(n_\nu)$

with $n_i \geq 0$ and $\nu \geq 0$. We define for $m \geq 0$ the homomorphism of R-modules

(3) $\chi : \tilde{X}^m \to \mathrm{Hom}_{\mathbf{Alg}_0}(H(m), X)$

which carries $\alpha_1 \otimes \cdots \otimes \alpha_\nu \in \tilde{X}^m$ with $\alpha_i \in \tilde{H}^{m_i}(n_i)$ to the composite

$$H(m) \xrightarrow{\mu} H(m_1) \otimes \cdots \otimes H(m_\nu) \xrightarrow{\bar{\alpha}_1 \otimes \cdots \otimes \bar{\alpha}_\nu} X.$$

For $\alpha \in \tilde{X}$ we write $\chi(\alpha) = \bar{\alpha}$. Now we define the morphisms $Y \to X$ in $\mathbf{T}_0$ with $Y = H(k_1) \otimes \cdots \otimes H(k_t)$ to be the maps

(4) $(\bar{\beta}_1, \dots, \bar{\beta}_t) : Y \to X \in \mathbf{T}_0$

with $\bar{\beta}_i : H(k_i) \to X$ given by an element $\beta_i \in \tilde{X}^{k_i}$ via χ above. Here we use the coproduct property of the tensor product in $\mathbf{Alg}_0$. Using Definition 2.7(4)(g) we see by an induction argument based on the other properties in Definition 2.7(4) that $\mathbf{T}_0$ is a well defined subcategory of $\mathbf{Alg}_0$. Of course the category $\mathbf{T}_0$ has coproducts.

Next we set $\Omega H(n) = H(n-1)$ and

(5) $\Omega X = H(n_1 - 1) \otimes \cdots \otimes H(n_\nu - 1).$

Using Ω in Definition 2.7(3) we define the degree (-1) homomorphism of graded R-modules

(6) $\Omega : \tilde{X} \to \widetilde{\Omega X}$

as follows. Take $\alpha_1 \otimes \cdots \otimes \alpha_\nu \in \tilde{X}$ with $\alpha_i \in H^{m_i}(n_i)$. Then we set

$$\Omega(\alpha_1 \otimes \cdots \otimes \alpha_\nu) = \Omega\alpha_i$$

if $\alpha_j \in R$ for $j \neq i$, $j = 1, \ldots, \nu$, and we set $\Omega(\alpha_1 \otimes \cdots \otimes \alpha_\nu) = 0$ otherwise. We now define a functor

(7) $\Omega : \mathbf{T}_0 \to \mathbf{T}_0$

by (5) and by

$$\Omega(\bar{\beta}_1, \ldots, \bar{\beta}_t) = (\overline{\Omega\beta_1}, \ldots, \overline{\Omega\beta_t}),$$

where we use (4) and (6) and (3). The functor Ω in (7) defines a functor $\Omega : \mathbf{T} \to \mathbf{T}$ by $\Omega(f)^{\mathrm{op}} = (\Omega f)^{\mathrm{op}}$ and this is a functor as in Definition 1.1(1). Next we consider the partial loop operation L in $\mathbf{T}$ which is obtained by the function

(8) $L : [B, X \otimes A]_X \to [\Omega B, X \otimes \Omega B]_X,$

where $[\ ,\ ]$ denotes the group of morphisms in $\mathbf{T}_0$ and X, A, B are objects in $\mathbf{T}_0$. Here L is compatible with the coproduct structure in B by Definition 1.1(5) so that it suffices to define L for $B = H(m)$. In this case we have

(9) $[H(m), X \otimes A]_X \overset{\chi}{=\!=} \mathrm{kernel}(p_X : (X \tilde{\otimes} A)^m \to \tilde{X}^m)\ ,$

where $X \tilde{\otimes} A$ is the augmentation ideal of $X \otimes A$. Hence we get

(10) $[H(m), X \otimes A]_X = \tilde{A}^m \oplus (\tilde{X} \otimes \tilde{A})^m.$

For $x \in \tilde{A}^m$ let $L(x) = \Omega x \in \widetilde{\Omega A}^{m-1}$ be given by (6) and for $x \otimes a \in \tilde{X} \otimes \tilde{A}$ let $L(x \otimes a) = x \otimes \Omega a \in \tilde{X} \otimes \widetilde{\Omega A}$. This defines via (10) the function (8). Now it is easy to check that $(\mathbf{T}, \Omega, L)$ is a graded Ω-theory in the sense of Definition 1.1. Finally let $\mu_{i,j}$ in $\mathbf{T}$ be the dual of $\mu^{i,j}$ in $\mathbf{T}_0$ given by $\mu^{i,j}$ in Definition 2.7(4). Then one can check that $\mathbf{T}_H = (\mathbf{T}, \mu)$ is an Ω-algebra over R. □

3. Ω-algebras over a Hopf Algebra

It is well known that the Steenrod algebra $\mathcal{A}_{(2)}$ is a Hopf algebra and that the cohomology ring $H^*(X;\mathbb{Z}/2)$ is an "algebra over the Hopf algebra $\mathcal{A}_{(2)}$". This implies a certain structure for the Eilenberg–Mac Lane Ω-algebra $\mathbf{K}(\mathbb{Z}/2)$ which we study in this section. This leads to a complete algebraic characterization of the Ω-algebra $\mathbf{K}(\mathbb{Z}/2)$ by use of a classical result of Serre [6] in the next section.

Let R be a commutative ring. Recall that the tensor product $A \otimes B$ of graded algebras over R is a graded algebra. We say that an augmented graded algebra $\mathcal{A}$ together with an algebra map

(3.1) $\quad \psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

is a *Hopf algebra* if the compositions

$$\mathcal{A} \xrightarrow{\psi} \mathcal{A} \otimes \mathcal{A} \xrightarrow{1\otimes\varepsilon} \mathcal{A} \otimes R \cong \mathcal{A}$$

and

$$\mathcal{A} \xrightarrow{\psi} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\varepsilon\otimes 1} R \otimes \mathcal{A} \cong \mathcal{A}$$

are both the identity. The Hopf algebra is *coassociative*, resp. *cocommutative,* if $(\psi \otimes 1)\psi = (1 \otimes \psi)\psi$, resp. $T\psi = \psi$.

Now let M be a left $\mathcal{A}$-module, where $\mathcal{A}$ is a Hopf algebra and assume that M is a connected graded commutative algebra (over R) with multiplication

(3.2) $\quad m : M \otimes M \to M.$

Here $M \otimes M$ is an $\mathcal{A} \otimes \mathcal{A}$-module and hence via (3.1) an $\mathcal{A}$-module. We say that M is an (connected graded commutative) *algebra over the Hopf algebra* $\mathcal{A}$ if m is a homomorphism of $\mathcal{A}$-modules.

Let $\mathbf{Alg}_0(\mathcal{A})$ be the category of such algebras over the Hopf algebra $\mathcal{A}$. Morphisms are maps in $\mathbf{Alg}_0$ which are also homomorphisms of $\mathcal{A}$-modules. The ring R is an $\mathcal{A}$-module via $\varepsilon : \mathcal{A} \to R$ and hence an object in $\mathbf{Alg}_0(\mathcal{A})$ which is the zero object of $\mathbf{Alg}_0(\mathcal{A})$.

DEFINITION 3.3. An *Ω-algebra* $\mathbf{T}$ *over the Hopf algebra* $\mathcal{A}$ is an Ω-algebra (over R) as in Definition 2.2 together with a map between algebras over R

(1) $\quad \mathcal{A} \to \mathcal{A}(\mathbf{T})$

with the following properties. Here $\mathcal{A}(\mathbf{T})$ is the algebra of stable operations of $\mathbf{T}$ and (1) extends the algebra map $R \to \mathcal{A}(\mathbf{T})_0$ in Definition 2.2, see (1.4).

Each object X in $\mathbf{T}$ yields the graded algebra $H^*(X)$ as in Definition 2.2 which via (1) is a left $\mathcal{A}$-module. In fact $\mathcal{A}$ acts on $\tilde{H}^*(X)$ by composition and on R by the augmentation ε. Now we assume that $H^*(X)$ in addition is an algebra over the Hopf algebra $\mathcal{A}$ so that

(2) $\quad \mu_* : H^*(X) \otimes H^*(X) \to H^*(X)$

in Definition 2.2(5) is a homomorphism of $\mathcal{A}$-modules. This completes the definition.

If the Künneth property holds in **T** we can characterize an Ω-algebra over a Hopf algebra $\mathcal{A}$ by the following data:

DEFINITION 3.4. An *Ω-sequence of algebras over the Hopf algebra $\mathcal{A}$* is an Ω-sequence of algebras as in Definition 2.7 with the following additional properties. Each $H(n)$ is an algebra over the Hopf algebra $\mathcal{A}$ in $\mathbf{Alg}_0(\mathcal{A})$. Moreover for each $\alpha \in \mathcal{A}_k$ the map

$$H(n) \to H(n+k), \quad x \mapsto \alpha x$$

is linear with respect to the Abelian cogroup structure Definition 2.7(1). This generalizes Definition 2.7(1)(d). Moreover χ in Definition 2.7(2) carries elements α to morphisms $\bar{\alpha}$ in $\mathbf{Alg}_0(\mathcal{A})$, that is $\bar{\alpha}$ is a homomorphism of $\mathcal{A}$-modules. Also Ω and $\mu^{i,j}$ are homomorphisms of $\mathcal{A}$-modules; see Definition 2.7(3) and Definition 2.7(4). This completes the definition.

As in Lemma 2.9 we see that each Ω-algebra **T** over the Hopf algebra $\mathcal{A}$ for which **T** has the Künneth property yields canonically a sequence as in Definition 3.4. In fact this yields the following specification of Theorem 2.10.

THEOREM 3.5. *There is a* 1-1-*correspondence between isomorphism classes of Ω-algebras over the Hopf algebra $\mathcal{A}$ which satisfy Künneth property, and isomorphism classes of Ω-sequences of algebras over the Hopf algebra $\mathcal{A}$.*

4. The Eilenberg–Mac Lane Ω-algebra $\mathbf{K}(\mathbb{Z}/2)$

Recall that the Eilenberg–Mac Lane Ω-algebra $\mathbf{K}(\mathbb{Z}/2)$ is the full subcategory of the homotopy category of pointed spaces consisting of products of Eilenberg–Mac Lane spaces

(4.1) $K(n) = K(\mathbb{Z}/2, n), \quad n \geq 1.$

It is clear that $\mathbf{K}(\mathbb{Z}/2)$ is a graded Ω-theory (see Section 1) and also an Ω-algebra with the Künneth property (see Section 2).

PROPOSITION 4.2. $\mathbf{K}(\mathbb{Z}/2)$ *is an Ω-algebra over the Steenrod algebra $\mathcal{A}_{(2)}$ in the sense of Definition* 3.3.

This follows readily from the fact that the cohomology ring $H^*(X;\mathbb{Z}/2)$ of a space is an algebra over the Hopf algebra $\mathcal{A}_{(2)}$, see [7]. Hence by Theorem 3.5 the Ω-algebra $\mathbf{K}(\mathbb{Z}/2)$ is completely determined by the associated Ω-sequence of algebras over the Hopf algebra $\mathcal{A}_{(2)}$

(4.3) $H(n) = H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2).$

These algebras are computed by Serre [6] as follows; compare [7], II, Section 5.

EXAMPLE 4.4. The Steenrod algebra $\mathcal{A}_{(2)}$ has an unstable structure (B, D) given as follows. The structure B consists of a sequence of left ideals ($n \geq 0$)

(1) $\cdots \subset B(n+1) \subset B(n) \subset \cdots \subset B(0) \subset \mathcal{A}_{(2)}$

in the algebra $\mathcal{A}_{(2)}$. Here $B(0) = \mathrm{kernel}(\varepsilon)$ is the augmentation ideal and $B(n)$ as a $\mathbb{Z}/2$-vector space is spanned by all admissible monomials of excess greater than n; see [7], II.5. An $\mathcal{A}_{(2)}$-module X is termed *unstable* if $B(n)X^n = 0$ for $n \geq 0$ or equivalently if for $x \in X$ one has

(2) $i > |x|$ implies $\mathrm{Sq}^i(x) = 0$.

Moreover for each unstable $\mathcal{A}_{(2)}$-module X the structure D consists of an ideal

(3) $D_X \subset T(X)$

where $T(X)$ is the tensor algebra of X. Here D_X is the ideal generated by the elements ($x, y \in X$)

$$\begin{cases} x \otimes y - (-1)^{|x||y|} y \otimes x, & \\ \mathrm{Sq}^n(x) - x \otimes x & \text{for } |x| = n \geq 0. \end{cases}$$

Compare [7], II.5.3. Accordingly an algebra H over the Steenrod algebra $\mathcal{A}_{(2)}$ is termed *unstable* if (2) holds and if for $x \in H$

(4) $i = |x|$ implies $\mathrm{Sq}^i(x) = x \cdot x$.

Here (2) and (4) correspond to axioms for the squaring operations Sq^i in [7], Section 1.

The unstable structure (B, D) of $\mathcal{A}_{(2)}$ has additional properties as described in the next definition.

DEFINITION 4.5. Let R be a commutative ring and let $\mathcal{A}$ be a Hopf algebra over R. We say that $\mathcal{A}$ is an *unstable Hopf algebra* if a structure (B, D) is given with the following properties. The structure B consists of a sequence of left ideals ($n \geq 0$)

(1) $\cdots \subset B(n+1) \subset B(n) \subset \cdots \subset B(0) \subset \tilde{A}$,

where $\tilde{\mathcal{A}}$ is the augmentation ideal of $\mathcal{A}$. We say that a (non-negatively) graded $\mathcal{A}$-module X is B-unstable if

(2) $B(n) \cdot X^n = 0$ for $n \geq 0$.

Moreover the structure B has the property that for B-unstable modules X, Y the tensor product $X \otimes Y$ with the $\mathcal{A}$-module structure given by $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is again a B-unstable $\mathcal{A}$-module. This shows that the tensor algebra $T(X)$ which is the direct sum of tensor powers $X^{\otimes n} = X \otimes \cdots \otimes X$, $n \geq 0$, is an algebra over the Hopf algebra $\mathcal{A}$ and $T(X)$ is B-unstable as an $\mathcal{A}$-module.

For each B-unstable module X the structure D yields an ideal

(3) $D_X \subset T(X)$,

where D_X is an $\mathcal{A}$-submodule and $x \otimes y - (-1)^{|x||y|} y \otimes x \in D_X$ for $x, y \in X$. Let H be an augmented graded commutative algebra over the Hopf algebra $\mathcal{A}$. Then we say that H is (B, D)-*unstable* if $\tilde{H}$ is a B-unstable module and if the canonical algebra map $T(\tilde{H}) \to H$ extending the inclusion $\tilde{H} \to H$ carries D_H to 0. The structure D_X in (3) is natural in X and D has the property that for (B, D)-unstable algebras H, H' also the tensor product $H \otimes H'$ is (B, D)-unstable. In particular we obtain for a B-unstable module X the (B, D)-unstable algebra generated by X

(4) $U(X) = T(X)/D_X$.

We define for a B-unstable module X the B-unstable modules $X/\sim$ and $X/\approx$ by

$$X/\sim = \text{image}(X \to \tilde{U}(X)),$$
$$X/\approx = \text{image}(X \to \tilde{U}(X) \to \tilde{U}(X)/\tilde{U}(X) \cdot \tilde{U}(X)) = \tilde{U}(X)/\tilde{U}(X) \cdot \tilde{U}(X).$$

We in particular have the B-unstable modules $(\mathcal{A}/B(n))[n]$ generated by a single element $[n]$ in degree n. Then (B, D) has the property that there is a map of $\mathcal{A}$-modules $(n \geq 1)$

(5) $\tilde{\Omega} : (\mathcal{A}/B(n))[n]/\approx \longrightarrow (\mathcal{A}/B(n-1))[n-1]/\sim$

which caries $[n]$ to $\Omega[n] = [n-1]$. Since the left-hand side is generated by $[n]$ as an $\mathcal{A}$-module, we set that $\tilde{\Omega}$ is uniquely determined. This completes the definition.

For an unstable Hopf algebra $(\mathcal{A}, B, D)$ we obtain the full subcategory

(4.6) $\mathbf{Alg}_0(\mathcal{A}, B, D) \subset \mathbf{Alg}_0(\mathcal{A})$

consisting of connected (B, D)-unstable algebras. This subcategory is closed under tensor products and hence has sums. Moreover the object $(n \geq 1)$

(1) $H(n) = U((\mathcal{A}/B(n))[n])$

has the following freeness property. For each object H in $\mathbf{Alg}_0(\mathcal{A}, B, D)$ and element $x \in H^n$ there is a unique morphism in $\mathbf{Alg}_0(A, B, D)$

(2) $\bar{x} : H(n) \to H$

which carries $[n]$ to x. Here $H(n)$ is termed the *completely free* (B, D)-unstable algebra generated by $[n]$.

The result of Serre [6] on $H(n)$ in (4.1) now can be stated as follows; compare [7].

PROPOSITION 4.7. *The Steenrod algebra* $\mathcal{A}_{(2)}$ *with the structure* (B, D) *in Example* 4.4 *is an unstable Hopf algebra and*

$$H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) = H(n)$$

is the completely free (B, D)*-unstable algebra generated by* $[n]$.

We are now able to determine algebraically the Eilenberg–Mac Lane Ω-algebra $\mathbf{K}(\mathbb{Z}/2)$ in (4.1). For this we need the following result.

PROPOSITION 4.8. *Let* $(\mathcal{A}, B, D)$ *be an unstable Hopf algebra. Then one gets an* Ω*-sequence of algebras over the Hopf algebra* $\mathcal{A}$ *by the completely free algebras* $H(n)$*,* $n \geq 1$*, in* (4.6)(1)*. For this let*

$$\begin{aligned} + : H(n) &\rightarrow H(n) \otimes H(n), \\ - : H(n) &\rightarrow H(n) \end{aligned}$$

be the maps in $\mathbf{Alg}_0(\mathcal{A}, B, D)$ *which carry* $[n]$ *to* $[n] \otimes 1 + 1 \otimes [n]$ *and* $-[n]$ *respectively. Moreover let*

$$\chi : \tilde{H}^m(n) \rightarrow \mathrm{Hom}_{\mathbf{Alg}_0}(H(m), H(n))$$

be the function which carries x *to* $\bar{x}$ *as in* (4.6)(2)*. Let* Ω *be the composite*

$$\begin{aligned} \Omega : \tilde{H}(n) &\longrightarrow (\mathcal{A}/B(n))[n]/{\approx} \xrightarrow{\tilde{\Omega}} (\mathcal{A}/B(n-1))[n-1]/{\approx} \\ &\longrightarrow \tilde{H}(n-1) \end{aligned}$$

where $\tilde{\Omega}$ *is defined in Definition* 4.5(5)*. Finally let*

$$\mu^{i,j} : H(i+j) \rightarrow H(i) \otimes H(j)$$

be the map in $\mathbf{Alg}_0(\mathcal{A}, B, D)$ *which carries* $[i+j]$ *to* $[i] \otimes [j]$.

Proof. One readily checks that the properties of an unstable Hopf algebra in Definition 4.5 imply that the data in the theorem satisfy all properties in Definitions 2.7 and 3.4. □

Hence we get by use of Theorem 3.5 the next result.

COROLLARY 4.9. *Each unstable Hopf algebra* $(\mathcal{A}, B, D)$ *yields canonically an* Ω*-algebra* $\mathcal{K}(\mathcal{A}, B, D)$ *over the Hopf algebra* $\mathcal{A}$ *for which the associated* Ω*-sequence is given by* $H(n)$ *in Proposition* 4.8.

Moreover we get:

THEOREM 4.10. *The Eilenberg–Mac Lane* Ω*-algebra* $\mathbf{K}(\mathbb{Z}/2)$ *over* $\mathcal{A}_{(2)}$ *of products of Eilenberg–Mac Lane spaces* $K(\mathbb{Z}/2, n)$*,* $n \geq 0$*, is isomorphic to the* Ω*-algebra* $\mathcal{K}(\mathcal{A}_{(2)}, B, D)$ *over* $\mathcal{A}_{(2)}$ *given as in Corollary* 4.9 *by the Steenrod algebra* $\mathcal{A}_{(2)}$ *with the unstable structure* (B, D) *in Example* 4.4.

Proof. Using Proposition 4.7 we see that the Ω-sequences for $\mathbf{K}(\mathbb{Z}/2)$ and $\mathbf{K}(\mathcal{A}_{(2)}, B, D)$ are isomorphic. □

A similar result is available for the Eilenberg–Mac Lane Ω-algebra $\mathbf{K}(\mathbb{Z}/p)$, where p is a prime. For this we need the computation of Cartan [4] of $H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$ in terms of the Steenrod algebra $\mathcal{A}_{(p)}$.

For an unstable Hopf algebra we get by Corollary 4.9 the Ω-algebra $\mathcal{K}(\mathcal{A}, B, D)$ for which the Ω-cohomology

$$H^*_\Omega(\mathcal{K}(\mathcal{A}, B, D))$$

is defined; see [3]. If $\mathcal{A} = \mathcal{A}_{(2)}$ is the Steenrod algebra then we know that the track category associated to $\mathbf{K}(\mathbb{Z}/2)$ is classified by the *universal Toda bracket*

$$\langle \mathbf{K}(\mathbb{Z}/2) \rangle \in H^3_\Omega(\mathcal{K}(\mathcal{A}_{(2)}, B, D)).$$

Here the right-hand side has a completely algebraic description in terms of $\mathcal{A}_{(2)}$. The Toda bracket $\langle \mathbf{K}(\mathbb{Z}/2) \rangle$ however, is not understood algebraically. It is the purpose of this paper to prepare the ground for the computation of this class.

5. Models of $\mathbf{K}(\mathbb{Z}/2)$

Recall that a *theory with products* is a small category $\mathbf{T}$ with a final object $*$ and with products denoted by $A \times B$. A *model* of $\mathbf{T}$ is a product preserving functor $M : \mathbf{T} \to \mathbf{Set}$ which carries $*$ to a point. Let $\mathbf{model}(\mathbf{T})$ be the category of such models. Morphisms are natural transformations.

For each commutative ring R we introduced the Eilenberg–Mac Lane Ω-algebra $\mathbf{K}(R)$ which is the full subcategory of the homotopy category of pointed spaces consisting of products of Eilenberg–Mac Lane spaces $K(R, n)$, $n \geq 1$. Hence in particular $\mathbf{K}(R)$ is a theory with products and we obtain the category $\mathbf{model}(\mathbf{K}(R))$ of models of $\mathbf{K}(R)$.

The Steenrod algebra $\mathcal{A}_{(2)}$ is an unstable Hopf algebra with the unstable structure (B, D) in Example 4.4 such that the category $\mathbf{Alg}_0(\mathcal{A}_{(2)}, B, D)$ is the category of connected commutative graded algebras H over the Hopf algebra $\mathcal{A}_{(2)}$ for which

$$\mathrm{Sq}^i(x) = \begin{cases} 0, & i > |x|, \\ x \cdot x, & i = |x|. \end{cases}$$

THEOREM 5.1. *There is an equivalence of categories*

$$\mathbf{model}(\mathbf{K}(\mathbb{Z}/2)) = \mathbf{Alg}_0(\mathcal{A}_{(2)}, B, D).$$

Proof. By Theorem 4.10 we know that $\mathbf{K}(\mathbb{Z}/2)$ coincides with $\mathcal{K}(\mathcal{A}_{(2)}, B, D)$ and it is easy to see that models of $\mathcal{K}(\mathcal{A}_{(2)}, B, D)$ coincide with objects in $\mathbf{Alg}_0(\mathcal{A}_{(2)}, B, D)$. □

The theorem shows that models of $\mathbf{K}(\mathbb{Z}/2)$ have a surprisingly simple description by use of the Steenrod algebra $\mathcal{A}_{(2)}$. In general a similar result for models of

$\mathbf{K}(R)$ where R is a commutative ring is not known. For $R = \mathbb{Z}/p$ with p an odd prime we get the following result.

For an odd prime p we have the Steenrod algebra $\mathcal{A}_{(p)}$ (over $\mathbb{Z}/p$) generated by the Bockstein operator β of degree 1 and the reduced powers P^i of degree $2i(p-1)$, $i \geq 0$, with $\mathrm{P}^0 = 1$. Moreover $\mathcal{A}_{(p)}$ is a (coassociative and cocommutative) Hopf algebra with an unstable structure (B, D) such that the category $\mathbf{Alg}_0(\mathcal{A}_{(p)}, B, D)$ consists of augmented commutative graded algebras H over the Hopf algebra $\mathcal{A}_{(p)}$ with

$$\beta(x \cdot y) = (\beta x) \cdot y + (-1)^{|x|} x \cdot (\beta y),$$
$$\mathrm{P}^i(x) = \begin{cases} 0, & 2i > |x|, \\ x^p, & 2i = |x| \end{cases}$$

for $x, y \in H$. Similarly as in Theorem 5.1 we get

THEOREM 5.2. *For an odd prime p there is an equivalence of categories*

$$\mathbf{model}(\mathbf{K}(\mathbb{Z}/p)) = \mathbf{Alg}_0(\mathcal{A}_{(p)}, B, D).$$

References

1. Adams, J. F.: *Infinite Loop Spaces*, Annals of Math. Studies 90, Princeton, 1978.
2. Baues, H.-J.: The algebra of secondary cohomology operations, Preprint, 2001.
3. Baues, H.-J. and Jibladze, M.: Suspension and loop objects in theories and cohomology, Preprint.
4. Cartan, H.: Sur les groupes d'Eilenberg–Mac Lane. II, *Proc. Nat. Acad. Sci. U.S.A.* **40** (1954), 704–707.
5. Lawvere, F. W.: Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–873.
6. Serre, J.-P.: Cohomologie modulo 2 des complexes d'Eilenberg–Mac Lane, *Comment. Math. Helv.* **27** (1953), 198–232.
7. Steenrod, N. E. and Epstein, D. B. A.: *Cohomology Operations*, Annals of Math. Studies 50, Princeton, 1962.